JOURNAL OF MATHEMATICAL PHYSICS VOLUME 42, NUMBER 4 APRIL 2001

Computing the inertia operator of a rigid body

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(Received 24 August 2000; accepted for publication 22 December 2000)

We prove that the inertia operator A of a rigid body is generically determined, up to a scalar multiple, by the curve Ω in $\mathbf{R}^3$ that describes its angular velocity in the body. The precise condition is that Ω not be contained in a two-dimensional subspace of $\mathbf{R}^3$. We derive two indirect methods to compute A from the values of Ω over an arbitrary interval, and a direct method to compute A from the second- and fourth-order moments of Ω. The direct method utilizes moment identities derived from symmetries in Euler's equation.
[DOI: 10.1063/1.1352051]

I. INTRODUCTION

In this section we review some of the kinematics and dynamics of rigid bodies as presented in Ref. 1. Consider a rigid body whose center of mass is fixed at $\mathbf{0}\in\mathbf{R}^3$, and let F be an orthonormal frame at $\mathbf{0}$ fixed relative to the body. At time t during the motion, F is an orthonormal frame $B(t)=(b_1(t),b_2(t),b_3(t))$ relative to ambient space $\mathbf{R}^3$. Suppose that $B(0)$ is the standard basis of $\mathbf{R}^3$. Identifying triples of column vectors with 3×3 matrices, $B(t)$ lies in the rotation group $SO(3)$, and $B(0)$ is the identity I. Relative to the standard basis, the position at time t of a point fixed relative to the body is then $q(t)=B(t)q(0)$, where $q(0)\in\mathbf{R}^3$ is the position at time 0. For $u\in\mathbf{R}^3$ define the skew-symmetric operator $\alpha(u):\mathbf{R}^3\to\mathbf{R}^3$ by $\alpha(u)v=u\times v$, $v\in\mathbf{R}^3$ where $\times$ is the cross product. The *angular velocity in the body* is the function $\Omega:\mathbf{R}\to\mathbf{R}^3$ defined by

$$\dot{B}=B\alpha(\Omega). \tag{1}$$

The *angular velocity in space* is $\omega=B\Omega$ and Eq. (1) implies that

$$\dot{\omega}=B\dot{\Omega}. \tag{2}$$

Inertial motion of the body is described by *Euler's equation*[2]

$$A\dot{\Omega}=(A\Omega)\times\Omega, \tag{3}$$

where A is the symmetric positive-definite 3×3 matrix that represents the inertia operator with respect to the standard basis (the inertia operator is defined from the mass distribution of the body). It follows from Eq. (1) and (3) that the *kinetic energy* $E=\frac{1}{2}\Omega^TA\Omega$ and the *angular momentum in space* $\mathbf{m}=BA\Omega$ are constant. Write $m=\|\mathbf{m}\|$. Evidently $\Omega^TA^2\Omega=m^2$, and $\omega^T\mathbf{m}=2E$. Notice that Ω is analytic, since it is bounded and its derivative is an analytic function of Ω. Suppose from now on that the rigid body is not fixed, namely q is a nonconstant function of t. Then $E,m>0$.

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There is an extensive classical literature including Refs. 2–4 that addresses the problem of computing the solution Ω of Euler's equation from A and the initial value of Ω. For the remainder of this paper we consider the inverse problem of computing A from a solution Ω. For future investigations we record the fact that Euler's equation describes uniform speed parametrized geodesics on the rotation group with a left-invariant Riemmanian metric induced by the inertia operator, and that it can be generalized to describe geodesics on Lie groups[1] and Riemannian manifolds.[5] An important objective of this paper, and one motivation for our detailed derivations of three distinct methods to compute the inertia operator, is to develop techniques for investigating more inverse problems in these more general contexts.

