Non-Null Lie Quadratics in E^3

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Abstract

Interpolation problems in the space $SO(3)$ of rotations of Euclidean 3-space E^3 are reviewed in §§1, 2, as background and motivation to a study of curves in E^3 called Lie quadratics [26]. *Except for a special class called* null *[27], Lie quadratics have resisted analysis until now. The rest of the present paper is devoted to new results showing non-null Lie quadratics have rich analytical, geometrical and asymptotic structures: rates of growth are studied using differential equations and inequalities, Lie quadratics are proved to be extendible over the whole of $\mathbb{R}$, and existence of axes is proved under fairly general conditions. Examples show sharpness of many results.*

1 Introduction

Interpolating in the group $SO(3)$ of rotations of Euclidean 3-space E^3, and in Riemannian manifolds generally, is a more significant task, in terms of applications and mathematics, than might at first be suspected. To place the new mathematical results of the present paper in context, we first say something about applications and previous mathematical work. Interpolation in $SO(3)$ is very different to the standard problem of interpolation in E^3.

Example 1 *A rigid body K, perhaps a camera, is free to rotate about the origin $\mathbf{0}$ in Euclidean 3-space E^3 . Configurations x_i, and possibly angular velocities v_i, are specified at times $i = 0, T$. The problem is to move K accordingly. At time t the configuration $x(t)$ is given by a positively oriented orthonormal frame $(x_1(t), x_2(t), x_3(t))$ fixed relative to K. Equivalently, $x(t)$ is the rotation matrix*

$$\left[\begin{array}{ccc} x_1(t) & x_2(t) & x_3(t) \end{array} \right] \in SO(3),$$

and so we have an interpolation problem for a curve $x : [0, T] \to SO(3)$. In the simplest case, where angular velocities are not specified at endpoints, the interpolation conditions are

$$x(0) = x_0, \qquad x(T) = x_T. \tag{1}$$

If our elementary problem was posed in E^3, instead of $SO(3)$, the affine line segment from x_0 to x_T would probably be chosen as interpolant. However, although $SO(3)$ is contained in the

Euclidean space $M_{3\times 3} \cong E^9$ *of real* 3×3 *matrices, an affine line in* E^9 *intersects* $SO(3)$ *in at most* 2 *points. So line segments are not available for interpolation in* $SO(3)$. *However* $SO(3)$ *is covered by open subsets* U *diffeomorphic to open subsets of* E^3. *Such covers must contain several open sets, because* U *cannot be the whole of* $SO(3)$. *For instance, rotations may be mapped to Euler angles* $(\psi,\theta,\phi) \in [0,2\pi)\times[0,\pi)\times[0,2\pi) \subset E^3$. Chart-based interpolation *proceeds by mapping* x_0, x_T *to points in* E^3, *interpolating in* E^3 *then mapping the interpolant back into* $SO(3)$. *This prescription is less straightforward than it sounds. For instance, if the Euler angle of* x_i *are* $(\psi_i,\theta_i,\phi_i) = \Phi(x_i)$ *for* $i = 0,T$, *small perturbations in* x_i *can give large changes in* ψ_i,ϕ_i *when* $\theta_i \approx 0$. *Also, whether the* θ_i *are small or not, the chart-based interpolant* x *may be a very unnatural choice. For example, taking*

$$x_{jT} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & (-1)^j/2 \\ 0 & (-1)^{j+1}/2 & \sqrt{3}/2 \end{bmatrix},$$

where $j = 0,1$, *the curve of directions of the camera lens takes the long way round:*

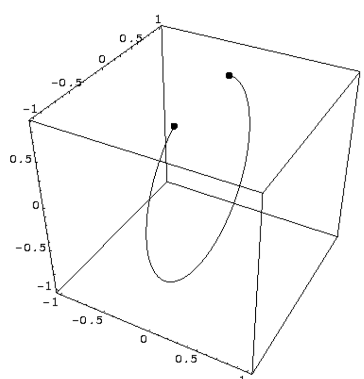

Figure 1. Lens directions with chart-based linear interpolation.

Although these difficulties can be ameliorated by switching between charts, this complicates implementation and has unwelcome side-effects: the interpolant from x_T *to* x_0 *may be different to that from* x_0 *to* x_T.

□

The geometrical difficulties in Example 1 suggest using a more geometrical interpolation scheme, like the following simple method from Riemannian geometry. A *Riemannian metric* on a C^∞ manifold M such as $SO(3)$ is a smooth assignment of inner products to the tangent spaces at each point in M. The *length* $L(x)$ and *energy* $J_1(x)$ of a smooth curve $x : [0,T] \to SO(3)$ are then defined as

$$L(x) = \int_0^T \|x^{(1)}(t)\| \, dt, \quad \text{and} \quad J_1(x) = \int_0^T \|x^{(1)}(t)\|^2 \, dt,$$

where the norm $\| \;\|$ is calculated using the Riemannian metric, and superscript (n) means n-fold derivative. Curves satisfying (1) of minimum length and uniform speed are called *minimal geodesics*. They also minimise energy. Geodesics on $SO(3)$ can sometimes be written down in closed form, notably when the metric is *bi-invariant*. In this case rotations are represented by unit quaternions, namely points in the 3-dimensional unit sphere $S^3 \subset E^4$, with geodesics reresented by arcs of great circles. Interpolation by geodesics effectively deals with the problems raised in Example 1, but is inadequate for most applications.

In Example 1 only two camera configurations $x_0, x_T \in SO(3)$ are prescribed, whereas in practice x may need to satisfy many such constraints. Piecewise-geodesics are inappropriate because, like piecewise-linear curves in E^3, they are usually nondifferentiable at junctions. A simple way to ensure this does not occur is to prescribe derivatives of x at junctions, replacing (1) by

$$x(0) = x_0, \quad x^{(1)}(0) = v_0, \quad x(T) = x_T, \quad x^{(1)}(T) = v_T. \tag{2}$$

If the problem was posed in E^3 a cubic polynomial could satisfy (2), but $SO(3)$ has no nonconstant polynomial curves, and chart-based cubic polynomials are problematic in the same ways as chart-based line segments. On the other hand, geodesics in $SO(3)$ are readily calculable, at least for a bi-invariant metric, and correspond to lines in E^3. This reminds us of the classical *deCastlejau algorithm* for generating polynomial curves in E^3 from line segments.

Replacing line segments by geodesic arcs in the cubic deCastlejau algorithm gives curves in $SO(3)$ capable of satisfying (2), as in [31], [14]. This elegant and effective method is frequently applied, and has been investigated further in [12]. A recursive form of the deCastlejau algorithm also adapts to Hermite interpolation in $SO(3)$ [21], [22], [23], [24], [25]. Unlike the Euclidean versions, the adapted nonrecursive and recursive deCastlejau schemes generate different curves in $SO(3)$. Yet another kind of curve results when we insist on an analogue of the important *variation diminishing property*, that cubics in E^3 minimise

$$\int_0^T \|x^{(2)}(t)\|^2 \, dt$$

among curves $x : [0, T] \to E^3$ satisfying (2).

2 Riemannian Cubics and Lie Quadratics

A Riemannian manifold comes equipped with a *Levi-Civita covariant derivative* ∇, which is a procedure for differentiating vector fields, whose associated *parallel translation* respects the Riemannian metric. *Riemannian cubics* are critical points of the functional J_2 given by

$$J_2(x) \;=\; \int_0^T \|\nabla_{d/dt} x^{(1)}\|^2 \, dt \tag{3}$$

where $x : [0, T] \to M$ satisfies (2). As shown in [18], [26], the Euler-Lagrange equation of J_2 is

$$\nabla^3_{d/dt} x^{(1)} + R(\nabla_{d/dt} x^{(1)}, x^{(1)}) x^{(1)} \;=\; \mathbf{0}, \tag{4}$$

where R is the *Riemannian curvature* of ∇. Let M be $SO(3)$ with a bi-invariant Riemannian metric. Then, as it stands, (4) amounts to 36 nonlinear first order ODEs for 36 scalar functions, with 24 equality constraints, and 36 scalar boundary conditions. The first step in solving this system is the reduction in [26] of (4) to a second order system of ODEs for $V : [0, T] \to E^3$

$$V^{(2)}(t) \;=\; V^{(1)}(t) \times V(t) \;+\; C \qquad \text{where } C \in E^3, \tag{5}$$

together with the first order equation

$$x^{(1)}(t) \;=\; x(t) B(V(t)), \tag{6}$$

where $B : E^3 \to so(3)$ is the linear isomorphism from E^3 onto the space of skew-symmetric 3×3 real matrices given by $B(v)(w) = v \times w$, and $\times$ denotes the vector product in E^3. In §4 of the present paper equations (5), (6) are shown to be solvable over the whole real line. In [28] equation (6) is solved using at most one quadrature. For any interval $S \subseteq \mathbb{R}$, a curve $V : S \to E^3$ satisfying (5) is called a *Lie quadratic* in E^3. Defining $F : S \to [0, \infty)$ by $F(t) = \|V(t)\|^2$, it follows easily from (5) that

$$F^{(2)}(t) = 6 < C, V(t) > + 2b, \tag{7}$$

$$\|V^{(1)}(t)\|^2 = 2 < C, V(t) > + b, \tag{8}$$

$$\|V^{(2)}(t)\|^2 = c, \tag{9}$$

where $b, c \in \mathbb{R}$ are constant. Set $d = 3(\|C\|^2 + c)$ and $\delta_\pm = 3(\|C\| \pm \sqrt{c})^2$.

Example 2 *Taking* v_0, v_T *as*

$$\begin{bmatrix} 0.0000 & 0.9280 & -0.3725 \\ -0.6175 & -0.1225 & 0.2121 \\ 0.7866 & -0.2121 & -0.1225 \end{bmatrix}, \quad \begin{bmatrix} 0.0000 & -0.2708 & -0.2380 \\ 0.1155 & -0.0224 & -0.0388 \\ 0.3415 & 0.0388 & -0.0224 \end{bmatrix},$$

the curve of lens directions for the Riemannian cubic shown (thick) in Figure 2 is less wavy in appearance than the curve for the adapted non-recursive deCastlejau algorithm of §1. These curves were generated in about 7 seconds on a 2GHZ PC running Mathematica.

