

# Duality and Riemannian Cubics

Lyle Noakes

School of Mathematics and Statistics  
The University of Western Australia  
Nedlands, WA 6009  
Perth, AUSTRALIA  
lyle@maths.uwa.edu.au

*To Charles Micchelli, with warm greetings and deep respect, on his 60th birthday.*

## Abstract

*Riemannian cubics* are curves used for interpolation in Riemannian manifolds. Applications in trajectory planning for rigid body motion emphasise the group  $SO(3)$  of rotations of Euclidean 3-space. It is known that a Riemannian cubic in a Lie group  $G$  with bi-invariant Riemannian metric defines a *Lie quadratic*  $V$  in the Lie algebra, and satisfies a *linking equation*. Results of the present paper include explicit solutions of the linking equation by quadrature in terms of the Lie quadratic, when  $G$  is  $SO(3)$  or  $SO(1,2)$ . In some cases we are able to give examples where the Lie quadratic is also given in closed form. A basic tool for constructing solutions is a new duality theorem. Duality is also used to study asymptotics of differential equations of the form

$$\dot{x}(t) = (\beta_0 + t\beta_1)x(t),$$

where  $\beta_0, \beta_1$  are skew-symmetric  $3 \times 3$  matrices, and  $x : \mathbb{R} \rightarrow SO(3)$ . This is done by showing that the dual of  $\beta_0 + t\beta_1$  is a null Lie quadratic. Then results on asymptotics of  $x$  follow from known properties of null Lie quadratics.

## 1 Introduction

Before stating our main results we briefly review some background. For  $m \geq 1$ , let  $M$  be a  $C^\infty$   $m$ -manifold equipped with an everywhere nondegenerate symmetric covariant tensor field  $\langle \cdot, \cdot \rangle$  of order 2. In the case where  $\langle \cdot, \cdot \rangle$  is everywhere positive definite,  $M$  is a Riemannian manifold. Let  $\nabla$  be the Levi-Civita covariant derivative on  $M$  associated with  $\langle \cdot, \cdot \rangle$ , and for  $i = 0, 1$ , choose  $y_i \in M$  and  $v_i \in (TM)_{y_i}$ . Fix  $T > 0$  and let  $\mathcal{C}_{v_0, v_1}$  be the space of  $C^\infty$  curves  $x : [0, T] \rightarrow M$  satisfying

$$x(iT) = y_i \quad \text{and} \quad \dot{x}(iT) = v_i,$$

for  $i = 0, 1$ . Define a functional  $\Phi : \mathcal{C}_{v_0, v_1} \rightarrow [0, \infty)$  by

$$\Phi(x) = \int_0^T \langle \nabla_{d/dt} \dot{x}, \nabla_{d/dt} \dot{x} \rangle dt,$$

where  $\langle \cdot, \cdot \rangle$  is the Riemannian inner product, and  $x \in \mathcal{C}_{v_0, v_1}$ . In Euclidean  $m$ -space  $E^m$ , cubic polynomials minimize  $\Phi$ : the *variation diminishing property* of cubic spline approximation [1]. Let  $R$  denote the Riemannian curvature tensor field [14] associated with  $\nabla$ . The proof of the following result in [12], [23] extends naturally from the Riemannian case to the present situation, where  $\langle \cdot, \cdot \rangle$  is permitted to be indefinite.

**Theorem 1**  $x \in \mathcal{C}_{v_0, v_1}$  is a critical point of  $\Phi$  if and only if, for all  $t \in [0, T]$ ,

$$\nabla_{d/dt}^3 \dot{x} + R_{x(t)}(\nabla_{d/dt} \dot{x}, \dot{x}(t))\dot{x}(t) = \mathbf{0}. \tag{1}$$

□

Let  $S \subseteq \mathbb{R}$  be an interval (possibly infinite). A curve  $x : S \rightarrow M$  satisfying (1) is called a *Riemannian cubic* in  $M$  or just a *cubic*. Conditions for a cubic to minimise  $\Phi$  are studied in [6].

Riemannian cubics have applications to the important engineering problem of interpolation in  $SO(3)$ , but the present paper is a self-contained study of their mathematical theory. For background, including alternatives to cubics, we refer to [16], [7], [9], [10], [24], [29], [30], [31], [32], [13], [5], [26], [18], [19], [20], [21], [22], [8], [23], [12]. The present paper has benefited from conversations with Professors Jerrold Marsden, Peter Crouch, Matthias Kawski, Ronald Goldman, Rida Farouki and Bahram Ravani. I am also grateful to the referees for constructive and thoughtful comments.

From now on we take  $M$  to be a Lie group with  $\langle \cdot, \cdot \rangle$  bi-invariant.

**Example 1** For  $n \geq 2$  let  $E^n$  be Euclidean  $n$ -space  $\mathbb{R}^n$  with the Euclidean inner product  $\langle \cdot, \cdot \rangle_E$ . Let  $SO(n)$  be the Lie group of orthogonal  $n \times n$  matrices with determinant 1. The Lie algebra  $so(n)$  is the space of  $n \times n$  skew-symmetric real matrices, with  $[V, W] = VW - WV$ . Defining  $[v, w]$  as the vector product  $v \times w$  for  $v, w \in E^3$ ,  $E^3$  is also a Lie algebra, and

$$B_E : E^3 \rightarrow so(3), \quad \text{given by } B_E(v)(w) = v \times w,$$

is a Lie algebra isomorphism.

□

**Example 2** For  $n \geq 2$  let  $L^n$  be  $\mathbb{R}^n$  with the indefinite inner product  $\langle \cdot, \cdot \rangle_L$  given by  $\langle v, w \rangle_L = \langle v, Jw \rangle_E$ , where  $J$  is the  $n \times n$  diagonal matrix whose entries along the diagonal are  $-1, 1, 1, \dots, 1$ . Let  $SO(1, n-1)$  be the Lie group of matrices  $A$  preserving  $\langle \cdot, \cdot \rangle_L$ , namely  $A^T J A = J$ . The Lie algebra is the space of  $n \times n$  matrices  $V$  for which  $V^T J + J V = \mathbf{0}$ , with the usual Lie bracket. Defining  $[v, w] = J(v \times w)$ ,  $L^3$  is a Lie algebra, and

$$B_L : L^3 \rightarrow so(1, 2), \quad \text{given by } B_L(v)(w) = J B_E(v) = [v, w],$$

is a Lie algebra isomorphism.

