Variational Interpolants: Riemannian Cubics and Lie Quadratics

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Context

Variational Interpolants

Reduction and Reconstruction

Lie Quadratics and Envelopes

Null Lie Quadratics

Classical Mechanics

Outline

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2. Variational Interpolants
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6. Classical Mechanics

This is about some mathematical research arising from a question, asked in 1988, by the mechanical engineers

G. Heinzinger (Berkeley) and B. Paden (Santa Barbara).

How best to interpolate finitely many rigid body configurations?

This was to be answered routinely and frequently, as a module for use in more complicated tasks in robotics (trajectory planning).

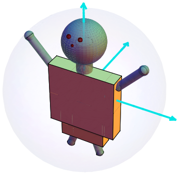
What's a Rigid Body Configuration?

A rigid body configuration is taken to be a reference point p together with a positively oriented orthonormal frame x, both fixed relative to the body and measured in an inertial frame.

So p is a point in Euclidean 3-space E^3, and x can be identified with the 3×3 determinant 1 orthogonal matrix whose columns make up the frame.

Natural cubic splines could be used for the reference points $p_i \in E^3$, since C^2 interpolants are needed in applications.

This leaves the problem of interpolating the rotations $x_i \in SO(3)$.



What's $SO(3)$?

Rotations are often represented by points on the unit 3-sphere S^3 in E^4 (unit quaternions).

Under this representation, a rotation corresponds an unordered pair $\{\pm y\}$ where $y \in S^3$. So $SO(3)$ is homeomorphic to the quotient space $\mathbb{R}P^3 := S^3/\pm$ (real projective 3-space).

The nicest (bi-invariant) metric on $SO(3)$ corresponds to the standard metric on S^3. This makes $SO(3)$ a Riemannian manifold whose geodesics (locally shortest curves) correspond to great circle arcs in S^3.

What's Best?

How best to interpolate finitely many rigid body configurations, given at times t_i?

Like piecewise-affine interpolation in E^3, piecewise-geodesic interpolation in the smooth 3-manifold $SO(3)$ is usually not C^1.

So it's reasonable to seek some sort of analogue of a cubic spline in $SO(3)$. This might be attempted in several ways.

Interpolation in Ambient Space

Although $SO(3) \subset E^9$, polynomials in E^9 joining distinct points in $SO(3)$ always move off into the ambient space. Similarly, there are no nonconstant polynomial curves in S^m.

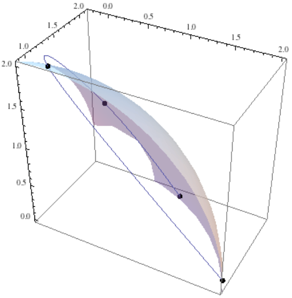


Figure: Cubic Spline Interpolant in E^3 of 4 Points on S^2

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Normalizations

This can be corrected by projecting back onto $SO(3)$. The fix works reasonably well for finely sampled data.

For coarse data one finds large distortions of distances.

This gives problems in applications, such as estimating configurations and angular velocities of moving rigid bodies (unreliable for coarse data).

Charts

An even more obvious (and more problematic) fix is to use a system of coordinate charts, interpolate in these using polynomial splines, then map back.

A system of book-keeping is needed to deal with overlaps of charts, leading to geometric distortion and other unattractive features (irreversibility, sudden switching).

Large numbers of charts are needed to reduce distortion, but this exacerbates the other book-keeping problems.

Geometrical Alternatives

- polynomial splines in homogeneous coordinates [B. Jüttler, M. Wagner]
- generalized Bézier constructions [K. Shoemake, LN, B. Ravani, F.C Park, Q.J. Ge, T. Popiel]
- generalized B-splines [LN, T. Popiel]
- variational interpolants [J. Kajiya, LN, G. Heinzinger, K. Krakowski, P. Crouch, M. Camarinha, F. Silva-Leite, M. Hofer, T. Popiel, M. Pauley, P. Schrader].

A lot can be said (there are many pros and cons) in favor of variational interpolants.

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With other applications in mind, it's convenient to work in a slightly more general setting: $x : \mathbb{R} \to M$ where M is some Riemannian manifold.

- $M = S^2$ (directed lines in E^2)
- $M = SO(3)$ (rotations of E^3, Heinzinger-Paden)
- $M = \mathbb{R}P^2$ (lines in E^2)
- $M = \mathbb{R}P^3$ (lines in E^3, same as $SO(3)$)
- $M = S^5$ (conics in E^2)
- $M = S^9$ (quadrics in E^3)
- $M = SL(2, \mathbb{R})$ (area-preserving linear transformations)
- any compact real algebraic set (mechanical linkages).