□

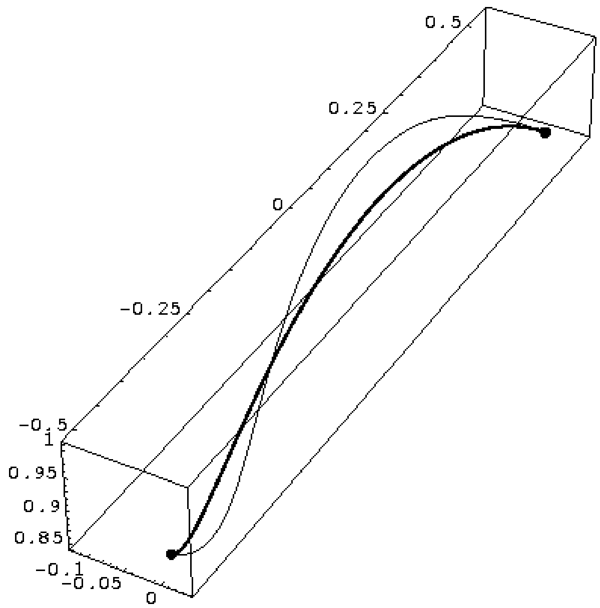

Figure 2. Lens directions for the Riemannian cubic.

There is a special class of Lie quadratics for which quite a lot is known: the Lie quadratic $V : S \to E^3$ is called *null* when its constant C is $\mathbf{0}$. In [27] null Lie quadratics in E^3 are shown to have constant (usually nonzero) curvature, linearly varying torsion, and two *axes*. The axes are rays through $\mathbf{0}$, to which V becomes C^0-close as $t \to \pm\infty$. The space of null Lie quadratics has rotational symmetry, and individual null quadratics have *internal symmetry*. A Riemannian cubic in $SO(3)$ associated with a null Lie quadratic also has internal symmetries and is asymptotic to a pair of geodesics.

Example 3 *Figure 3 shows the null Lie quadratic* $V : [0, 26] \to E^3$ *with* $V(0)$, $V^{(1)}(0)$ *taken as*

$$\begin{bmatrix} 3.2734 & 0.6697 & -5.1300 \end{bmatrix}^{\mathbf{T}}, \qquad \begin{bmatrix} -0.2320 & -0.0935 & 0.3427 \end{bmatrix}^{\mathbf{T}}$$

where $\mathbf{T}$ *means transpose. The Lie quadratic starts in the lower right, spiralling outwards along* *an axis pointing upwards and to the left. Around* $V(14)$ *the curve spirals inwards along the other axis, which points to the left and slightly downwards.*

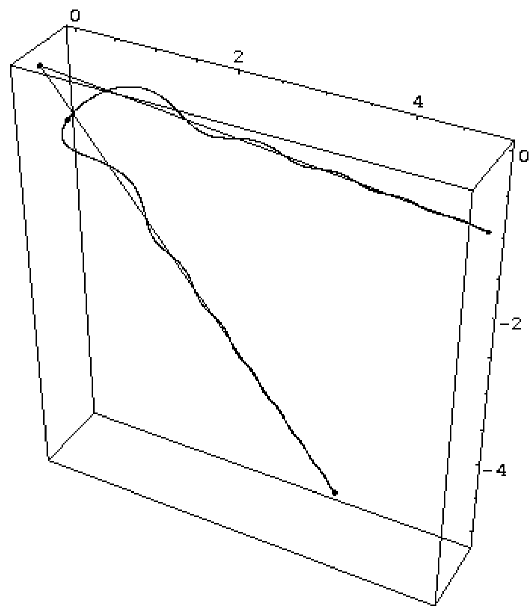

Figure 3. Null Lie quadratic V in E^3, showing $V(0), V(14.3), V(26)$, and axes.

Figure 4 shows the curve x_1 *in* S^2 *of first columns of* $x : [0, 26] \to SO(3)$ *associated with the null Lie quadratic* V.

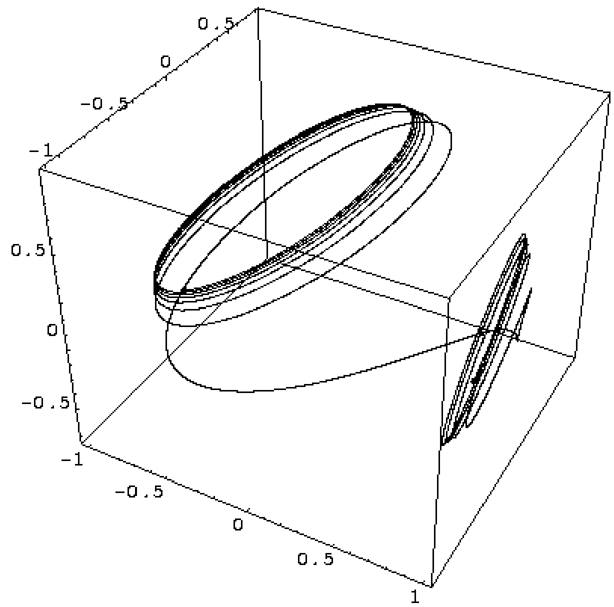

Figure 4. $x_1 : [0, 26] \to S^2, \; x_1(0), \; x_1(14.3), \; x_1(26).$

x_1 spirals downwards from a closed curve in the upper left, switches, then spirals upwards towards a closed curve in the lower right (note the change in sense of spiralling). The limiting curves are projections of geodesics in $SO(3)$, namely circles in S^2. Self-symmetry of x is not evident on casual inspection.

□

In addition to references already cited, for further background on null Lie quadratics, Riemannian cubics, variational problems, reduction to Lie quadratics, generalizations and alternatives see [6], [30], [15], [16], [34], [35], [36], [10], [20], [8], [9], [11], [13] and [2]. For engineering applications see [19], [4], [29], [5], [3], [32], [33].

For the larger class of non-null Lie quadratics V we note that if $A \in SO(3)$ then $t \mapsto AV(t)$ is a non-null Lie quadratic with constant AC, if $t_0 \in \mathbb{R}$ then $t \mapsto V(t - t_0)$ is a non-null Lie quadratic with constant C, and if $0 \neq a_1 \in \mathbb{R}$ then $t \mapsto a_1 V(a_1 t)$ is a non-null Lie quadratic with constant $a_1^3 C$. So after a rotation, change of origin, and change of scale a non-null Lie quadratic $V : \mathbb{R} \to E^3$ can be made to satisfy

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\mathbf{T}}, \qquad V_1(0) = 0, \qquad < V(0), V^{(1)}(0) > \;=\; 0.$$

The geometry of Lie quadratics is much more complicated in the non-null case.

Example 4 *Taking* $V(0) = \begin{bmatrix} 1.0 & 2.0 & 2.0 \end{bmatrix}^{\mathbf{T}}$, $V^{(1)}(0) = \begin{bmatrix} -1.0 & 1.0 & 1.0 \end{bmatrix}^{\mathbf{T}}$, *and* $C = -\begin{bmatrix} 0.4 & 0.5 & 0.75 \end{bmatrix}^{\mathbf{T}}$, *we find*

$$b = 8.80000, \quad c = 20.4742, \quad d = 64.34, \quad \delta_- = 37.5669, \quad \delta_+ = 91.1131.$$

Numerical simulation with Mathematica's default 16 *significant figures working precision gives Figure* 5 *for* $V : [-45, 30] \to E^3$. *The curve appears to spiral backwards and forwards along rays through the origin.*

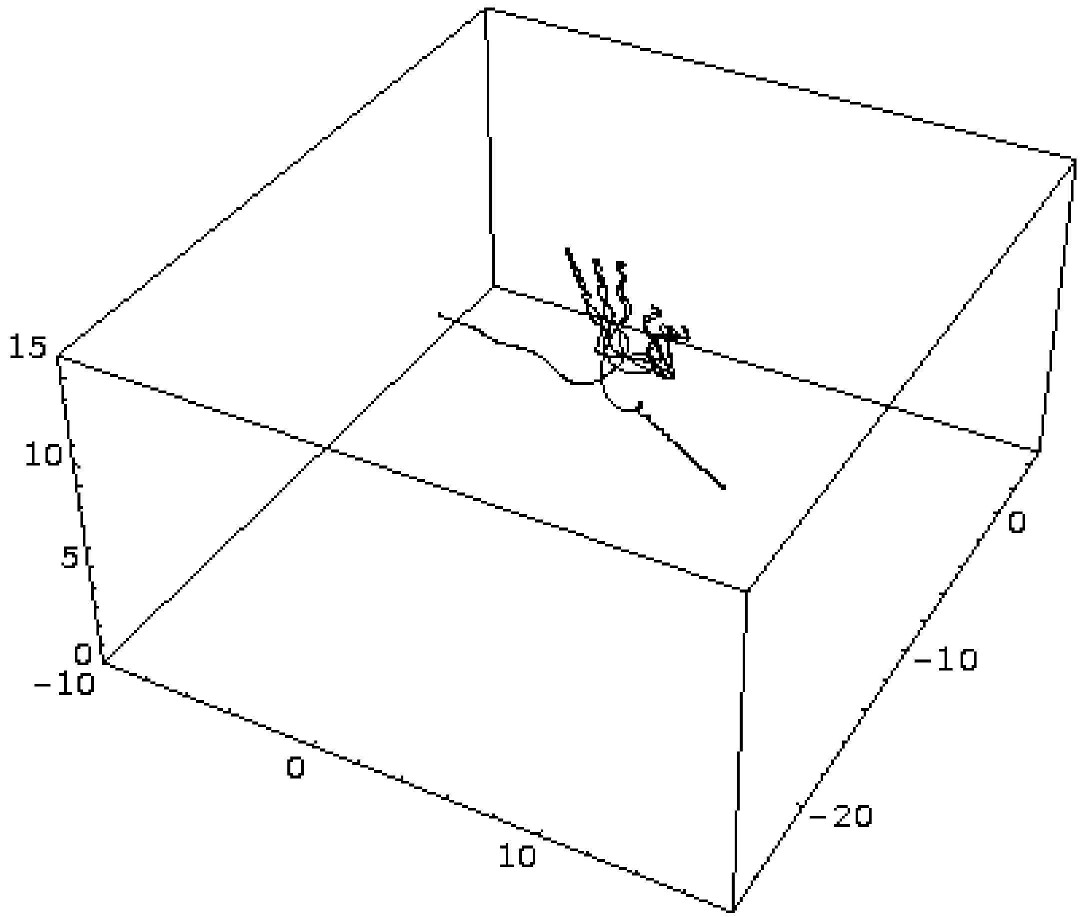

Figure 5. $V : [-45, 30] \to E^3$ computed with 16 significant figures in Example 4.