□

The case where  $M$  is the group  $SO(3)$  of rotations of  $E^3$  is important in applications in computer graphics, and trajectory planning for rigid body motion, as in [3], [11], [26], and [2], [4], [25], [27], [28] respectively. Cubics in  $SO(1, 2)$  have not been used in this context, so far as we are aware: their theory is equally interesting, different, and more intricate than for  $SO(3)$ .

Theorem 2 of [17] when  $M$  is a Lie group  $G$  with bi-invariant Riemannian metric (1) reduces to a second-order system

$$\ddot{V} = [\dot{V}, V] + C \tag{2}$$

of ordinary differential equations for a curve  $V : S \rightarrow \mathcal{G}$  in the Lie algebra  $\mathcal{G}$  of  $G$ , together with a *linking equation* of the form

$$\dot{x}(t) = dL(x(t))_1 V(t), \tag{3}$$

where  $L(g) : G \rightarrow G$  denotes left multiplication by  $g \in G$ . The curve  $V$  is called a *Lie quadratic*, and said to be *null* when  $C = \mathbf{0}$ . Symmetries and asymptotics of null Lie quadratics in  $so(3)$  are studied in [17], together with properties of associated Riemannian cubics in  $SO(3)$ . In §2 we briefly recall and slightly extend some known results on Lie reductions of Riemannian cubics. Then in subsequent sections we prove our main results, as follows.

- Nonconstant null Lie quadratics  $V : \mathbb{R} \rightarrow \mathcal{G}$  are unbounded when  $\langle \cdot, \cdot \rangle$  is positive definite (Proposition 1), and also when  $\mathcal{G}$  is  $L^3$  with the indefinite inner product  $\langle \cdot, \cdot \rangle_L$  of Example 2 (Theorem 3). Non-null Lie quadratics may be bounded however (Examples 8, 10).
- Corresponding to a Lie quadratic  $V : S \rightarrow \mathcal{G}$ , where  $\mathcal{G}$  is the Lie algebra of a Lie group  $G$ , there is a *dual Lie quadratic*  $V^* : S \rightarrow \mathcal{G}$ . The dual of a null Lie quadratic is affine, and  $Ad(x(t))\ddot{V}(t)$  is the constant  $C^*$  of the dual  $V^*$  of  $V$  (Theorem 4).
- Applying duality to affine Lie quadratics, in combination with results from [17], we obtain asymptotic properties of solutions of the differential equation  $\dot{x}(t) = x(t)(\beta_0 + t\beta_1)$ , for  $x : \mathbb{R} \rightarrow SO(3)$ , where  $\beta_0, \beta_1$  are constant skew-symmetric real  $3 \times 3$  matrices (Corollary 2 and Example 11).
- Applying duality to nonaffine Lie quadratics  $V$  in  $so(3)$ , solutions  $x$  to the linking equation (3) are found in terms of  $V$  and quadratures (Theorem 5). When  $V$  is null there is no need for quadratures.
- Analogous constructions and arguments are made for Lie quadratics in  $so(1, 2)$ , depending on whether  $V$  is null (Theorem 6), and whether  $\ddot{V}$  is spacelike (Theorem 7), timelike (Theorem 8) or lightlike (Theorem 9).

## 2 Reduction of Cubics to Lie Quadratics

Let  $\mathcal{G}$  be a finite-dimensional real Lie algebra. For  $S \subseteq \mathbb{R}$  a nonempty open interval, a  $C^2$  map  $V : S \rightarrow \mathcal{G}$  is a *Lie quadratic* [17] when, for some *constant*  $C \in \mathcal{G}$  and all  $t \in S$ ,

$$\ddot{V}(t) = [\dot{V}(t), V(t)] + C. \tag{4}$$

When  $C = \mathbf{0}$   $V$  is said to be *null*. Equivalently,  $V$  is a Lie quadratic when

$$\frac{d}{dt}\ddot{V} = [\ddot{V}, V]. \quad (5)$$

**Example 3** For  $\beta_0, \beta_1, \beta_2 \in \mathcal{G}$ , and  $S = \mathbb{R}$ ,  $t \mapsto \beta_0 + t\beta_1 + t^2\beta_2$  is a Lie quadratic when

$$[\beta_2, \beta_0] = [\beta_2, \beta_1] = \mathbf{0}, \quad \text{and then } C = [\beta_0, \beta_1] + 2\beta_2.$$

In particular, let  $V$  be affine, namely  $V(t) = \beta_0 + t\beta_1$  where  $\beta_0, \beta_1 \in \mathcal{G}$ . Then  $V$  is a Lie quadratic with constant  $C = [\beta_0, \beta_1]$ .

□

If  $t \mapsto V(t)$  is a Lie quadratic in  $\mathcal{G}$  with constant  $C$ , then

- $V$  is  $C^\infty$ ,
- $t \mapsto V(t + a_0)$  is a Lie quadratic in  $\mathcal{G}$  with constant  $C$ , for any  $a_0 \in \mathbb{R}$ ,
- $s \mapsto a_1 V(a_1 s)$  is a Lie quadratic in  $\mathcal{G}$  with constant  $a_1^3 C$ , for any nonzero  $a_1 \in \mathbb{R}$ ,
- $B \circ V$  is a Lie quadratic in  $\mathcal{H}$ , for any Lie algebra homomorphism  $B : \mathcal{G} \rightarrow \mathcal{H}$ .

Let  $G$  be a finite-dimensional Lie group, with identity  $\mathbf{1}$  and Lie algebra  $\mathcal{G} = TG_{\mathbf{1}}$ . Denote the group of Lie algebra automorphisms of  $\mathcal{G}$  by  $Aut(\mathcal{G})$ . For  $g \in G$  define left and right translations  $L(g), R(g) : G \rightarrow G$  by  $L(g)(h) = gh$ ,  $R(g)(h) = hg$ , and an inner automorphism  $I(g) = L(g) \circ R(g^{-1})$ . Then the derivative at  $\mathbf{1}$  of  $I(g)$  is a Lie algebra automorphism  $Ad(g)$  of  $\mathcal{G}$ .

**Example 4** If  $t \mapsto V(t)$  is a Lie quadratic with constant  $C$  then  $Ad(g) \circ V$  is a Lie quadratic with constant  $Ad(g)C$ .

□

The assignment  $g \mapsto Ad(g)$  is a group homomorphism  $Ad : G \rightarrow Aut(\mathcal{G})$ , whose derivative  $ad$  at  $\mathbf{1}$  is given by  $ad(v)w = [v, w]$  for  $v, w \in \mathcal{G}$ . A symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathcal{G}$  is *ad-invariant* when

$$\langle ad(u)v, w \rangle + \langle v, ad(u)w \rangle = 0 \quad \text{for all } u, v, w \in \mathcal{G}. \quad (6)$$

For instance when  $\mathcal{G}$  is abelian any symmetric bilinear form is *ad-invariant*.