II. DEGENERACY

Let $\Omega:\mathbf{R}\rightarrow\mathbf{R}^3$ be a solution of Euler's equation (3). Call Ω *degenerate* when its range is contained in a two-dimensional subspace of $\mathbf{R}^3$. Otherwise Ω is *nondegenerate*. The distinction turns out to be useful in determining A from Ω in Sec. III. Euler's equation is invariant with respect to SO(3) in the sense that for any $O\in\mathrm{SO}(3)$, $\tilde{\Omega}=O\Omega$ satisfies Euler's equation where A is replaced by $\tilde{A}=OAO^T$. Evidently $\tilde{\Omega}$ is degenerate if and only if Ω is degenerate. Choose $O\in\mathrm{SO}(3)$ such that

$$OAO^T=\mathrm{diag}(I_1,I_2,I_3), \tag{4}$$

where $0<I_1\leqslant I_2\leqslant I_3$ are the eigenvalues of A (Arnold lists the eigenvalues in the opposite order).

Proposition 1: The function Ω is degenerate if and only if $m^2/2E\in\{I_1,I_2,I_3\}$. If Ω is nondegenerate then $\|\dot{\Omega}\|$ is bounded below by a positive number and Ω is periodic.

Proof: It suffices to prove these assertions for $\tilde{\Omega}$. Define $\alpha_j\equiv I_j-m^2/(2E)$, $j=1,2,3$. Since the range of $\tilde{\Omega}$ is contained in both the energy ellipsoid $\epsilon_E\equiv\{x\in\mathbf{R}^3: I_1x_1^2+I_2x_2^2+I_3x_3^2=2E\}$ and the momentum (or co-adjoint) ellipsoid $\epsilon_m\equiv\{x\in\mathbf{R}^3: I_1^2x_1^2+I_2^2x_2^2+I_3^2x_3^2=m^2\}$, it is contained in $\epsilon_h\equiv\{x\in\mathbf{R}^3:\alpha_1I_1x_1^2+\alpha_2I_2x_2^2+\alpha_3I_3x_3^2=0\}$ and $I_1\leqslant m^2/2E\leqslant I_3$. If $m^2/2E\in\{I_1,I_3\}$ then $\tilde{\Omega}$ is degenerate since it is constant. If $I_1<m^2/2E=I_2<I_3$ then the range of $\tilde{\Omega}$ is contained in the union $\{x\in\mathbf{R}^3:\sqrt{-\alpha_1 I_1}\,x_1=\sqrt{\alpha_3 I_3}\,x_3=0\}\cup\{x\in\mathbf{R}^3:\sqrt{-\alpha_1 I_1}\,x_1=\sqrt{\alpha_3 I_3}\,x_3=0\}$ of two two-dimensional subspaces of $\mathbf{R}^3$. Therefore, since $\tilde{\Omega}$ is analytic, its range is entirely contained in one of these two subspaces and it is degenerate (the orbits consist of two fixed points and four heteroclinic connections between them). To complete the proof, observe that if $m^2/2E\notin\{I_1,I_2,I_3\}$ then the ellipsoids ϵ_E and ϵ_m intersect transversally. Hence their intersection is diffeomorphic to the disjoint union of two circles and Euler's equation implies that $\|\dot{\Omega}\|$ is bounded below by a positive number. Furthermore, the range of $\tilde{\Omega}$ is contained in a half-space $H\subset\mathbf{R}^3$. Therefore, $\tilde{\Omega}$ is nondegenerate and periodic.

III. *A* FROM Ω: METHODS 1 AND 2

Consider the problem of determining the inertia matrix A from the solution Ω of Euler's equation (3). Clearly A is at most determined up to a positive scalar multiple. When this happens we say A is *almost-determined* by Ω.

Theorem 2: *A is almost-determined by Ω if and only if Ω is nondegenerate.*

Proof: Assume Ω is degenerate. Then there is a nonzero $v\in\mathbf{R}^3$ such that $v^T\Omega=v^T\dot{\Omega}=0$. Define a symmetric positive definite matrix $A_v=A+vv^T$. Then Ω satisfies Eq. (3) with A replaced by A_v, and A_v is not a scalar multiple of A. This proves the only if part. To prove the if part choose any $s_2>s_1$ and define three matrices and one vector

$$M_\Omega=\int_{s_1}^{s_2}\Omega(t)\,\Omega(t)^T\,dt,\quad M_\omega=\int_{s_1}^{s_2}\omega(t)\,\omega(t)^T\,dt,$$

$$M_{\mathbf{m}}=\int_{s_1}^{s_2}\Omega(t)\,\mathbf{m}^T B(t)\,dt,\quad u_\omega=\int_{s_1}^{s_2}\omega(t)dt.$$

Then $M_{\mathbf{m}}$ is linear in $\mathbf{m}$, and A and $\mathbf{m}$ satisfy

$$M_\Omega A=M_{\mathbf{m}},\tag{5}$$

$$M_\omega \mathbf{m}=2E\,u_\omega.\tag{6}$$

Proposition 3: If Ω is nondegenerate then the matrices M_Ω and M_ω are nonsingular.