$\square$

From this point onwards the results are new, emphasising the squared-norm function F of a Lie quadratic V. There are 7 theorems, all central to the paper. These emphasise asymptotic properties of V, through a detailed analysis of F and its relationship to the constants $b, c, d, \delta_\pm$.

- Lie quadratics in E^3 (whether null or not) extend to Lie quadratics defined on the whole of $\mathbb{R}$ (Corollary 5), as do Riemannian cubics (Theorem 2), and can be sensitive to initial data (Example 5)

- F satisfy various differential inequalities and differential equations (Theorem 1)

- F satisfies a nonlinear third-order differential equation (11) from which follow further inequalities (Corollaries 1, 2), and a second order equation (14) for $G \equiv (F^{(1)})^2$

- when $b < 0$ F is strictly convex, with $F(t)$ increasing as t^4 for $t \to \pm\infty$ (Theorem 3)

- for $b > 0$ V can be periodic (Example 6) and sometimes F is multimodal (Example 5)

- for $b \geq 0$ V is often unbounded (Theorem 5) and then $F(t)$ increases as t^2 or t^4

- for $b \geq 0$ there are relationships between critical values of F (Theorem 5, Example 9)

- when V is unbounded the angular part $U(t)$ of $V(t)$ converges to a limit $\alpha_\pm(V)$ as $t \to \pm\infty$ (Theorem 6) with $a_\pm \equiv < C, \alpha_\pm(V) > \ \geq \ 0$ (Corollary 9)

- when V is unbounded $F(t) = O(t^2)$ as $t \to \pm\infty$ if and only if $a_\pm = 0$ (Theorem 7)

- the asymptotic directions $\alpha_\pm(V)$ may be difficult to determine (Example 11).

Properties of F, as well as being of interest in their own right, are the key to the rest of our analysis. Whereas in the null case F is a quadratic polynomial, the possibilities when V is non-null are more various and take a little longer to unfold, starting with some differential equations and inequalities.

3 Differential Equations and Inequalities

Let $V : S \rightarrow E^3$ be a Lie quadratic defined on an open interval S, with constant vector C, $F : S \rightarrow [0, \infty)$, and associated constants $b, c, d, \delta_\pm$ as defined in §2. I thank a referee for pointing out an error in Corollary 3 below, and for advice that, after submission of the present paper, a result similar to the following theorem appeared independently in [1].

Theorem 1 $F \geq 0, \; F^{(2)} + b = 3\|V^{(1)}\|^2 \geq 0,$

$$\delta_- \leq (F^{(2)} + b)F - \frac{3}{4}(F^{(1)})^2 = d - F^{(4)} \leq \delta_+, \quad \textit{and} \tag{10}$$

$$\begin{aligned} &48F(F^{(3)})^2 - 48F^{(1)}(F^{(2)} - 2b)F^{(3)} + 64b^3 + 48d^2 - 96bdF + 48b^2F^2 + 72d(F^{(1)})^2 - 72bF(F^{(1)})^2 \\ &+27(F^{(1)})^4 - 96dFF^{(2)} + 96bF^2F^{(2)} - 72F(F^{(1)})^2F^{(2)} - 48b(F^{(2)})^2 + 48F^2(F^{(2)})^2 + 16(F^{(2)})^3 \\ &-576d\|C\|^2 + 1728\|C\|^4 = 0. \end{aligned} \tag{11}$$

Proof: Eliminating $< C, V >$ between (7), (8), $\|V^{(1)}\|^2 = \frac{1}{3}(F^{(2)} + b)$. By (9), (5), $\|V^{(1)}\|^2F - < V^{(1)}, V >^2 + 2 < C, V^{(1)} \times V > + \|C\|^2 = c$. Then

$$\frac{1}{3}(F^{(2)} + b)F - \frac{1}{4}(F^{(1)})^2 + 2 < C, V^{(1)} \times V > + \|C\|^2 = c.$$

Differentiating (7) twice, and by (5), $F^{(4)} = 6 < C, V^{(1)} \times V + C >$ and consequently

$$< C, V^{(1)} \times V > = \frac{1}{6}F^{(4)} - \|C\|^2. \tag{12}$$

For (10), it remains to prove $\delta_- \leq d - F^{(4)} \leq \delta_+$: by (7), $|F^{(4)}| = 6| < C, V^{(2)} > | \leq 6\|C\|\sqrt{c}$.

So

$$\delta_- = 3(\|C\| - \sqrt{c})^2 \leq d - F^{(4)} \leq 3(\|C\| + \sqrt{c})^2 = \delta_+,$$

and this proves (10). Now $F^{(4)} = 6 < C, C > + 6 < C, V^{(1)} \times V >$ by (5), and then

$$(\frac{F^{(4)}}{6} - \|C\|^2)^2 = \|C\|^2(\|V^{(1)}\|^2\|V\|^2 - < V^{(1)}, V >^2) - \| < C, V > V^{(1)} - < C, V^{(1)} > V\|^2 =$$

$$\|C\|^2((\frac{F^{(2)} + b}{3})F - \frac{F^{(1)})^2}{4}) - (\frac{F^{(2)} - 2b}{6})^2(\frac{F^{(2)} + b}{3}) - (\frac{F^{(3)}}{6})^2 F + (\frac{F^{(2)} - 2b}{6})\frac{F^{(3)}}{6}F^{(1)}.$$

Eliminating $F^{(4)}$ with (10) gives (11).

□

Corollary 1 $64b^3F + 48d^2F - 96bdF^2 + 48b^2F^3 - 48b^2(F^{(1)})^2 + 72dF(F^{(1)})^2 - 72bF^2(F^{(1)})^2$

$+27F(F^{(1)})^4 - 96dF^2F^{(2)} + 96bF^3F^{(2)} + 48b(F^{(1)})^2F^{(2)} - 72F^2(F^{(1)})^2F^{(2)} - 48bF(F^{(2)})^2 + 48F^3(F^{(2)})^2$

$-12(F^{(1)})^2(F^{(2)})^2 + 16F(F^{(2)})^3 - 576d\|C\|^2F + 1728\|C\|^4F \leq 0.$

Proof: Equation (11) is quadratic in $F^{(3)}$, with discriminant -192 times the left hand side.

□

Corollary 2 *At a critical point* t_0 *of* F *where* $F(t_0) > 0$,

$$(F^{(2)})^3 - 3b(F^{(2)})^2 + 3(d - bF - FF^{(2)})^2 - 108\|C\|^2c + 4b^3 \leq 0.$$

Proof: Set $F^{(1)} = 0$ in Corollary 1 and divide both sides by $16F$.

□

Writing $G = (F^{(1)})^2$ for $F^{(1)} \neq 0$,

$$F^{(2)} \;=\; \frac{1}{2}\frac{dG}{dF}, \qquad F^{(3)} \;=\; \frac{\epsilon}{2}G^{1/2}\frac{d^2G}{dF^2}, \qquad F^{(4)} \;=\; \frac{1}{2}G\frac{d^3G}{dF^3} \;+\; \frac{1}{4}\frac{dG}{dF}\frac{d^2G}{dF^2},$$

where ϵ is the sign of $F^{(1)}$. Then, from (10), (11),

Corollary 3 *In any open interval where* $F^{(1)} \neq 0$,

$$2G^2\frac{d^3G}{dF^3} \;+\; \frac{dG}{dF}\frac{d^2G}{dF^2} \;+\; 2F\frac{dG}{dF} - 3G + 4bF - 4d \;=\; 0, \qquad \textit{and} \tag{13}$$

$$12\frac{d^2G}{dF^2}(4bG - G\frac{dG}{dF} + FG\frac{d^2G}{dF^2}) \;+2(\frac{dG}{dF})^3 - 12\frac{dG}{dF}(4F - 4bF^2 + b\frac{dG}{dF} + 3FG - F^2\frac{dG}{dF}) \;+$$

$$9G(8d - 8bF + 3G) \;+\; 48bF(bF - 2d) \;+\; 16(4b^3 + 3d^2 - 36d\|C\|^2 + 108\|C\|^4) \;=\; 0. \tag{14}$$

□

Despite extreme sensitivity of solutions of (5) to numerical measurements, the evidence suggests these inequalities are sharp.

Example 5 *In Example* 4*, numerical simulation with* 16 *significant figures working precision gives Figure* 5 *for* $V : [-45, 30] \to E^3$*. Figure* 6*, using* 25 *significant figures (our default from now on), is noticeably different, although the two figures share some similarities in general appearance.*