**Example 5** Let  $\langle \cdot, \cdot \rangle$  be a bi-invariant Riemannian metric on a Lie group  $G$ . The restriction of  $\langle \cdot, \cdot \rangle$  to the Lie algebra  $\mathcal{G}$  is *ad-invariant*.

□

**Example 6** Let  $\langle \cdot, \cdot \rangle$  be the Killing form [14] on an arbitrary finite-dimensional real Lie algebra  $\mathcal{G}$ . Then  $\langle \cdot, \cdot \rangle$  is *ad-invariant*, and is nondegenerate if and only if  $\mathcal{G}$  is semisimple.

□

**Theorem 2** Let  $\langle \cdot, \cdot \rangle$  be an ad-invariant symmetric bilinear form on  $\mathcal{G}$ . Given a Lie quadratic  $V : S \rightarrow \mathcal{G}$ , there are constants  $b, c \in \mathbb{R}$  such that

$$\frac{d^2}{dt^2} \langle V(t), V(t) \rangle = 6 \langle C, V(t) \rangle + 2b, \quad (7)$$

$$\langle \dot{V}(t), \dot{V}(t) \rangle = 2 \langle C, V(t) \rangle + b, \quad (8)$$

$$\langle \ddot{V}(t), \ddot{V}(t) \rangle = c. \quad (9)$$

**Proof:** By (4), (6),  $\langle \ddot{V}(t), \dot{V}(t) \rangle = \langle C, \dot{V}(t) \rangle$ , and (8) follows by symmetry of  $\langle \cdot, \cdot \rangle$ . We have

$$\frac{d^2}{dt^2} \langle V(t), V(t) \rangle = 2 \langle \ddot{V}(t), V(t) \rangle + 2 \langle \dot{V}(t), \dot{V}(t) \rangle.$$

Again by (4), (6),  $\langle \ddot{V}(t), V(t) \rangle = \langle C, V(t) \rangle$ , and (7) then follows from (8). Finally,  $\frac{d}{dt} \langle \ddot{V}(t), \dot{V}(t) \rangle = 2 \langle [\ddot{V}(t), \dot{V}(t)], \ddot{V}(t) \rangle = 0$  by (4), (6).

□

**Corollary 1** Let  $V : S \rightarrow \mathcal{G}$  be a null Lie quadratic, where  $\mathcal{G}$  has an ad-invariant symmetric bilinear form  $\langle \cdot, \cdot \rangle$ . For some  $d_0, d_1 \in \mathbb{R}$ , and all  $t \in S$ ,  $\langle V(t), V(t) \rangle = bt^2 + d_1t + d_0$ .

□

### 3 Boundedness

A Lie quadratic  $V : S \rightarrow \mathcal{G}$  is said to be *bounded* when it is bounded with respect to some positive definite norm on  $\mathcal{G}$ . Because Lie quadratics are  $C^\infty$ , if  $S$  is bounded and  $V$  extends to a Lie quadratic whose domain contains the closure of  $S$ , then  $V$  is bounded.

**Proposition 1** Let  $V : S \rightarrow \mathcal{G}$  be a nonconstant null Lie quadratic, where  $S$  is unbounded and  $\mathcal{G}$  has an ad-invariant definite symmetric bilinear form. Then  $V$  is unbounded.

**Proof:** Because  $V$  is nonconstant and by (8),  $b \neq 0$ . So by Corollary 1,  $V$  is unbounded.

□

**Example 7** Let  $\mathcal{G}$  be semisimple with definite Killing form, for instance  $\mathcal{G} = so(n)$  with  $n \geq 3$ .

□

**Example 8** The Euclidean inner product  $\langle \cdot, \cdot \rangle_E$  is ad-invariant for the Lie algebra  $E^3$ . So a nonconstant null Lie quadratic in  $E^3$  defined on an unbounded interval is unbounded. Lie quadratics in  $E^3$  correspond to Lie quadratics in  $so(3)$ , by means of the Lie algebra isomorphism  $B_E : E^3 \rightarrow so(3)$ . In [17] nonconstant null Lie quadratics  $V : \mathbb{R} \rightarrow so(3)$  are shown to have constant curvature, and as  $t \rightarrow \pm\infty$ ,  $V$  becomes  $C^0$  close to a pair of rays  $\alpha_\pm$  through  $\mathbf{0}$ . The torsion of  $V$  turns out to be linear in  $t$ , and so  $V$  twists in opposite directions around  $\alpha_\pm$  as  $t \rightarrow \pm\infty$ . Nonconstant null Lie quadratics in  $so(3)$  have interesting symmetries: for some  $t_0 \in \mathbb{R}$ ,  $V(t) = A \circ V(2t_0 - t)$  where  $A$  is reflection in the hyperplane orthogonal to  $\dot{V}(t_0)$ . In particular  $A(\alpha_\pm) = \alpha_\mp$ . On the other hand, define a non-null Lie quadratic  $V_E : \mathbb{R} \rightarrow E^3$  by

$$V_E(t) = \begin{bmatrix} -1 & \cos t & \sin t \end{bmatrix}^{\mathbf{T}},$$

where  $\mathbf{T}$  means matrix transposition. Then  $V_E$  is periodic, and has constant  $C_E = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\mathbf{T}}$ .  
 $\square$

**Theorem 3** Let  $V : \mathbb{R} \rightarrow L^3$  be a nonconstant null Lie quadratic. Then  $V$  is unbounded. Write  $V(t) = \begin{bmatrix} v_1(t) & v_2(t) & v_3(t) \end{bmatrix}^{\mathbf{T}}$ . If  $b \leq 0$ , then  $v_1$  is unbounded.