Geodesics

An m-manifold M is Riemannian when it's equipped with a smoothly varying positive-define inner product $< \ , \ >$ (the Riemannian metric) on tangent spaces. The associated norm is denoted $\| \ \|$.

Given a smooth curve $x : [t_0, t_1] \to M$ satisfying $x(t_0) = x_0, \ x(t_1) - x_1,$ we can form

$$J_1(x) \ := \ \int_{t_0}^{t_1} \|\dot{x}(t)\|^2 \ dt.$$

A critical point of J_1 is called a geodesic. Equivalently x is a geodesic when

$$\nabla_{d/dt}\dot{x} \ = \ \mathbf{0}$$

($\nabla_{d/dt}$ the covariant derivative given by the metric.)

- Geodesics in E^m are line segments.
- Geodesics in S^m are great-circle arcs.
- Geodesics in $SO(m)$ have the form

$$x(t) = \exp(tv)x_0$$

where v is a constant skew-symmetric $m \times m$ matrix.

Geodesics in a Riemannian manifold M are generalized line segments.

Riemannian cubics are the first higher order geodesics.

Riemannian Cubics

A smooth curve $x : \mathbb{R} \to M$ is a Riemannian cubic when it is a critical point of

$$J_2(x) \; = \; \int_{t_0}^{t_1} \|\nabla_{d/dt}\dot{x}\|^2 \; dt$$

among all curves with $x(t_0), \; \dot{x}(t_0), \; x(t_1), \; \dot{x}(t_1)$ prescribed.

Equivalently x is a Riemannian cubic when

$$\nabla^3_{d/dt}\dot{x} + R_{x(t)}(\nabla_{d/dt}\dot{x}, \dot{x})\dot{x} \; = \; \mathbf{0} \qquad (1)$$

where R is Riemannian curvature. For $M = S^m$, $R(u, v)w =< v, w > u - < u, w > v$.

- Riemannian cubics in E^m are cubic polynomials.
- Riemannian cubics in S^m are solutions of

$$\frac{d}{dt}\left(\frac{d^3x}{dt^3} + 2 < \dot{x}, \dot{x} > \dot{x}\right) = \lambda(t)x(t) \qquad (2)$$

where

$$\lambda(t) := -4 < \frac{d^3x}{dt^3}, \dot{x} > -3 < \ddot{x}, \ddot{x} > -2 < \dot{x}, \dot{x} >^2$$

(Except for $m = 1$ (trivial), and planar circles (periodic), closed-form solutions are unknown. Solving (2) for $m = 3$ would give all solutions for all m.)

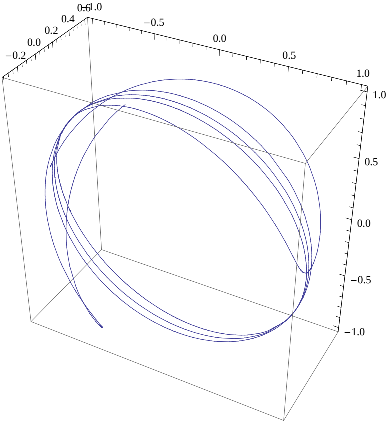

Figure: a Riemannian Cubic in S^2

A natural Riemannian cubic spline interpolating $x_0, x_1, \ldots, x_n$ at $t_0, t_1, \ldots, t_n$ is the C^2 track-sum of Riemannian cubics satisfying the interpolation conditions, together with

$$\nabla_{d/dt}\dot{x}(t_0) \;=\; \mathbf{0}, \quad \nabla_{d/dt}\dot{x}(t_n) \;=\; \mathbf{0}.$$

- A natural Riemannian cubic spline in E^m is a natural cubic spline in the sense of classical approximation theory.
- Natural Riemannian cubic splines are usually not calculated from the Euler-Lagrange equation (1). Instead there are efficient numerical methods using SQP [M. Kobilarov, J. Marsden, LN].
- Mathematical properties of Riemannian cubics are found by reducing (1) to a pair of ODEs for x and a curve V in a vector space.