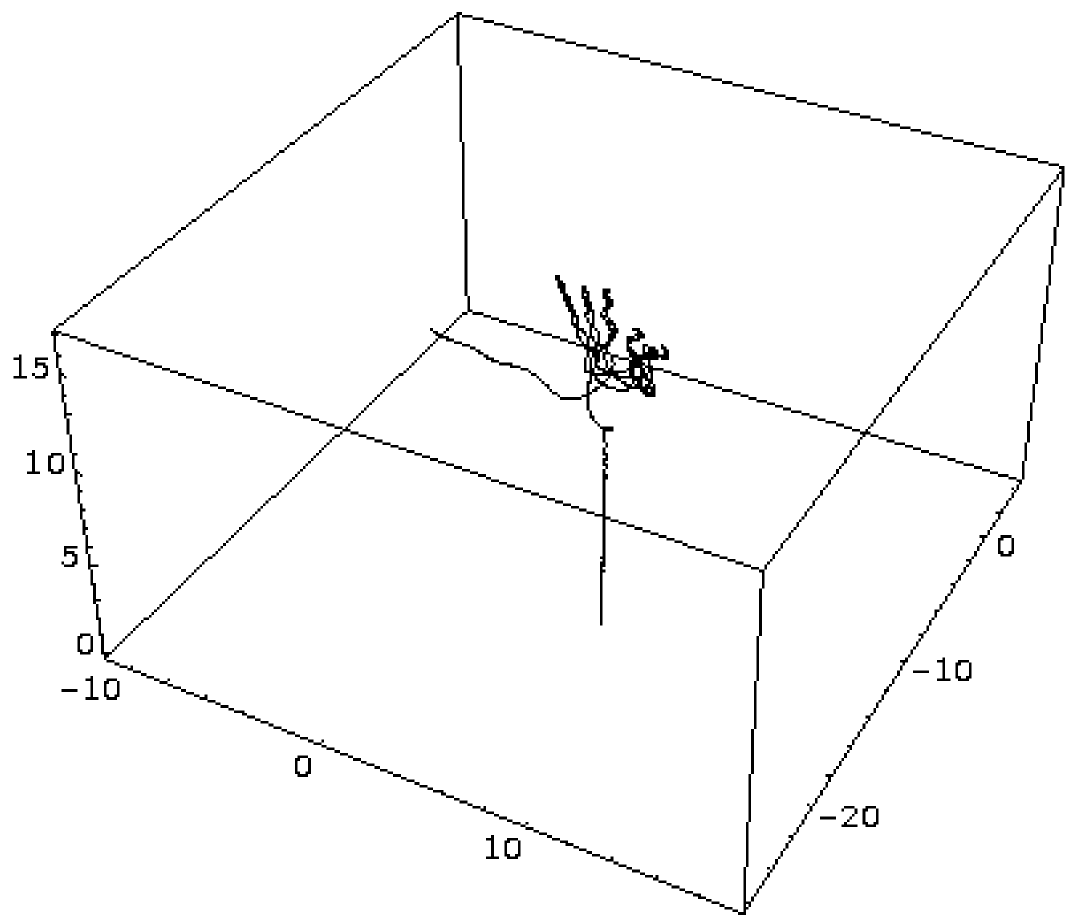

Figure 6. $V : [-45, 30] \to E^3$ computed with 25 significant figures in Example 5.

In particular, both Figures 5, 6 *suggest* F *is multimodal. The graph of* $F|[-45, 30]$ *in Figure* 7 *shows more detail:* 8 *points of local minimum and* 7 *of local maximum. In Figure* 6*, the curve starts from around the right of the front panel, spirals outwards then, after sporadic spiralling in the middle range, spirals outwards to the panel on the left.*

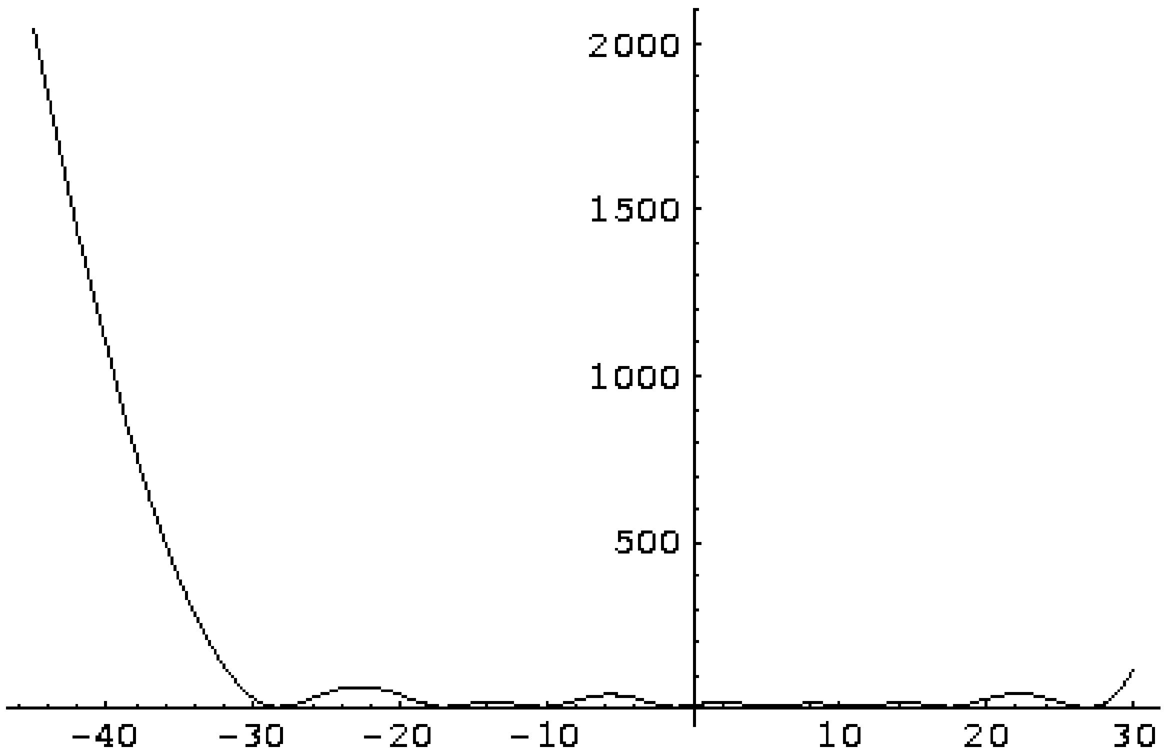

Figure 7. F in Example 5**.**

In Figure 8 $d - F^{(4)}$ *is plotted together with the horizontal lines through* $\delta_{\pm}$. $F^{(4)}$ *appears is highly oscillatory, and the inequalities in* (10) *seem sharp.*

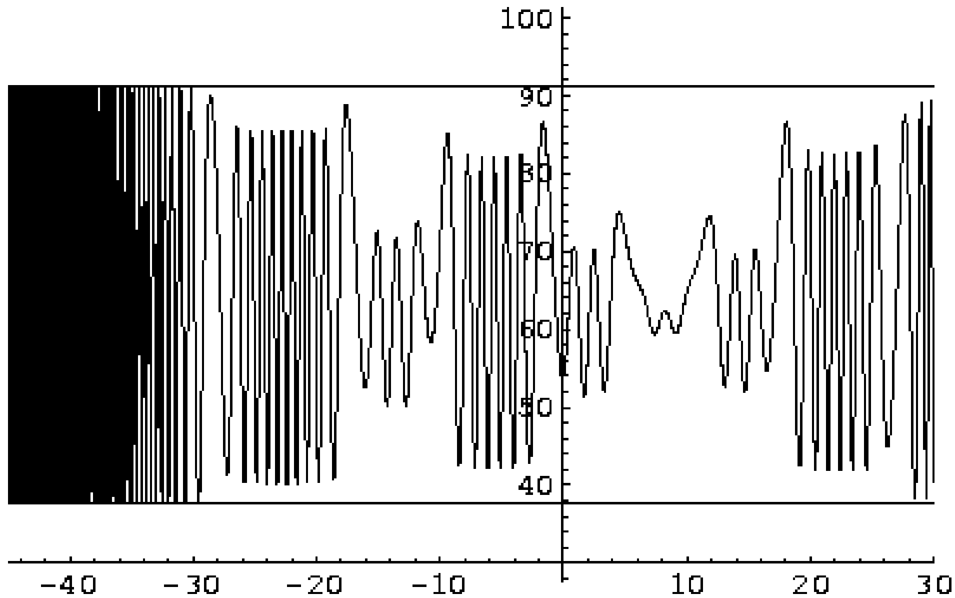

Figure 8. $d - F^{(4)}$ and $\delta_{\pm}$ in Example 5.

Corollary 1 *is also sharp, as illustrated by the plot in Figure* 9 *of the left hand side of the inequality.*

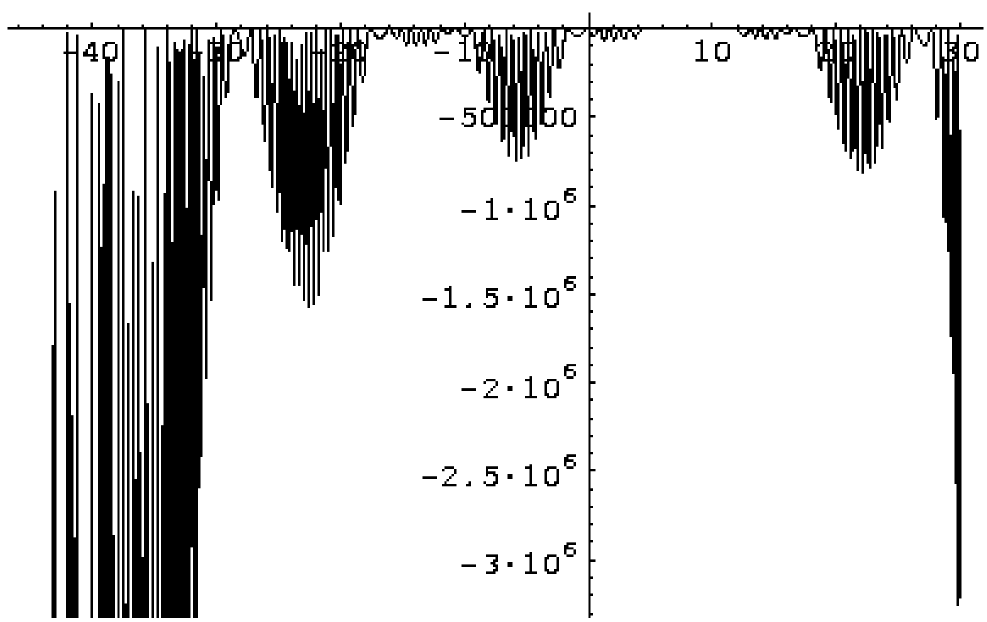

Figure 9. Corollary 1 in Example 5.

However, for a null Lie quadratic F *is a quadratic polynomial* [27].

□

Another basic fact is the extendibility of Riemannian cubics and Lie quadratics from intervals S to the whole of $\mathbb{R}$. The following inequality is needed for the proof in §4.

Corollary 4 *Given* $t_0 \in S$*, there are constants* $\kappa_0, \kappa_2 > 0$ *depending only on* $V(t_0), V^{(1)}(t_0), V^{(2)}(t_0)$ *such that, for all* $t \in S$*,*

$$\|V(t)\|,\ \|V^{(1)}(t)\|^2 \ \le\ \kappa_2^2 t^2 + \kappa_0^2.$$

Proof: For $t \in S$, $|F^{(4)}(t)| \le 6\|C\|\sqrt{c}$. So when $t \ge t_0$, $|F^{(3)}(t)| \le |F^{(3)}(t_0)| + 6\|C\|\sqrt{c}(t - t_0)$. Then $|F^{(2)}(t)| \le k_0 + k_1(t - t_0) + k_2(t - t_0)^2$ where k_0, k_1, k_2 depend only on $V(t_0), V^{(1)}(t_0), V^{(2)}(t_0)$, and so on: $F(t)$ is bounded by a quartic in t, at least for $t \ge t_0$, and a similar argument applies for $t < t_0$. So $\|V\|$ is bounded by a quadratic. Because $F^{(2)}$ is bounded by a quadratic so is $\|V^{(1)}\|^2$.