**Proof:** The Lorentz inner product  $\langle \cdot, \cdot \rangle_L$ , is  $ad$ -invariant. By Corollary 1, if  $b > 0$   $V$  is unbounded. For  $b \leq 0$  suppose  $v_1$  is bounded. Then  $b = d_1 = 0$ , and

$$-v_1(t)^2 + v_2(t)^2 + v_3(t)^2 = d_0. \quad (10)$$

If  $\dot{v}_1(t_0) = 0$  for any  $t_0$  then  $\dot{V}(t_0) = \mathbf{0}$  by (8), and  $V$  is constant, by Picard uniqueness for ordinary differential equations. So  $\dot{v}_1(t) \neq 0$  for all  $t \in \mathbb{R}$ , and

$$\dot{v}_j(t) = \dot{v}_1(t)u_j(t) \quad \text{for } j = 2, 3, \quad \text{where } u_2(t)^2 + u_3(t)^2 = 1. \quad (11)$$

Differentiating (10) and substituting for the  $\dot{v}_j$  using (11),  $-2\dot{v}_1(v_1 - v_2u_2 - v_3u_3) = 0$ , and

$$v_1(t) = v_2(t)u_2(t) + v_3(t)u_3(t). \quad (12)$$

Eliminating  $v_1$  between (12), (10),  $(v_2(t)u_3(t) - v_3(t)u_2(t))^2 = d_0$ . So  $d_0 = \delta_0^2$ , where

$$v_2(t)u_3(t) - v_3(t)u_2(t) = \delta_0.$$

By (4), (11),  $-\ddot{v}_1 = \dot{v}_2v_3 - \dot{v}_3v_2 = -\delta_0\dot{v}_1$ . So, for some  $c_0 \in \mathbb{R}$ ,  $\dot{v}_1 = \delta_0v_1 + c_0$  where  $(\delta_0, c_0) \neq \mathbf{0}$ . If  $\delta_0 = 0$ ,  $v_1$  is unbounded. So  $\delta_0 \neq 0$  and, for some  $b_0$ ,

$$v_1(t) = -\frac{c_0}{\delta_0} + b_0e^{\delta_0 t}.$$

Because  $v_1$  is nonconstant  $b_0 \neq 0$ , contradicting the hypothesis that  $v_1$  is bounded.  
 $\square$

**Example 9** Given  $a_0, b_0, c_0 \in \mathbb{R}$  with  $(b_0, c_0) \neq \mathbf{0}$ , define a nonconstant null Lie quadratic  $V : \mathbb{R} \rightarrow so(1, 2)$  by  $v_1(t) = a_0, v_2(t) = b_0t, v_3(t) = c_0t$ . Then  $b = b_0^2 + c_0^2 > 0$  and  $v_1$  is constant. So the condition on  $b$  in Theorem 3 is needed.  
 $\square$

**Example 10**  $t \mapsto V_L(t) = \begin{bmatrix} -1 & \cos t & \sin t \end{bmatrix}^{\mathbf{T}}$  is a periodic Lie quadratic in  $L^3$  with constant  $C_L = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^{\mathbf{T}}$ . So  $V$  needs to be null in Theorem 3.  
 $\square$

## 4 Duality

Fix  $t_0 \in S$ , where  $S \subseteq \mathbb{R}$  is an open interval. Let  $G$  be a Lie group, with identity  $\mathbf{1}$  and Lie algebra  $\mathcal{G} = TG_{\mathbf{1}}$ . Given a Lie quadratic  $V : S \rightarrow \mathcal{G}$ , call  $x : S \rightarrow G$  the *associated Riemannian cubic* when  $x(t_0) = \mathbf{1}$  and, for all  $t \in S$ ,

$$\dot{x}(t) = dL(x(t))_{\mathbf{1}}V(t). \quad (13)$$

Of course the solution of (13) satisfying  $x(t_0) = x_0$  is given by premultiplying the associated Riemannian cubic by  $x_0$ . The *dual*  $V^* : S \rightarrow \mathcal{G}$  of  $V$  is defined by  $V^*(t) = -Ad(x(t))V(t)$ .

**Theorem 4**  $V^*$  is a Lie quadratic, with constant  $C^* = -\ddot{V}(t_0)$ , and associated Riemannian cubic  $x^*$  given by  $x^*(t) = x(t)^{-1}$ . For all  $t \in \mathbb{R}$ ,

- $V^{**}(t) = V(t)$ ,
- $\ddot{V}(t) = Ad(x(t)^{-1})\ddot{V}(t_0)$ ,
- $V$  is constant if and only if  $V^*$  is constant,
- $V$  is null if and only if  $V^*$  is affine.

**Proof:** From (13), and (4),

$$\begin{aligned} \dot{V}^*(t) &= -Ad(x(t))(ad(V(t))V(t) + \dot{V}(t)) = -Ad(x(t))\dot{V}(t), \\ \ddot{V}^*(t) &= -Ad(x(t))(ad(V(t))\dot{V}(t) + \ddot{V}(t)) = -Ad(x(t))C, \\ d^3V^*/dt^3 &= -Ad(x(t))(ad(V(t))C) = -Ad(x(t))[V(t), C] = [\ddot{V}^*(t), V^*(t)], \end{aligned}$$

so that  $V^*$  is a Lie quadratic. Setting  $x^*(t) = x(t)^{-1}$ ,  $x^*(t_0) = \mathbf{1}$  and

$$dL(x(t))_{x^*(t)}\dot{x}^*(t) + dR(x^*(t))_{x(t)}\dot{x}(t) = \mathbf{0}, \quad \text{namely}$$

$$\begin{aligned} \dot{x}^*(t) &= -dL(x(t))_{x^*(t)}^{-1} \circ dR(x^*(t))_{x(t)} \circ dL(x(t))_{\mathbf{1}}V(t) \\ &= -dL(x^*(t))_{\mathbf{1}} \circ Ad(x(t))V(t) = dL(x^*(t))_{\mathbf{1}}V^*(t). \end{aligned}$$

So  $x^*$  is the Lie quadratic associated with  $V^*$ , and  $V^{**} = V$ . Also

$$\ddot{V}^*(t) = [\dot{V}^*(t), V^*(t)] - Ad(x(t))\ddot{V}(t) \quad \text{and so} \quad Ad(x(t))\ddot{V}(t) = -C^*.$$

□

**Corollary 2** Given  $\beta_0, \beta_1 \in so(3)$ , let  $x : \mathbb{R} \rightarrow SO(3)$  be a solution of  $\dot{x}(t) = x(t)(\beta_0 + t\beta_1)$ . There are geodesics  $y_{\pm} : \mathbb{R} \rightarrow SO(3)$  such that, given  $\epsilon > 0$ , for some  $k_{\epsilon}$  and all  $t > k_{\epsilon}$ ,

$$d(x(\pm t), y_{\pm}(\mathbb{R})) < \epsilon, \quad \text{where } d \text{ is Riemannian distance.}$$

**Proof:** Fix  $t_0 \in \mathbb{R}$ . Without loss of generality we can suppose  $x(t_0) = \mathbf{1}$ . Then, because  $x$  is associated with an affine Lie quadratic,  $x^*$  is associated with a null Lie quadratic. Apply Theorem 7 of [17] to  $x^*$ .