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Lie Reduction

Let M be a simple connected Lie group G. Its Lie algebra $\mathcal{G}$ is the real vector space of tangents to G at the identity, equipped with an anticommutative bilinear transformation (Lie bracket) $[\ ,\ ] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ satisfying (Jacobi identity)

$$[[u, v,], w] + [[w, u], v] + [[v, w], u] \ = \ \mathbf{0} \quad \text{for all } u, v, w \in \mathcal{G}.$$

For a matrix Lie group, the Lie reduction $V : \mathbb{R} \to \mathcal{G}$ of $x : \mathbb{R} \to G$ is defined by

$$V(t) \ := \ x(t)^{-1}\dot{x}(t) \ \in \ \mathcal{G}.$$

The Lie reduction of a Riemannian cubic is a Lie quadratic.

The Lie algebra of S^1 is $\mathbb{R}$, with $[\ ,\ ]$ identically 0.

The Lie algebra $so(3)$ of $SO(3)$ is the 3-dimensional vector space of skew-symmetric 3×3 matrices, with

$$[u, v] \ = \ uv - vu \quad \text{(matrix multiplications)}.$$

$so(3)$ is Lie-isomorphic to Euclidean 3-space E^3 with

$$[u, v] \ := \ u \times v \quad \text{(cross-product)}.$$

Theorem (LN, G. Heinzinger, B. Paden)

x is a Riemannian cubic $\iff$ V satisfies

$$\ddot{V}(t) = [\dot{V}(t), V(t)] + C \qquad (3)$$

where $C \in \mathcal{G}$ is constant. $\square$

Notice

- (3) is a quadratic ODE for a curve V in a vector space, whereas (1) is a 4th order nonlinear ODE in a curved space.
- x can be found from V by solving the linear reconstruction ODE with variable coefficients

$$\dot{x}(t) = x(t)V(t) \qquad (4)$$

furthermore ...

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The Reconstruction ODE is Integrable

Theorem (LN 2003)

Let x *be a generic Riemannian cubic. Then (4) can be solved explicitly in terms of* V*, derivatives, algebraic operations, and integrals.* $\square$

Analogous results hold when x is

- a cubic in tension
- an elastic curve
- a higher order geodesic of order > 2.

Algebraic Integrability for Null Lie Quadratics

The Lie quadratic V (and the corresponding Riemannian cubic x) are said to be null when $C = \mathbf{0}$.

Theorem (LN2003)

For a generic null Lie quadratic V, the associated null Riemannian cubic x can be found algebraically in terms of V, $\dot{V}$. □

Example $M = SO(3)$

For a canonical null Riemannian cubic x in $SO(3)$,

$$x(t) = \frac{1}{\sqrt{c}} \left[\ V(t) - t\dot{V}(t) \quad \sqrt{c}\dot{V}(t) \quad -\dot{V}(t) \times V(t) \ \right]$$

where $V : \mathbb{R} \to \mathcal{G} \cong E^3$ is the Lie reduction, $c > 0$ is constant, and $\times$ is the cross-product.

Other cases where x can be found algebraically in terms of V, $\dot{V}$ are found using a connection between Lie quadratics and the theory of envelopes.

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Lie Quadratics and Symmetric Spaces

If x is a Riemannian cubic in a totally geodesic submanifold of a bi-invariant Lie group G then it is also a Riemannian cubic in G. So x reduces to a Lie quadratic.

So a Riemannian cubic x in a symmetric space M also reduces to a Lie quadratic.

In particular this holds for $M = S^2 \subset S^3 = G$.

Example $M = S^2 \subset G = S^3$

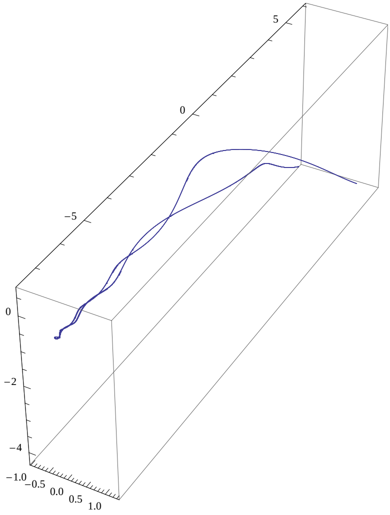


Figure: Lie quadratic of a Riemannian cubic in S^2

- There is a one-to-one correspondence

$$y = [\pm x] \;\mapsto\; S^2 \cap x^\perp \subset S^2 \;\mapsto\; \hat{y} \subset \mathbb{R}P^2$$

between points and geodesics of $\mathbb{R}P^2$.