□

4 Extending Riemannian Cubics and Lie Quadratics

We prove an extendibility result for Riemannian cubics in $SO(3)$, then use it to prove extendibility for Lie quadratics in E^3. I thank a referee for advice that [7] also contains results on extendibility of cubics.

Lemma 1 *For some* $\delta > 0$, *given* $t_0 \in \mathbb{R}$, $x_0 \in SO(3)$, *and* $v_i \in TSO(3)_{x_0}$ *with* $\|B^{-1}(x_0^{-1}v_i)\| \leq 1$ *for* $0 \leq i \leq 2$, *there is a unique Riemannian cubic* $x : (t_0 - \delta, t_0 + \delta) \rightarrow SO(3)$ *satisfying*

$$x(t_0) = x_0, \quad x^{(1)}(t_0) = v_0, \quad \nabla_{d/dt} x^{(1)}|_{t_0} = v_1, \quad \textit{and} \quad \nabla^2_{d/dt} x^{(1)}|_{t_0} = v_2. \tag{15}$$

Proof: Picard's Theorem on local unique solvability of ordinary differential equations almost asserts this, but with δ depending on x_0, v_0, v_1, v_2. However $SO(3)$ is compact. Restricting v_0, v_1, v_2 also to lie in a compact set permits a uniform choice of δ.

□

Lemma 2 *For* δ *as in Lemma 1, given* $t_0 \in \mathbb{R}$, $x_0 \in SO(3)$, *and* $v_i \in TSO(3)_{x_0}$ *for* $0 \leq i \leq 2$, *if* $\lambda \geq 1 + \max_i \|v_i\|$ *then there is a unique Riemannian cubic* $x : (t_0 - \delta/\lambda, t_0 + \delta/\lambda) \rightarrow SO(3)$ *satisfying* (15).

Proof: Using Lemma 1, let $\hat{x} : (t_0 - \delta, t_0 + \delta) \rightarrow SO(3)$ be the Riemannian cubic satisfying

$$\hat{x}(t_0) = x_0, \quad \hat{x}^{(1)}(t_0) = v_0/\lambda, \quad \nabla_{d/dt} \hat{x}^{(1)}|_{t_0} = v_1/\lambda^2, \quad \text{and} \quad \nabla^2_{d/dt} \hat{x}^{(1)}|_{t_0} = v_2/\lambda^3.$$

Then set $x(t) = \hat{x}(t_0 + \lambda(t - t_0))$.

□

Theorem 2 *Given* $t_0 \in \mathbb{R}$, $x_0 \in SO(3)$, *and* $v_0, v_1, v_2 \in TSO(3)_{x_0}$, *there is a unique Riemannian cubic* $x : \mathbb{R} \rightarrow SO(3)$ *satisfying* (15).

Proof: By Lemma 1, for some $\tilde{\delta} > 0$ there is a (Riemannian) cubic $\tilde{x} : (t_0 - \tilde{\delta}, t_0 + \tilde{\delta}) \to SO(3)$ satisfying (15). If $\tilde{x}$ is not extendible to a cubic on $(t_0 - \tilde{\delta}, \infty)$, let $\mathcal{T}$ be the set of real numbers T for which $\tilde{x}$ extends to a cubic on $(t_0 - \tilde{\delta}, T]$. Set $\bar{T} = \sup \mathcal{T}$ and

$$\lambda = (\kappa_2^2 \bar{T}^2 + \kappa_0^2 + 1)^2 + \|V^{(2)}(t_0)\| + 1.$$

Choose $T \in \mathcal{T}$ with $T > \bar{T} - \frac{\delta}{2\lambda}$ and δ as in Lemma 1. By (9) and Corollary 4, the Lie quadratic V associated with a cubic extension $x : (t_0 - \delta_0, T] \to SO(3)$ of $\tilde{x}$ satisfies

$$\|V(t)\| \le \kappa_2^2 t^2 + \kappa_0^2, \quad \|V^{(1)}(t)\| \le \sqrt{\kappa_2^2 t^2 + \kappa_0^2}, \quad \|V^{(2)}(t)\| = \|V^{(2)}(t_0)\|, \quad \text{for} \quad t \in (t_0 - \delta_0, T),$$

where κ_0, κ_2 depend only on $\tilde{x}$. By [27] Lemma 2,

$$\begin{aligned}
\|B^{-1}(x(t)^{-1} x^{(1)}(t))\| &= \|V(t)\| \le \lambda - 1, \\
\|B^{-1}(x(t)^{-1} \nabla_{d/dt} x^{(1)})\| &= \|V^{(1)}(t)\| \le \lambda - 1, \\
\|B^{-1}(x(t)^{-1} \nabla_{d/dt}^2 x^{(1)})\| &= \|V^{(2)}(t) + \frac{1}{2} V(t) \times V^{(1)}(t)\| \le \|V^{(2)}(t_0)\| + \frac{1}{2}(\kappa_2^2 t^2 + \kappa_0^2)^{3/2} \le \lambda - 1.
\end{aligned}$$

Applying Lemma 2, with $\hat{T} \equiv T - \frac{\delta}{4\lambda}$ in place of t_0, $x|(\hat{T} - \frac{\delta}{4\lambda}, T)$ extends to a cubic defined on $(\hat{T} - \frac{\delta}{\lambda}, \hat{T} + \frac{3\delta}{4\lambda})$. So x, and therefore $\tilde{x}$, extends to a Riemannian cubic on $(t_0 - \tilde{\delta}, \bar{T} + \frac{\delta}{4\lambda})$, namely $\bar{T} + \frac{\delta}{4\lambda} \in \mathcal{T}$, contradicting $\bar{T} = \sup \mathcal{T}$. So $\tilde{x}$ is right extendible after all. Left extendibility is proved similarly. Uniqueness follows from Picard's local uniqueness.

□

Corollary 5 *Given* $C \in E^3$, $t_0 \in \mathbb{R}$ *and* $w_0, w_1 \in E^3$, *there is a unique Lie quadratic* $V : \mathbb{R} \to E^3$ *with constant* C, *satisfying*

$$V(t_0) \ = \ w_0 \quad \textit{and} \quad V^{(1)}(t_0) \ = \ w_1.$$

Proof: Set $w_2 = w_1 \times w_0 + C$, $x_0 = \mathbf{1}$, $v_0 = B^{-1}w_0$, $v_1 = B^{-1}w_1$, $v_2 = B^{-1}w_2$, apply Theorem 2, and let V be the Lie quadratic associated with the Riemannian cubic $x : \mathbb{R} \to SO(3)$.

□

So there is no loss in restricting attention to Lie quadratics in E^3 defined on the whole of $\mathbb{R}$. We may then ask about rates of growth of $V(t)$, or equivalently of $F(t)$, as $t \to \pm\infty$.

5 Rates of Growth I

By Theorem 1, for a Lie quadratic, $V : \mathbb{R} \to E^3$,

$$F^{(4)} + (F^{(2)} + b)F \ = \ \frac{3}{4}(F^{(1)})^2 + d, \qquad \text{where } d \geq 0. \tag{16}$$

When $d = 0$ V is null of the form $V(t) = (t - t_0)\beta$, where $t_0 \in \mathbb{R}$ and $\beta \in E^3$. For $d > 0$, $\tilde{F}$ given by $F(t) = d^{1/3}\tilde{F}(d^{1/6}t)$ satisfies (16) with b replaced by $d^{-2/3}b$ and d by 1. So there would be no real loss of generality in taking $d = 1$, but we continue with only the assumption $d > 0$. By Corollary 4, $\limsup_{t\to\pm\infty} \frac{F(t)}{t^4} < \infty$ and $\limsup_{t\to\pm\infty} \frac{\|V^{(1)}(t)\|^2}{t^2} < \infty$. Sharper results follow from Theorem 1:

Corollary 6 *We have* $\max\{0, -\frac{b}{2}\} \leq \liminf_{t\to\pm\infty} \frac{F(t)}{t^2}$ *and* $\limsup_{t\to\pm\infty} \frac{F(t)}{t^4} \leq \frac{\|C\|\sqrt{c}}{4}$.

Proof: Because $F^{(2)}(t) \geq -b$, $F^{(1)}(t)-F^{(1)}(t_0) \geq -b(t-t_0)$ for any $t \geq t_0 \in \mathbb{R}$. Integrating again,

$$F(t) \geq F(t_0) + (t-t_0)F^{(1)}(t_0) - \frac{b}{2}(t-t_0)^2. \tag{17}$$

For $t < t_0$, $F^{(1)}(t) - F^{(1)}(t_0) \leq -b(t-t_0)$, and again (17) holds on second integration. Similarly, because

$$F^{(4)}(t) \leq d - \delta_- = 6\|C\|\sqrt{c}, \quad \text{we have} \quad \limsup_{t\to\pm\infty} \frac{F(t)}{t^4} \leq \frac{\|C\|\sqrt{c}}{4}.$$

□

Corollary 7 *Let t_0 be a point of local minimum of F. Then $bF(t_0) \leq \delta_+$.*

1. *If $F(t_0) = 0$ then $\delta_- = 0$.*

2. *If t_0 is a degenerate critical point then $bF(t_0) \geq \delta_-$.*

Proof: By (10), $\delta_- \leq (F^{(2)}(t_0) + b)F(t_0) \leq \delta_+$, where $F^{(2)}(t_0) \geq 0$.

□

By Theorem 1, for $F^{(1)}F \neq 0$, we have $2\delta_- F^{-5/2} \leq 2bF^{-3/2} + \frac{d}{dF}(F^{-3/2}G) \leq 2\delta_+ F^{-5/2}$ and, integrating over $[F(t_1), F(t_2)]$ where $F^{(1)} > 0$ on $[t_1, t_2]$,

$$\frac{4\delta_-}{3}(\rho^{3/2}-1) \leq 4bF(t_1)\rho(\rho^{1/2}-1) + G(t_2) - \rho^{3/2}G(t_1) \leq \frac{4\delta_+}{3}(\rho^{3/2}-1), \tag{18}$$

where $\rho \equiv F(t_2)/F(t_1) > 1$.