□

**Example 11** Let  $x : \mathbb{R} \rightarrow SO(3)$  be the solution of

$$\dot{x}(t) = x(t) \begin{bmatrix} 0 & 0 & -t \\ 0 & 0 & -1 \\ t & 1 & 0 \end{bmatrix} \quad \text{with } x(0) = \mathbf{1}.$$

Representing  $x$  locally by a curve in  $E^3$  is problematic, especially near singularities. So we plot the second column of  $x$  as  $x_2 : [-10, 10] \rightarrow E^3$  in Figure 1. As  $t$  increases, the resulting curve in the unit sphere  $S^2 \subset E^3$  spirals out downwards from a closed curve in the upper left, then spirals inwards towards a closed curve in the lower right. The change in sense of spiralling corresponds to reversal of sign of the torsion of the dual null Lie quadratic at  $t = 0$ , as noted in §IV of [17]. The limiting curves are projections of geodesics in  $SO(3)$ , by Theorem 7 of [17].

□

## 5 Cubics in $SO(3)$

Fix  $t_0$  in an open interval  $S$ . For a Riemannian cubic  $x : S \rightarrow SO(3)$ , associated with a Lie quadratic  $V : S \rightarrow so(3)$ , define  $V_E : S \rightarrow E^3$  by  $B_E \circ V_E = V$  and set  $C_E = \ddot{V}_E - \dot{V}_E \times V_E$ . Let  $V$  be nonaffine, namely  $c > 0$ . Theorem 5 says that if  $V$  is null  $x$  can be expressed in terms of  $V_E$  and its derivatives. Otherwise a quadrature is also needed.

Write  $\dot{V}_E(t) = \sqrt{c}W_3(t)$ ,  $\langle \cdot, \cdot \rangle$  for the Euclidean inner product  $\langle \cdot, \cdot \rangle_E$ , and denote by  $S^2$  the unit sphere in  $E^3$  with respect to the Euclidean norm. There are many  $C^\infty$  maps  $W_1 : S \rightarrow S^2$  such that  $\langle W_1(t), W_3(t) \rangle_E$  is identically zero. To see this, note that

$$\{(t, v) \in S \times S^2 : \langle W_3(t), v \rangle = 0\}$$

is a  $C^\infty$  fibre bundle over  $S$ , with fibre the unit circle  $S^1$  in  $E^2$ . Because  $S$  is contractible the bundle is trivial, and  $W_1$  can be defined using any cross-section. In practice it seems simpler to choose  $W_1$  by inspecting  $W_3$ . For  $t \in S$ , define  $W_2(t) = W_3(t) \times W_1(t)$ ,

$$\theta(t) = \int_{t_0}^t \langle W_1(s), \dot{W}_2(s) + V_E(s) \times W_2(s) \rangle ds, \quad \text{and}$$

$$U(t) = \begin{bmatrix} W_1(t) \cos \theta(t) + W_2(t) \sin \theta(t) & -W_1(t) \sin \theta(t) + W_2(t) \cos \theta(t) & W_3(t) \end{bmatrix} \in SO(3).$$

**Theorem 5** Let  $V : S \rightarrow so(3)$  be a non-affine Lie quadratic, with associated Riemannian cubic  $x$ . For all  $t \in S$ ,  $x(t) = U(t_0)U(t)^\mathbf{T}$ . If  $V$  is null,

$$x(t) = \begin{bmatrix} V_E(t_0) + (t - t_0)\dot{V}_E(t_0) & \dot{V}_E(t_0) & \dot{V}_E(t_0) \times V_E(t_0) \end{bmatrix} \begin{bmatrix} V_E(t) & \dot{V}_E(t) & \dot{V}_E(t) \times V_E(t) \end{bmatrix}^{-1}.$$

**Proof:**  $\dot{\theta}(t) = \langle W_1(t), \dot{W}_2(t) + V_E(t) \times W_2(t) \rangle$ , and  $\dot{W}_3 = W_3 \times V_E$  by (5). In particular  $\dot{U}_3 = -V_E \times U_3 = -B_E(V_E)U_3$ . Also

$$\begin{aligned} \langle \dot{U}_1, W_1 \rangle &= -\langle W_1, V_E \times W_2 \rangle \sin \theta = -\langle B_E(V_E)U_1, W_1 \rangle, \\ \langle \dot{U}_1, W_2 \rangle &= \langle W_1, V_E \times W_2 \rangle \cos \theta = -\langle B_E(V_E)U_1, W_2 \rangle, \\ \langle \dot{U}_1, W_3 \rangle &= \langle \dot{U}_1, U_3 \rangle = -\langle U_1, \dot{U}_3 \rangle = -\langle U_1, W_3 \times V_E \rangle \\ &= -\langle B_E(V_E)U_1, W_3 \rangle. \end{aligned}$$

So  $\dot{U}_1 = -B_E(V_E)U_1$ . Now, because  $U_2 = U_3 \times U_1$ ,

$$\dot{U}_2 = -(V_E \times U_3) \times U_1 - U_3 \times (V_E \times U_1) = -B(V_E)U_2.$$

So  $\dot{U}(t) = -B_E(V_E(t))U(t)$ . By (3),  $\dot{U}(t) = -x(t)^{-1}\dot{x}(t)U(t)$ . So  $x(t)U(t)$  is constant, and  $x(t) = U(t_0)U(t)^{-1}$ . Suppose now that  $V$  is null. Then  $\ddot{V}_E = \dot{V}_E \times V_E$ . Because  $V$  is nonaffine the constant  $\langle \ddot{V}_E, \dot{V}_E \rangle = b$  is nonzero. So  $V_E(t), \dot{V}_E(t)$  are linearly independent for all  $t$ . Because  $V^*$  is affine,  $V^*(t) = -V(t_0) - (t - t_0)\dot{V}(t_0)$ . So  $x(t)V_E(t) = V_E(t_0) + (t - t_0)\dot{V}_E(t_0)$ ,  $x(t)\dot{V}_E(t) = \dot{V}_E(t_0)$ , and  $x(t)\ddot{V}_E(t) = \dot{V}_E(t_0)$ .