- So a curve $y = [\pm x] : \mathbb{R} \to \mathbb{R}P^2$ corresponds to a family $t \mapsto \hat{y}(t)$ of geodesics in $\mathbb{R}P^2$
- whose envelope y^* is given by

$$y^*(t) \;=\; [\frac{W(t)}{\|W(t)\|}] \;\in\; \mathbb{R}P^2 \quad (W(t) \;:=\; \dot{x}(t) \times x(t))$$

- and $y^{**} = y$.
- So y (x up to sign) is expressible in terms of y^*,
- hence in terms of W.

As noted before, S^2 is totally geodesic in $G = S^3$, whose Lie algebra is E^3, but (unlike $SO(3)$) with

$$[v, w] := 2v \times w.$$

So a Lie quadratic W for a cubic x in S^3 (hence for a cubic in S^2) satisfies

$$\ddot{W} = 2\dot{W} \times W + C \tag{5}$$

namely

$V(t) := W(2t)$ is a Lie quadratic for a cubic in $SO(3)$ (non-null).

Envelopes and Integrals

The reduction of a Riemannian cubic x in $S^2 \subset S^3$ is $W := \dot{x} \times x$. Therefore:

- the 4th-order nonlinear ODE defining the cubic x in S^2 reduces to the 2nd order quadratic ODE (5) for W in E^3
- Lie quadratics V of cubics in $\mathbb{R}P^2 \subset SO(3)$ are non-null and their Riemannian cubics can be expressed algebraically in terms of V, $\dot{V}$.

Similar things hold for Riemannian cubics in the 2-sheeted hyperboloid $H^2 \subset SL(2,\mathbb{R})$.

Example: Silhouettes

Even when non-affine constraints can be easily avoided, Riemannian cubics are sometimes preferable to cubic polynomials. Silhouettes of a convex planar object give a finite number of directed planar lines x_i at times t_i. The problem is to estimate a curve whose tangent lines at t_i are x_i.

If we had a continuum of lines we'd just take the envelope.

To obtain a continuum of lines we can interpolate the given data (t_i, x_i). Then we take the envelope.

How to interpolate the given lines?

Interpolating 4 Lines in E^2

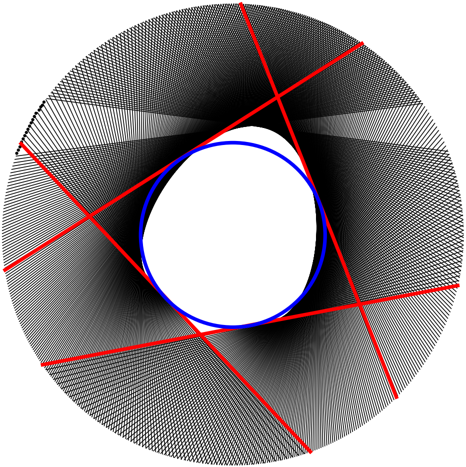
Method A: A directed line x_i not passing through $(0,0)$ has the form

$$\{a \in E^2 :< b_i, a > \; = -1\}$$

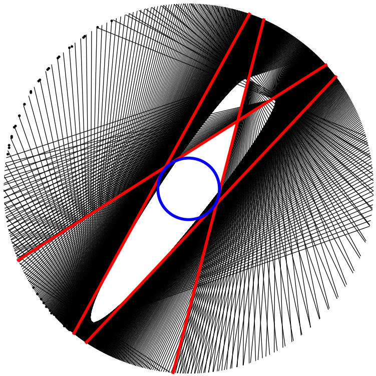
where $b_i \in E^2$ is constant and unique. Given 4 such lines at given times, interpolate the b_i by a cubic polynomial, to get a continuum of lines.

Method B: Form $\tilde{b}_i := \frac{(b_i,1)}{\sqrt{\|b_i\|^2+1}} \in S^2$ and interpolate these by a Riemannian cubic in S^2.

Method A



Method A: 4 slightly different lines



Method B (the Riemannian cubic) gives the circle in both cases.

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We've seen that nothing is lost by reducing the 4th order nonlinear system (1) to the second-order ODE

$$\ddot{V} = [\dot{V}, V] + C$$

for a curve V (the Lie quadratic) in a vector space (the Lie algebra).