Theorem 3 *Suppose* $b \leq 0$. *Then* F *is convex. If* $b < 0$ *or* $\delta_- > 0$ *then* F *is strictly convex and* $\liminf_{t\to\pm\infty} \frac{F(t)}{t^4} > 0$.

Proof: By Theorem 1, $F^{(2)} + b \geq 0$, and F is convex because $b \leq 0$. Similarly, if $b < 0$ then F is strictly convex. For $b \leq 0$ and $\delta_- > 0$, $(F^{(2)} + b)F \geq \delta_- + \frac{3}{4}(F^{(1)})^2 > 0$ by (10), and again F is strictly convex.

Because F is strictly convex, it is unbounded, has a point t_0 of global minimum, and t_0 is the only critical point. Take $[t_1, t_2] \subset (t_0, \infty)$ in (18), let $t_1 \to t_0^+$, and write $t_2 = t$:

$$G(t) \geq 4(\rho^{1/2} - 1)((\frac{\delta_-}{3} - bF(t_0))\rho + \frac{\delta_-}{3}\rho^{1/2} + \frac{\delta_-}{3}).$$

Given $0 < \epsilon < 1$, choose $t_3 > t_0$ so large that $\rho > \epsilon^2$ for all $t \geq t_3$. Then $\rho^{1/2} - 1 \geq (1-\epsilon)\rho^{1/2}$, and

$$F^{(1)}(t) = \sqrt{G(t)} \geq \gamma\rho^{3/4}, \quad \text{where} \quad \gamma \equiv 2\sqrt{(1-\epsilon)(\frac{\delta_-}{3} - bF(t_0))}.$$

So $F(t)^{-3/4}\frac{dF}{dt} \geq \gamma F(t_0)^{-3/4}$ and, integrating again, $F(t)^{1/4} - F(t_3)^{1/4} \geq 4\gamma F(t_0)(t - t_0)$, so that $\liminf_{t\to\infty} \frac{F(t)}{t^4} \geq \gamma$. Now if $F(t_0) = 0$ we have $V(t_0) = V^{(1)}(t_0) = \mathbf{0}$, and the unique solution of (5) satisfying these conditions is $V(t) = \frac{1}{2}(t - t_0)^2 C$, for which $b = 0$. So either $F(t_0) > 0$ or $b = 0$, and either $b > 0$ or $\delta_- > 0$ by hypothesis. So in any case we have $\gamma > 0$, completing the proof for $t \to \infty$. For $t \to -\infty$ consider the Lie quadratic $s \mapsto -V(-s)$.

□

6 Polynomial Solutions for F

Comparing Corollary 6 and Theorem 3, if $b\delta_- < 0$ then $F(t) = O(t^4)$ but $F(t) \neq O(t^3)$. On the other hand, F is sometimes bounded when $b > 0$.

Example 6 *Given* $a_0, c_0, t_0 \in \mathbb{R}$, $A \in SO(3)$, *define a Lie quadratic*

$$V(t) \;=\; a_0 A(-c_0, \cos a_0 c_0(t-t_0), \sin a_0 c_0(t-t_0)). \tag{19}$$

Then $C = a_0^3 c_0 A(1,0,0)$, F *is constant with value* $a_0^2(1+c_0^2)$,

$$b = 3a_0^4 c_0^2, \qquad c = a_0^6 c_0^4, \qquad d = 3a_0^6 c_0^2(1+c_0^2) \qquad \textit{and} \qquad \delta_\pm = 3a_0^6 c_0^2 (1 \pm c_0)^2.$$

If $a_0 c_0 = 0$ *then* V *is constant. Otherwise* V *is periodic and non-null (the only bounded null Lie quadratics in* E^3 *are constants).*

Conversely, let V *be any Lie quadratic in* E^3 *with* F *constant. If* $C = \mathbf{0}$ *then* $b = 0$ *by (7) and then* $\|V^{(1)}\| = 0$ *by (8), namely* V *is constant. If* $C \neq \mathbf{0}$ *then, after rotation in* E^3 *and time dilation, we can suppose* $C = (1,0,0)$, *so that the first component* V_1 *of* V *is* $-c_0$. *By (5), the other components satisfy*

$$V_2^{(2)} = -c_0 V_3^{(1)}, \qquad V_3^{(2)} = c_0 V_2^{(1)},$$

giving $V_2 = \cos c_0(t-t_0)$, $V_3 = \sin c_0(t-t_0)$ *for some* $t_0 \in \mathbb{R}$. *So, in any case where* F *is constant,* V *has the form* (19) *after rotation and dilation.*

□

Example 7 *For* $b > 0$, $c \geq 0$, $A \in SO(3)$ *and* $t_0 \in \mathbb{R}$, *let* V *be the null quadratic for which*

$$V(t_0) = A(\sqrt{c}, 0, 0), \quad V^{(1)}(t_0) = A(0, \sqrt{b}, 0).$$

Then $F(t) = b(t - t_0)^2 + c$, *and* $d = \delta_+ = \delta_- = 3c$. *Alternatively,* F *is also realised by the affine Lie quadratic*

$$t \mapsto -A(\sqrt{b}(t - t_0), \sqrt{c}, 0),$$

which is non-null for $c > 0$.

□

Example 8 *For* $a_0 > 0$, $c_0 \in \mathbb{R}$, $A \in SO(3)$ *and* $t_0 \in \mathbb{R}$, *set*

$$V(t) = (a_0(t - t_0)^2 + c_0)A(1, 0, 0).$$

Then V *is a Lie quadratic in* E^3 *with* $C = 2a_0 A(1, 0, 0)$, $F(t) = (a_0(t - t_0)^2 + c_0)^2$,

$$b = -4a_0c_0, \quad c = 4a_0^2, \quad d = 24a_0^2, \quad \delta_+ = 48a_0^2 \quad \textit{and} \quad \delta_- = 0.$$

□

In particular, Examples 6, 7 and 8 give constant, quadratic and quartic solutions of (16) with $F \geq 0$, $F^{(2)} + b \geq 0$ and $d \geq 0$. There are no other examples: all polynomial solutions of (16) have the form

$$a_0(t - t_0)^2 + c_0 \quad \text{or} \quad (b_0(t - t_0)^2 + d_0)^2, \quad \text{where} \tag{20}$$

- $a_0 = b$ and $3bc_0 = d$, or $a_0 = 0$ and $bc_0 = -d$,

- $24b_0^2 = d$ and $4b_0d_0 = -b$, or $b_0 = 0$ and $bd_0^2 = d$.

When $b < 0$ the solution $F(t) = b(t - t_0)^2 + d/(3b)$ of (16) satisfies neither $F \geq 0$ nor $F^{(2)} + b \geq 0$. In Examples 6, 7, 8, t_0 is a point of global minimum of F. In Example 6, and in Example 8 when $c_0 = 0$, t_0 is degenerate.

Theorem 4 *Unless V is one of the periodic Lie quadratics in Example 6,*

1. *the critical points of F are isolated, and*

2. *if $F(t_0) > 0$ and $F^{(1)}(t_0) = F^{(3)}(t_0) = 0$ then $F(t) = (a_0(t-t_0)^2 + c_0)^2$ where $a_0, c_0 \in \mathbb{R}$,*

3. *if t_0 is a degenerate critical point of F either $F^{(3)}(t_0) \neq 0$ or $F(t) = \frac{d}{24}(t - t_0)^4$.*

Proof: For *2.*, $F(t_0)$, $F^{(2)}(t_0)$ determine F uniquely as a solution of (16), and all $(F(t_0), F^{(2)}(t_0)) \in (0, \infty) \times \mathbb{R}$ are realised in Example 8. For *3.*, F is determined by $F(t_0) \geq 0$. Positive values are realised by taking F constant, and 0 is achieved in Example 8 by setting $c_0 = 0$. If F is constant V appears in Example 6. For *1.*, if F is nonconstant a critical point then, by *3.*, t_0 satisfies $F^{(i)}(t_0) \neq 0$ for some $i = 2, 3, 4$. So t_0 is isolated.

□

Corollary 8 *If F is nonconstant its points of local maximum are nondegenerate.*

□

7 Rates of Growth II: $b \geq 0$

Theorem 5 *Suppose* $b \geq 0$. *If* t_0 *is a point of local maximum of* F, *then* $\delta_- \leq bF(t_0)$. *If* t_0 *is a point of local minimum of* F *then* $bF(t_0) \leq \delta_+$, *and*

1. *if* $F|(t_0, \infty)$ *has no critical points and is unbounded, then* $3bF(t_0) \leq \delta_+$, *and either*

$$\lim_{t \to \infty} \frac{F(t)}{t^2} = b \quad \text{or} \quad \liminf_{t \to \infty} \frac{F(t)}{t^4} > 0.$$

2. *if* $F|(-\infty, t_0)$ *has no critical points and is unbounded, then* $3bF(t_0) \leq \delta_+$, *and either*

$$\lim_{t \to -\infty} \frac{F(t)}{t^2} = b \quad \text{or} \quad \liminf_{t \to -\infty} \frac{F(t)}{t^4} > 0.$$

3. *if* $3bF(t_0) \leq \delta_-$ *and* $\delta_- > 0$, *then* t_0 *is the only critical point of* F, *is nondegenerate, and* F *is unbounded on* $[0, \infty)$ *and on* $(-\infty, 0]$.