□

**Example 12** In Example 8, the Riemannian cubic  $x : \mathbb{R} \rightarrow SO(3)$  associated with  $V$  is given by

$$x(t) = \begin{bmatrix} \cos t & -\sin^2 t & \sin t \cos t \\ 0 & \cos t & \sin t \\ -\sin t & -\cos t \sin t & \cos^2 t \end{bmatrix}.$$

$$\text{Also } V^*(t) = B \begin{bmatrix} \cos t & -1 & -\sin t \end{bmatrix}^{\mathbf{T}} \quad \text{and} \quad C^* = B \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^{\mathbf{T}}.$$

□

**Example 13** Let  $V$  be affine, namely  $V(t) = \beta_0 + (t - t_0)\beta_1$  where  $\beta_0, \beta_1 \in so(3)$ . We say nothing about the associated Riemannian cubic, except when  $[\beta_0, \beta_1] = \mathbf{0}$ . In this case write  $\beta_0 = b_0\beta, \beta_1 = b_1\beta$ , where  $b_0, b_1 \in \mathbb{R}$  and  $\beta \in so(3)$ . Then  $x(t) = y(b_0(t - t_0) + b_1(t - t_0)^2/2)$ , where  $y$  is the geodesic in  $SO(3)$  with  $y(t_0) = \mathbf{1}$  and  $y'(t_0) = \beta$ .

□

## 6 Cubics in $SO(1, 2)$

Fix  $t_0$  in an open interval  $S$ . For a Riemannian cubic  $x : S \rightarrow SO(1, 2)$  associated with a Lie quadratic  $V : S \rightarrow so(1, 2)$ , define  $V_L : S \rightarrow L^3$  by  $B_L \circ V_L = V$  and set  $C_L = \ddot{V}_L - J(\dot{V}_L \times V_L)$ . Let  $V$  be nonaffine. If  $V$  is null  $x$  can be expressed in terms of  $V_L$  and its derivatives. For the remainder of this Section, we denote the inner product and Lie bracket in  $L^3$  by  $\langle \cdot, \cdot \rangle$  and  $[\cdot, \cdot]$ .

**Theorem 6** Let  $V : S \rightarrow so(1, 2)$  be a nonaffine null Lie quadratic, and  $x : S \rightarrow SO(1, 2)$  the associated Riemannian cubic. For all  $t \in S$ ,

$$x(t) = \begin{bmatrix} V_L(t_0) + (t - t_0)\dot{V}_L(t_0) & \dot{V}_L(t_0) & [\dot{V}_L(t_0), V_L(t_0)] \end{bmatrix} \begin{bmatrix} V_L(t) & \dot{V}_L(t) & [\dot{V}_L(t), V_L(t)] \end{bmatrix}^{-1}.$$

**Proof:** Because  $V_L$  is nonaffine,  $\ddot{V}_L = J(\dot{V}_L \times V_L) \neq \mathbf{0}$ , and  $V_L(t), \dot{V}_L(t)$  are linearly independent for all  $t$ . Because  $V^*$  is affine,  $V^*(t) = -V(t_0) - (t - t_0)\dot{V}(t_0)$ . So  $x(t)V_L(t) = V_L(t_0) + (t - t_0)\dot{V}_L(t_0)$ ,  $x(t)\dot{V}_L(t) = \dot{V}_L(t_0)$ , and  $x(t)\ddot{V}_L(t) = \ddot{V}_L(t_0)$  by Theorem 4.

□

For  $V$  nonaffine and non-null, the cases where  $\ddot{V}_L$  is spacelike, timelike and lightlike are treated separately.

### 6.1 $\ddot{V}_L$ Spacelike

Suppose  $c > 0$ . Write  $\ddot{V}_L(t) = \sqrt{c}W_3(t)$ , and let  $S_L^2$  be the unit sphere in  $L^3$  with respect to the Lorentz norm. Because  $W_3$  is everywhere spacelike, there are  $C^\infty$  maps  $W_2 : S \rightarrow S_L^2$  with  $\langle W_2(t), W_3(t) \rangle$  identically zero. For instance, writing coordinates of  $W_3(t)$  as  $W_{3,1}(t), W_{3,2}(t), W_{3,3}(t)$ , set

$$W_2(t) = \frac{\begin{bmatrix} 0 & -W_{3,3}(t) & W_{3,2}(t) \end{bmatrix}}{\sqrt{W_{3,2}(t)^2 + W_{3,3}(t)^2}}.$$

Then  $W_1(t) = -[W_2(t), W_3(t)]$  satisfies  $\langle W_1(t), W_1(t) \rangle = -1$ , and

$$[W_1(t), W_2(t)] = W_3(t), \quad [W_3(t), W_1(t)] = W_2(t), \quad \text{and} \quad [W_2(t), W_3(t)] = -W_1(t), \quad (14)$$

for all  $t \in S$ . Define

$$\theta(t) = \int_{t_0}^t \langle W_1(s), \dot{W}_2(s) + [V_L(s), W_2(s)] \rangle ds,$$

and let  $U(t) \in SO(1, 2)$  be the matrix

$$\begin{bmatrix} W_1(t) \cosh \theta(t) + W_2(t) \sinh \theta(t) & W_1(t) \sinh \theta(t) + W_2(t) \cosh \theta(t) & W_3(t) \end{bmatrix}.$$

**Theorem 7** *Let  $V : S \rightarrow so(1, 2)$  be a Lie quadratic with  $c > 0$ , and associated Riemannian cubic  $x$ . For all  $t \in S$ ,  $x(t) = U(t_0)JU(t)^T J$ .*

**Proof:** By (5),  $\dot{U}_3 = \dot{W}_3 = [W_3, V_L] = [U_3, V_L] = -B_L(V_L)U_3$ . Also  $\dot{\theta}(t) = \langle W_1(t), \dot{W}_2(t) + [V_L(t), W_2(t)] \rangle$ . So

$$\begin{aligned} \langle \dot{U}_1, W_1 \rangle &= (-\dot{\theta} + \langle \dot{W}_2, W_1 \rangle) \sinh \theta = -\langle W_1, [V_L, W_2] \rangle \sinh \theta = -\langle W_1, [V_L, U_1] \rangle \\ &= -\langle B_L(V_L)U_1, W_1 \rangle, \\ \langle \dot{U}_1, W_2 \rangle &= (\dot{\theta} + \langle \dot{W}_1, W_2 \rangle) \cosh \theta = \langle W_1, [V_L, W_2] \rangle \cosh \theta \\ &= -\langle [V_L, W_1], W_2 \rangle \cosh \theta = -\langle [V_L, U_1], W_2 \rangle = -\langle B_L(V_L)U_1, W_2 \rangle, \\ \langle \dot{U}_1, W_3 \rangle &= -\langle U_1, \dot{W}_3 \rangle = -\langle U_1, [W_3, V_L] \rangle = -\langle [V_L, U_1], W_3 \rangle \\ &= -\langle B_L(V_L)U_1, W_3 \rangle. \end{aligned}$$

So  $\dot{U}_1 = -B_L(V_L)U_1$ . Because  $U_2 = [U_3, U_1]$ ,

$$\dot{U}_2 = -[[V_L, U_3], U_1] - [U_3, [V_L, U_1]] = -[V_L, [U_3, U_1]] = -B(V_L)U_3.$$

So  $\dot{U}(t) = -B_L(V_L(t))U(t)$ . By (3),  $\dot{U}(t) = -x(t)^{-1}\dot{x}(t)U(t)$ , and therefore  $x(t)U(t)$  is constant.