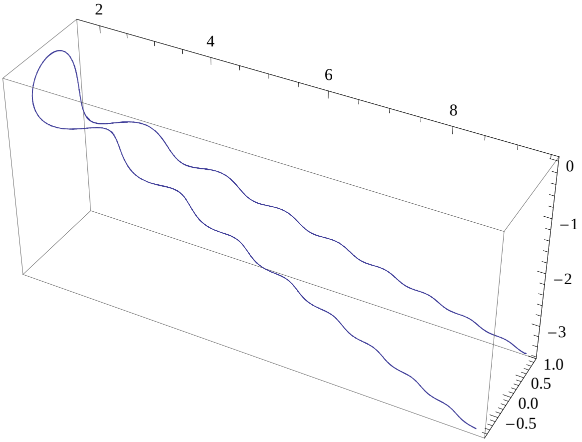
The theory of Lie quadratics is simplest when $C = \mathbf{0}$ (V is null).

Null Lie quadratics V can be rewritten in canonical form:

- $\|V((t)\|^2 \;=\; t^2 + d_0$ for some $d_0 \geq 0$
- $\|\dot{V}(t)\| \;=\; 1$
- $< \ddot{V}(t), \dot{V}(t) > \;=\; 0$
- $\|\ddot{V}(t)\|^2 \;=\; c$ for some c
- there exist axes, depending continuously on c,

$$\alpha_{\pm} \;:=\; \lim_{t \to \pm\infty} \frac{V(t)}{\|V(t)\|}.$$

Example: a Null Lie Quadratic in E^3



Null Lie Quadratics in E^3

A (canonical) null Lie quadratic $V : \mathbb{R} \to E^3$ has $c = d_0$,

- constant curvature $\sqrt{d_0}$
- linear torsion $-t$
- internal symmetry

and satisfies the 3rd-order linear ODE

$$\frac{d^3 V}{dt^3} = tV - (t^2 + d_0)\dot{V}.$$

Definitions

- $f(t) := \sqrt{t^2 + c}$
- $g(t) := \frac{1}{2}(tf(t) + c\ln(t + f(t)) - (c/2)\ln c)$
- $I : \alpha^\perp \to \alpha^\perp$ by $I(v) := [\alpha, v]$ where $\alpha := \alpha_\pm$ according as $t \to \pm\infty$.

Then

- $I^2 = -\mathbf{1}$
- so I defines a complex structure on $\alpha^\perp$.

Asymptotics of V, $\dot{V}$, $\ddot{V}$ as $t \to \infty$

Theorem (LN)

For a canonical null Lie quadratic in E^3,

$$
\begin{aligned}
V(t) &= f\alpha - \frac{\sqrt{c}}{f(t)^2}\exp(-g(t)I)\beta + O(t^{-3}) \\
\dot{V}(t) &= \frac{t}{f(t)}\alpha + \frac{\sqrt{c}}{f(t)} I \exp(-g(t)I)\beta + O(t^{-2}) \\
\ddot{V}(t) &= \sqrt{c}\exp(-g(t)I)\beta + O(t^{-1})
\end{aligned}
$$

where $\beta \in S^2$. □

Ongoing related work includes

- higher order asymptotics
- asymptotics for null Lie quadratics in $sl(2, \mathbb{R})$ (richer)
- self-similarity and asymptotics.

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There are connections between the theory of Lie quadratics and classical mechanics.

An example is the (nonholonomic) motion of a spherically symmetric ball rolling on an inclined plane.

- The angular velocity of the ball in body coordinates is a null Lie quadratic in E^3.
- The ball's trajectory can be written in terms of the associated null Riemannian cubic $x : \mathbb{R} \rightarrow SO(3)$.

Zorba is strapped inside a plastic sphere (the zorb) ....

… and rolled down a hill

For a planar hillside

- Zorba's angular momentum Ω in in body coordinates is a null Lie quadratic in E^3, and
- his configuration x in space is given by the corresponding Riemannian cubic.
- Using the previous theorem, and integrability of the reconstruction ODE, asymptotically accurate closed form expressions can be given for both.

When the hillside is curved the null Lie quadratic and associated Riemannian cubic can be used to interpolate Ω and x.

Conclusion

Riemannian cubics and Lie quadratics arose in mathematical studies of an elementary problem in approximation theory, where interpolants are required to satisfy nonaffine (usually quadratic) constraints.

We have touched on some of the mathematical theory, and looked at the simplest of several applications in classical mechanics. A little has been said (envelopes) about the (richer) theory of non-null Lie quadratics.

This is an active area of research, with promising applications in approximation theory and mechanics.

I thank the conference organizers for arranging this very successful meeting,

and for the invitation to speak.

Thank you for coming to my talk.

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ZORBA: THE MOVIE