4. *if* $3bF(t_0) < \delta_-$ *then*

$$\liminf_{t \to \pm\infty} \frac{F(t)}{t^4} > 0.$$

5. *for* $\delta_- > 0$, *let* t_1 *be another critical point of* F, *where* F *has no critical points between* t_0, t_1. *Then* $3bF(t_0) > \delta_-$ *and*

$$F(t_1) \geq \left(\frac{\mu F(t_0)^{1/2} + \sqrt{4\mu F(t_0)^2 - 3\mu^2 F(t_0)}}{2(F(t_0) - \mu)}\right)^2 \quad \text{where } \mu = \frac{\delta_-}{3b}. \tag{21}$$

If also $3bF(t_0) > \delta_+$ *then*

$$F(t_1) \leq \left(\frac{\mu F(t_0)^{1/2} + \sqrt{4\mu F(t_0)^2 - 3\mu^2 F(t_0)}}{2(F(t_0) - \mu)}\right)^2 \quad \text{where } \mu = \frac{\delta_+}{3b}. \tag{22}$$

Proof: For t_0 a critical point of F, $\delta_- \leq (F^{(2)}(t_0)+b)F(t_0) \leq \delta_+$, by (10). If t_0 is a point of local maximum then $F^{(2)}(t_0) \leq 0$ and therefore $\delta_- \leq (F^{(2)}(t_0)+b)F(t_0) \leq bF(t_0)$. Similarly, if t_0 is a point of local minimum $F^{(2)}(t_0) \geq 0$ and $bF(t_0) \leq \delta_+$. For *1.*, by (18), with $[t_1, t_2] \subset (t_0, \infty)$,

$$\rho^{3/2}(4bF(t_1) - G(t_1) - \frac{4\delta_+}{3}) - 4bF(t_1)\rho \;\leq\; -(G(t_2) + \frac{4\delta_+}{3}), \tag{23}$$

$$\rho^{3/2}(4bF(t_1) - G(t_1) - \frac{4\delta_-}{3}) - 4bF(t_1)\rho \;\geq\; -(G(t_2) + \frac{4\delta_-}{3}). \tag{24}$$

By hypothesis, $\limsup_{t_2\to\infty} \rho = \infty$. By (23), and because $G(t_2) + \frac{4\delta_+}{3} \geq 0$,

$$4bF(t) - G(t) \;\leq\; \frac{4\delta_+}{3} \quad \text{for all} \quad t > t_0. \quad \text{So } 3bF(t_0) \;\leq\; \delta_+.$$

By (24) if, for any $t_1 \geq t_0$, $k(t_1) \equiv 4bF(t_1) - G(t_1) - \frac{4\delta_-}{3} < 0$, then for all $t \geq t_1$,

$$G(t) \;\geq\; -\,k(t_1)\rho^{3/2} \;+\; 4bF(t_1)\rho \;-\; \frac{4\delta_-}{3}, \qquad \text{where } \rho = F(t)/F(t_1).$$

Then, as in the proof of Theorem 3, $\liminf_{t\to\infty} \frac{F(t)}{t^4} > 0$. Alternatively if $k(t) \geq 0$ for all $t \geq t_0$, then $G(t) < 4bF(t)$ and $0 \leq F^{(1)}(t) < 2\sqrt{b}F(t)^{1/2}$. So

$$0 \;\leq\; F(t) \;<\; (F(t_0)^{1/2} + \sqrt{b}(t - t_0))^2,$$

and $\limsup_{t\to\infty} \frac{F(t)}{t^2} \leq b$. When $b = 0$ this proves *1.*, and when $b > 0$ we argue as follows. By (23), since ρ is unbounded, $G(t) \geq 4bF(t) - \frac{4\delta_+}{3}$, for all $t \geq t_0$. Eventually the right hand side is positive, when $t = t_1$ say, and

$$F^{(1)}(t) \;\geq\; 2(bF - \delta_+)^{1/2},$$

for all $t \geq t_1$. Integrating, $(bF(t) - \delta_+)^{1/2} \geq b(t-t_1) + (bF(t_1) - \delta_+)^{1/2}$, which completes the proof of *1.*

Now *2.* follows by applying *1.* to the Lie quadratic W where $W(s) = -V(2t_0 - s)$. The parameters $b, c, d, \delta_\pm$ are the same for W, and C is replaced by $-C$.

For *3.* $F(t_0) \neq 0$ and t_0 is nondegenerate, by Corollary 7. Taking $t_1 \to t_0^+$ in (24),

$$G(t) \geq \frac{4}{3}(\rho^{1/2} - 1)((\delta_- - 3bF(t_0))\rho + \delta_-\rho^{1/2} + \delta_-) \geq \frac{4}{3}(\rho - 1)\delta_-, \tag{25}$$

where $\rho(t) = F(t)/F(t_0)$. In particular $F^{(1)}(t) \neq 0$ for $t \in (t_0, t_1]$ and, since t_0 is a point of local minimum of F, $F^{(1)}$ is positive on $(t_0, t_1]$. So $F|(t_0, t_1]$ is increasing, and $F(t_1)F^{(1)}(t_1) > 0$ So $F(t)F^{(1)}(t) > 0$ for all $t > t_0$, and (25) still holds. Since F is strictly increasing on (t_0, ∞), so is ρ. Also

$$F^{(1)}(t) \geq 2\sqrt{\frac{\delta_-}{3}}\,(\rho - 1)^{1/2} = 2\sqrt{\frac{\delta_-}{3}}\,(F(t)/F(t_0) - 1)^{1/2}.$$

Then integration gives $\liminf_{t\to\infty} \frac{F(t)}{t^2} > 0$, and *3.* follows by applying this to W in place of V.

For *4.*, given $\lambda \in (0,1)$, choose ρ so large (and t accordingly) so that (25) gives

$$\rho(t)^{-3/4}F^{(1)}(t) \geq \lambda\sqrt{\frac{4(\delta_- - 3bF(t_0))}{3}} \quad \text{namely} \quad F(t)^{-3/4}F^{(1)}(t) \geq \lambda\sqrt{\frac{4(\delta_- - 3bF(t_0))}{3}}F(t_0)^{-3/4}.$$

Integration then proves *4.* for $t \to \infty$, and applying this to W proves the rest of *4.*.

For *5.*, $bF(t_0) > 0$ by *3.*. Also (18) gives $\delta_-(\rho + \rho^{1/2} + 1) \leq 3bF(t_1) \leq \delta_+(\rho + \rho^{1/2} + 1)$, where

$\rho = F(t_1)/F(t_0)$. Writing $f_i = F(t_i)^{1/2}$,

$$f_1^2(\mu - f_0^2) + f_1(\mu f_0) + \mu f_0^2 \ \leq \ 0, \tag{26}$$

where $\mu = \frac{\delta_-}{3b}$ and $\mu - f_0^2 \ < \ 0$ by *3.*. So either

$$f_1 \ \leq \ \frac{\mu f_0 - \sqrt{4\mu f_0^4 - 3\mu^2 f_0^2}}{2(f_0^2 - \mu)} \quad \text{or} \quad f_1 \ \geq \ \frac{\mu f_0 + \sqrt{4\mu f_0^4 - 3\mu^2 f_0^2}}{2(f_0^2 - \mu)}.$$

Now $(\mu f_0)^2 - (4\mu f_0^4 - 3\mu^2 f_0^2) = 4\mu f_0^2(\mu - f_0^2) < 0$ and, since $f_1 \geq 0$, this proves (21). Taking $\mu = \frac{\delta_+}{3b}$, the direction of the inequality in (26) reverses. When also $3bF(t_0) > \delta_+$, namely $\mu - f_0^2 \ < \ 0$, (22) follows in similar fashion to (21).

□

Notice that when $3bF(t_0) \ < \ 2\delta_-$, (21) in *5.* of Theorem 5 is stronger than $F(t_0) \ < \ F(t_1)$.

Example 9 *Figure* 10 *plots* F *in Example* 5, *with horizontal lines of heights* $\frac{\delta_\pm}{3b}$. *As in, Theorem* 5, $F(t_0) \leq \frac{\delta_+}{3b}$ *for every point* t_0 *of local minimum of* F. *Also notice that* $F(t_0) \leq \frac{\delta_+}{3b}$ *only for the first and last observed points* t_0 *of local minimum. Even if we only know* $F|[-20, 20]$, *it follows from Part 1 of Theorem* 5 *that the restriction of* F *to each of* $(-\infty, -20)$ *and* $(20, \infty)$ *is either bounded or has a point of local minimum.*

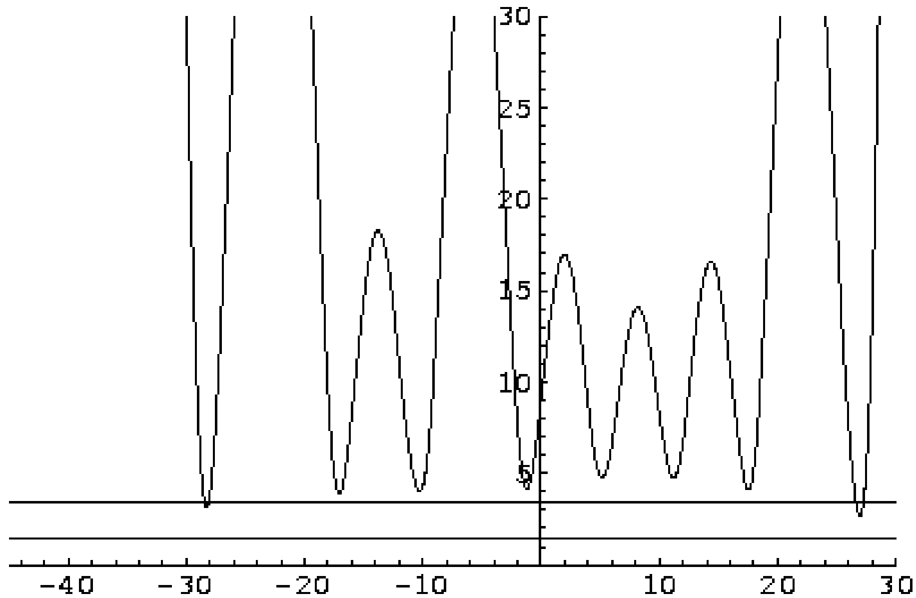

Figure 10. F in Examples 5, 9.

□

Under fairly general conditions $F(t)$ grows faster than linearly and V can be shown to possess *axes*, generalising a result for null Lie quadratics. The axes of a non-null Lie quadratic V have non-negative components in the direction of the constant vector C.

8 Superlinearity

Whether $b > 0$ or not, for $V(t) \neq \mathbf{0}$ write $U(t) = \frac{V(t)}{\|V(t)\|}$. Then $U^{(1)} = \frac{V^{(1)}}{F^{1/2}} - \frac{F^{(1)}U}{2F}$.

Lemma 3 $\|U^{(1)}\| = \frac{\sqrt{(d-F^{(4)})/3}}{F} \leq \frac{\|C\|+\sqrt{c}}{F}$.

Proof: $\|U^{(1)}\|^2 = \frac{4F\|V^{(1)}\|^2-(F^{(1)})^2}{4F^2} = \frac{F(F^{(2)}+b)-\frac{3}{4}(F^{(1)})^2}{3F^2} = \frac{d-F^{(4)}}{3F^2} \leq \frac{\delta_+}{3F^2}$, by Theorem 1.