□

**Example 14** In Example 10, the Riemannian cubic in  $SO(1,2)$  associated with  $V$  is given by

$$x(t) = \begin{bmatrix} \cosh t & \sin t \sinh t & -\cos t \sinh t \\ 0 & \cos t & \sin t \\ -\sinh t & -\sin t \cosh t & \cos t \cosh t \end{bmatrix}.$$

- $V^*(t) = B_L \begin{bmatrix} \cosh t & -1 & -\sinh t \end{bmatrix}^{\mathbf{T}}$  and  $C^* = B_L \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^{\mathbf{T}}$ ,
- $V$  is everywhere spacelike,  $C$  timelike and  $\ddot{V}$  spacelike,
- $V^*$  is everywhere lightlike,  $C^*$  spacelike and  $\ddot{V}^*$  timelike.

□

## 6.2 $\ddot{V}$ Timelike

Suppose  $c < 0$ , write  $\ddot{V}(t) = \sqrt{-c}W_1(t)$ , and let  $W_2 : S \rightarrow S_L^2$  be a  $C^\infty$  map satisfying  $\langle W_1(t), W_2(t) \rangle = 0$  for all  $t \in S$ . For instance, writing coordinates of  $W_1$  as  $W_{1,1}, W_{1,2}, W_{1,3}$ , define

$$W_2(t) = \frac{\begin{bmatrix} W_{1,2}(t) & W_{1,1}(t) & 0 \end{bmatrix}}{\sqrt{W_{1,1}(t)^2 - W_{1,2}(t)^2}}.$$

Set  $W_3(t) = [W_1(t), W_2(t)]$ , so that (14) holds. Define

$$\theta(t) = \int_{t_0}^t \langle W_2(s), \dot{W}_3(s) + [V_L(s), W_3(s)] \rangle ds, \quad \text{and}$$

$$U(t) = \begin{bmatrix} W_1(t) & W_2(t) \cos \theta(t) + W_3(t) \sin \theta(t) & -W_2(t) \sin \theta(t) + W_3(t) \cos \theta(t) \end{bmatrix}.$$

**Theorem 8** Let  $V : S \rightarrow so(1,2)$  be a non-affine Lie quadratic, with  $c < 0$  and associated Riemannian cubic  $x$ . For all  $t \in S$ ,  $x(t) = U(t_0)JU(t)^{\mathbf{T}}J$ .

**Proof:** We have  $\dot{\theta}(t) = \langle W_2(t), \dot{W}_3(t) + [V_L(t), W_3(t)] \rangle$ , and  $\dot{W}_1 = [W_1, V_L]$  by (5). In particular  $\dot{U}_1 = -B_L(V_L)U_1$ . Also

$$\begin{aligned} \langle \dot{U}_2, W_1 \rangle &= -\langle U_2, \dot{U}_1 \rangle = -\langle U_2, [W_1, V_L] \rangle = -\langle B_L(V_L)U_2, W_1 \rangle, \\ \langle \dot{U}_2, W_2 \rangle &= (-\dot{\theta} + \langle \dot{W}_3, W_2 \rangle) \sin \theta = -\langle W_2, [V_L, W_3] \rangle \sin \theta \\ &= -\langle B_L(V_L)U_2, W_2 \rangle, \\ \langle \dot{U}_2, W_3 \rangle &= (\dot{\theta} + \langle \dot{W}_2, W_3 \rangle) \cos \theta = \langle W_2, [V_L, W_3] \rangle \cos \theta = -\langle B_L(V_L)U_2, W_3 \rangle. \end{aligned}$$

So  $\dot{U}_2 = -B_L(V_L)U_2$ . Because  $U_3 = [U_1, U_2]$ ,

$$\dot{U}_3 = -[[V_L, U_1], U_2] - [U_1, [V_L, U_2]] = -B(V_L)U_3.$$

So  $\dot{U}(t) = -B_L(V_L(t))U(t)$ ,  $\dot{U}(t) = -x(t)^{-1}\dot{x}(t)U(t)$ , and  $x(t)U(t)$  is constant.

□

### 6.3 $V$ Nonaffine and $\ddot{V}$ Lightlike

Suppose  $c = 0$ . Because  $V$  is nonaffine,  $\ddot{V}(t) \neq \mathbf{0}$  for all  $t$ . Because  $\ddot{V}$  is lightlike, the coordinates  $\dot{V}_i$  of  $\ddot{V}_L$  satisfy  $\dot{V}_1(t) \neq 0$  for all  $t$ , and  $\dot{V}_1(t)^2 = \dot{V}_2(t)^2 + \dot{V}_3(t)^2$ . Let  $\sigma = \pm 1$  be the sign of  $\dot{V}_1$ ,  $W_1(t) = \begin{bmatrix} \sigma & 0 & 0 \end{bmatrix}$ ,  $\lambda(t) = |\dot{V}_1(t)| = -\langle \ddot{V}_L(t), W_1 \rangle$ , and define  $W_2(t)$  by  $\ddot{V}_L = \lambda(W_1 + W_2)$ . By (5),  $\lambda\dot{W}_2 + \dot{\lambda}(W_1 + W_2) = \lambda[W_1 + W_2, V_L]$ ,

$$\dot{\lambda} = \lambda \langle [V_L, W_2], W_1 \rangle, \quad (15)$$

$$\dot{W}_2 = -[V_L, W_1 + W_2] - \langle [V_L, W_2], W_1 \rangle (W_1 + W_2). \quad (16)$$

Set  $W_3(t) = [W_1(t), W_2(t)]$ , so that (14) holds,  $W(t) = \begin{bmatrix} W_1(t) & W_2(t) & W_3(t) \end{bmatrix}$ ,

$$\omega(t) = \frac{1}{\sqrt{2}} \int_{t_0}^t \frac{\langle V_L(s), W_2(s) \rangle}{\lambda(s)} ds, \quad \text{and}$$

$$U(t) = W(t) \begin{bmatrix} \frac{1}{2\lambda} + \lambda(1 + \sqrt{2}\omega + \omega^2) & -\frac{1}{2\lambda} - \lambda\omega(\sqrt{2} + \omega) & \lambda(1 + \sqrt{2}\omega) \\ -\frac{1}{2\lambda} + \lambda(1 + \sqrt{2}\omega + \omega^2) & \frac{1}{2\lambda} - \lambda\omega(\sqrt{2} + \omega) & \lambda(1 + \sqrt{2}\omega) \\ 1 + \sqrt{2}\omega & -1 - \sqrt{2}\omega & 1 \end{bmatrix} \in SO(1, 2).$$

**Theorem 9** *Let  $V : S \rightarrow so(1, 2)$  be a non-affine Lie quadratic, with  $c = 0$  and associated Riemannian cubic  $x$ . For all  $t \in S$ ,  $x(t) = U(t_0)JU(t)^T J$ .*

**Proof:** Since  $\dot{W}_1(t) = 0$ ,  $\langle \dot{W}_2(t), W_1(t) \rangle = 0 = \langle \dot{W}_3(t), W_1(t) \rangle$ . We also have

$$\sqrt{2}\lambda(t)\dot{\omega}(t) = \langle [V_L(t), W_3(t)], W_1(t) \rangle.$$

Then

$$\begin{aligned} \langle \dot{U}_1, W_1 \rangle &= -\left(-\frac{\dot{\lambda}}{2\lambda^2} + \dot{\lambda}(1 + \sqrt{2}\omega + \omega^2) + \sqrt{2}\lambda\dot{\omega}(1 + \sqrt{2}\omega)\right) \\ &= -\lambda \langle [V_L, W_2], W_1 \rangle \left(-\frac{1}{2\lambda^2} + 1 + \sqrt{2}\omega + \omega^2\right) - \langle [V_L, W_3], W_1 \rangle (1 + \sqrt{2}\omega) \\ &= -\langle [V_L, \left(-\frac{1}{2\lambda} + \lambda(1 + \sqrt{2}\omega + \omega^2)\right)W_2 + (1 + \sqrt{2}\omega)W_3], W_1 \rangle \\ &= -\langle B_L(V_L)U_1, W_1 \rangle. \\ \langle \dot{U}_1, W_2 \rangle &= \frac{\dot{\lambda}}{2\lambda^2} + \dot{\lambda}(1 + \sqrt{2}\omega + \omega^2) + \sqrt{2}\lambda\dot{\omega}(1 + \sqrt{2}\omega) + (1 + \sqrt{2}\omega) \langle \dot{W}_3, W_2 \rangle \\ &= \lambda \langle [V_L, W_2], W_1 \rangle \left(\frac{1}{2\lambda^2} + 1 + \sqrt{2}\omega + \omega^2\right) \\ &\quad + (1 + \sqrt{2}\omega)(\sqrt{2}\lambda\dot{\omega} + \langle [V_L, W_1 + W_2], W_3 \rangle) \\ &= -\langle [V_L, W_1], W_2 \rangle \left(\frac{1}{2\lambda} + \lambda(1 + \sqrt{2}\omega + \omega^2)\right) \\ &\quad + (1 + \sqrt{2}\omega)(\langle [V_L, W_3], W_1 \rangle - \langle [V_L, W_1 + W_2], W_3 \rangle) \\ &= -\langle [V_L, \left(\frac{1}{2\lambda} + \lambda(1 + \sqrt{2}\omega + \omega^2)\right)W_1 + (1 + \sqrt{2}\omega)W_3], W_2 \rangle \end{aligned}$$

$$\begin{aligned}
&= - \langle B_L(V_L)U_1, W_2 \rangle . \\
\langle \dot{U}_1, W_3 \rangle &= \sqrt{2}\dot{\omega} + \left(-\frac{1}{2\lambda} + \lambda(1 + \sqrt{2}\omega + \omega^2)\right) \langle \dot{W}_2, W_3 \rangle \\
&= \frac{1}{\lambda} \langle [V_L(t), W_3(t)], W_1(t) \rangle \\
&\quad - \left(-\frac{1}{2\lambda} + \lambda(1 + \sqrt{2}\omega + \omega^2)\right) \langle [V_L, W_1 + W_2], W_3 \rangle \\
&= - \langle [V_L, \left(\frac{1}{2\lambda} + \lambda(1 + \sqrt{2}\omega + \omega^2)\right)W_1 + \left(-\frac{1}{2\lambda} + \lambda(1 + \sqrt{2}\omega + \omega^2)\right)W_2], W_3 \rangle \\
&= - \langle B(V_L)U_1, W_3 \rangle .
\end{aligned}$$

So  $\dot{U}_1 = -B_L(V_L)U_1$ . However  $U_1 + U_2 = \dot{V}$ . Therefore, by (5),  $\dot{U}_2 = -B_L(V_L)U_2$ . We also have  $U_3 = [U_1, U_2]$ . Then

$$\dot{U}_3 = -[[V_L, U_1], U_2] - [U_1, [V_L, U_2]] = -B(V_L)U_3.$$

So  $\dot{U}(t) = -B_L(V_L(t))U(t)$ ,  $\dot{U}(t) = -x(t)^{-1}\dot{x}(t)U(t)$ , and  $x(t)U(t)$  is constant.

□

## 7 Conclusion

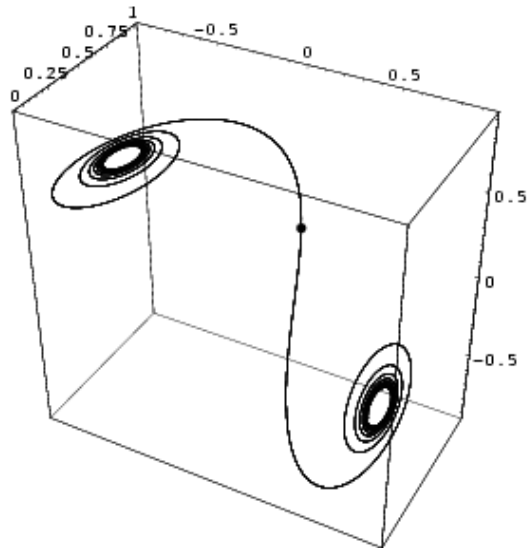
Duality is introduced for Riemannian cubics in Lie groups, especially  $SO(3)$  and  $SO(1, 2)$ . Using this theory we give explicit formulae for Riemannian cubics in terms of associated Lie quadratics and quadratures (Theorems 5, 6, 7, 8, 9). For null cubics it is even possible to dispense with quadratures. The use of these theoretical results in practical engineering problems is a topic for future investigation.

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**Figure 1.**  $x_2 : [-10, 10] \rightarrow S^2$  in Example 11.