□

Definition 1 *For* $\sigma = \pm$, F *is said to be* σ-superlinear *when, for some* $\epsilon > 0$, *we have* $0 < \liminf_{t \to \sigma\infty} \frac{F(t)}{|t|^{1+\epsilon}} < \infty$.

□

In Example 6, V is not superlinear. In general,

- if F is superlinear then $\epsilon \leq 3$, by Corollary 6.
- if $b < 0$ then F is $\pm$-superlinear with $\epsilon = 3$, by Theorem 3.
- for $b \geq 0$, if $F(t)$ is unbounded for $\sigma t > 0$, with finitely many critical points t_i satisfying $\sigma t_i > 0$, then F is σ-superlinear with ϵ either 1 or 3, by Theorem 5.
- if V is null and nonconstant then F is $\pm$-superlinear with $\epsilon = 1$.

Definition 2 *Let* $\sigma = \pm$, $p \in \mathbb{R}$ *and* $h : \mathbb{R} \to \mathbb{R}$ *be given. A function* $g : \mathbb{R} \to \mathbb{R}$ *is said to be* $O_\sigma(h)$ *when* $\limsup_{t \to \sigma\infty} \frac{g(t)}{h(t)} < \infty$.

□

Theorem 6 *If* F *is* σ*-superlinear then* $\alpha_\sigma(V) \equiv \lim_{t \to \sigma\infty} U(t)$ *exists, and*

$$U(t) \;=\; \alpha_\sigma(V) + O_\sigma(|t|^{-\epsilon}).$$

Proof: For $r < s$,

$$\|U(s) - U(r)\| \;\leq\; \int_r^s \|U^{(1)}(t)\| \; dt \;\leq\; \int_r^s \frac{\|C\| + \sqrt{c}}{F(t)} \; dt.$$

For $\sigma = +$, the right hand side is bounded above by $k(r^{-\epsilon} - s^{-\epsilon})$ for some constant $k > 0$, at least for r sufficiently large. So $\{U(j) : j \geq 1\} \subset S^2$ is Cauchy, therefore convergent, then $\lim_{s\to\infty} U(s)$ exists, in the limit as $s \to \infty$ $\|\alpha_+(V) - U(r)\| \leq kr^{-\epsilon}$, and similarly for $\sigma = -$.

□

Example 10 *Let* V *be a null Lie quadratic* V *with* $b = 1$, $c > 0$, *and* $F(t) = c + t^2$. *Set* $\tau(t) = \int_0^t \frac{\sqrt{c}}{F(t)}\, dt = \frac{1}{\sqrt{c}} \arctan \frac{t}{c}$ *and* $W_2(\tau) \equiv U(t) = (c + t^2)^{-1/2} V(t)$. *Differentiating with respect to* τ, *set* $W_1(\tau) \equiv W_2'(\tau) = c^{-1/2}((c + t^2)^{1/2} V^{(1)} - t(c + t^2)^{-1/2} V)$, *and* $W_3(\tau) \equiv W_1(\tau) \times W_2(\tau)$. *Then* $W \equiv \begin{bmatrix} W_1 & W_2 & W_3 \end{bmatrix} \in SO(3)$, *and*

$$W'(\tau) = \begin{bmatrix} 0 & -1 & c\ \sec^3(\sqrt{c}\tau) \\ 1 & 0 & 0 \\ -c\ \sec^3(\sqrt{c}\tau) & 0 & 0 \end{bmatrix} W(\tau),$$

where $\tau \in (-\pi/(2\sqrt{c}), \pi/(2\sqrt{c}))$. *So* U *and therefore* V *can be found by solving the third order homogeneous linear differential equation with variable coefficients*

$$\cos^3(\sqrt{c}\tau)\ \frac{d}{d\tau}((y''(\tau) + y(\tau))\ \cos^3(\sqrt{c}\tau)) \ + \ c^2 y'(\tau) \ = \ 0,$$

for $y : (-\pi/(2\sqrt{c}), \pi/(2\sqrt{c}) \to \mathbb{R}$.

□

Theorem 7 *Let* F *be* σ*-superlinear. Then*

$$\frac{f(t)}{t^2} \ = \ \frac{a_\sigma}{2} \ + \ O_\sigma(|t|^{-1}),$$

where $f \equiv F^{1/2}$ *and* $a_\sigma \equiv < C, \alpha_\sigma(V) >$.

Proof: Suppose $\sigma = +$. By (5), $fU^{(2)} + 2f^{(1)}U^{(1)} + f^{(2)}U = f^2U^{(1)} \times U + C$, and, since $\|U\| \equiv 1$, taking inner products of both sides with U gives

$$0 \le f^{(2)} - < C, U > = f < U^{(1)}, U^{(1)} > \le \frac{\delta_+}{3f^3} = O_+(t^{-3(1+\epsilon)/2}),$$

by Lemma 3 and because F is +-superlinear. Then $f^{(2)}(t) = a_+ + O_+(t^{-\epsilon})$, by Theorem 6. Integrating, for large $t_0 < t$,

$$f^{(1)}(t) = a_+(t - t_0) + f^{(1)}(t_0) + O_+(K_1(t) - K_1(t_0)),$$

where $K_1(t)$ is $\ln t$ or $t^{1-\epsilon}/(1-\epsilon)$, according as $\epsilon = 1$ or not. Integrating again,

$$f(t) = \frac{a_+(t-t_0)^2}{2} + (t - t_0)f^{(1)}(t_0) + f(t_0) + O_+(K_2(t) - (t - t_0)K_1(t_0)),$$

where $K_2(t) = t\ln t - t_0 \ln t_0 - t + t_0$, $t_0^{-1} - t^{-1}$, or $(t^{2-\epsilon} - t_0^{2-\epsilon})/((1-\epsilon)(2-\epsilon))$, according as ϵ is 1, 2 or neither. In any case, $\frac{f(t)}{t^2} = \frac{a_+}{2} + O_+(t^{-1})$. For $\sigma = -$ apply what has already been proved to the Lie quadratic W given by $W(s) = -V(-s)$, noting that W has constant $-C$ and $\alpha_+(W) = -\alpha_-(V)$.

□

Corollary 9 *Let F be σ-superlinear. Then $< C, \alpha_\sigma > \ \ge 0$, and either*

- $\epsilon = 1$, $< C, \alpha_\sigma > \ = 0$, *and* $b \ge 0$, *or*
- $\epsilon = 3$, *or*

- $< C, \alpha_\sigma > \;=\; 0, \quad b \geq 0,$ *and* F *has critical points* t_0 *with* σt_0 *arbitrarily large.*

Proof: If $a_+ > 0$ then $\epsilon = 3,$ by Theorem 7. Alternatively, if $a_\sigma = 0$ then $b \geq 0$ by Theorem 3. Then, by Theorem 5, either $\epsilon = 1$ or F has critical points t_0 with σt_0 arbitrarily large.

□

Unlike the null case, where convergence to axes is more or less steady, unbounded non-null Lie quadratics in E^3 may explore numerous possibilities before settling on an asymptotic direction, at least when $b > 0$.

Example 11 *In Example* 5, $< C, U(-45) > \;=\; -0.03255$ *and* $< C, U(30) > \;=\; 0.2915$. *So, by Corollary* 9, $U(-45)$ *is far from* $\alpha_-(V)$, *which is not apparent from Figures* 6, 7. *Solving* (5) *numerically for* V *over the larger domain* $[-240, 30]$, $U : [-240, 30] \to S^2$ *is shown in Figure* 11, *together with the line segment from* $(0,0,0)$ *to* C *and labels when* $t = -240,\ -45,\ 30$. *We find* $< C, U(-240) > \;=\; 0.62375$, *and Corollary* 9 *permits* $\alpha_-(V) \approx U(-240), \quad \alpha_+(V) \approx U(30)$.

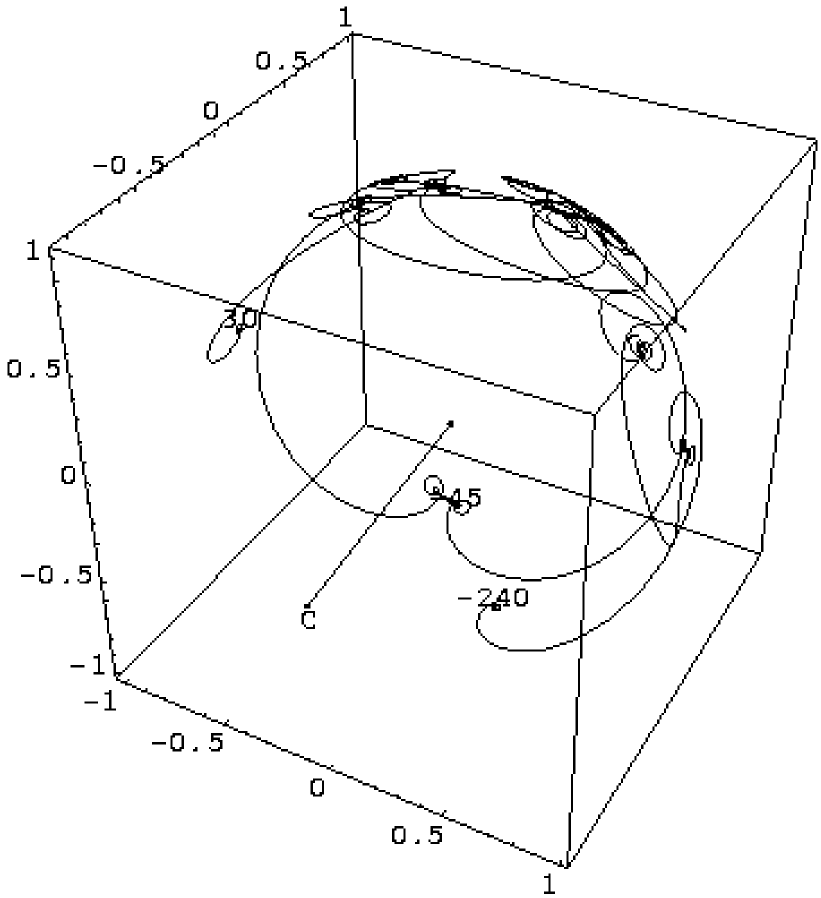

Figure 11. U in Examples 5, 11.

□

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