Proof: Suppose that M_Ω is singular and choose a nonzero vector $w\in\mathbf{R}^3$ such that $w^TM_\Omega w=0$. Then $w^T\Omega(t)\,\Omega(t)^Tw=0$ for all $t\in[s_1,s_2]$, namely $\|\Omega(t)^Tw\|=0$. Then $\Omega(t)$ lies in the plane V orthogonal to w for all $t\in[s_1,s_2]$ and hence for all $t\in\mathbf{R}$ since Ω is analytic, therefore Ω is degenerate. To show that M_ω is nonsingular it suffices to show that the range of ω is not contained in any two-dimensional subspace of $\mathbf{R}^3$. Assume to the contrary that the range of $\omega([s_1,s_2])$ is contained in a two-dimensional subspace V of $\mathbf{R}^3$ and hence in the line $V\cap\{x:x^T\mathbf{m}=2E\}$. By Proposition 1 $\|\dot{\Omega}\|=\|\dot{\omega}\|$ is bounded below by a positive number, thus the range of ω is unbounded. This is impossible since $\|\omega\|=\|\Omega\|$ and $\|\Omega\|$ is bounded above and the proposition is proved.

Define $C(t),t\in[s_1,s_2]$ by the nonautonomous linear differential equation $\dot{C}=C\alpha(\Omega)$ and initial value $C(s_1)=I$. Then $B(t)=B(s_1)C(t)$, $\omega(t)=B(s_1)C(t)\Omega(t)$, $u_\omega=B(s_1)\int_{s_1}^{s_2}C(t)\Omega(t)dt$, and Eq. (6) implies that

$$\left(\int_{s_1}^{s_2}C(t)\Omega(t)\Omega(t)^TC(t)^T\,dt\right)B(s_1)^T\mathbf{m}=2E\int_{s_1}^{s_2}C(t)\Omega(t)dt.$$

Proposition 3 implies that $B(s_1)^T\mathbf{m}$ is determined up to a positive scalar multiple by Ω, hence $\mathbf{m}^TB(t)$ and $M_{\mathbf{m}}$ are also determined up to a positive scalar multiple. Finally, Eq. (5) and Proposition 3 show that A is determined up to a positive scalar multiple from Ω. This proves Theorem 2 and provides a constructive method (Method 1) for almost-determining A from Ω.

Method 1 requires Ω to be observed on a continuous interval $[s_1,s_2]$, and may be difficult to apply in cases where Ω is known only on finite subsets of $[s_1,s_2]$. When sampling is coarse or noise contaminated, solving the ordinary differential equation for C is especially problematic. Notice also that Method 1 uses the associations between t and $\Omega(t)$: It is not enough to observe the image of Ω over $[s_1,s_2]$. Alternatively, we can apply Theorem 2 in a less direct way to almost-determine A as follows. Let Ω be a nondegenerate solution of Eq. (3), and let $s_2>s_1$. Denote the space of symmetric 3×3 matrices by S, and define a quadratic form $Q:S\to\mathbf{R}$ by

$$Q(D)=\int_{s_1}^{s_2}\|D\dot{\Omega}(t)-(D\Omega(t))\times\Omega(t)\|^2dt,\quad D\in S.$$

Corollary 4: Q has co-rank 1 and A is a null-eigenvector of Q. So Q defines A up to a positive scalar multiple.

Proof: Clearly $A\in S$ and $Q(A)=0$. Let $D\in S$ with $Q(D)=0$. It suffices to show that there exists $\mu\in\mathbf{R}$ such that $D=\mu A$. Choose $\epsilon>0$ such that $A_\epsilon=A+\epsilon D$ is positiive definite. Clearly $Q(A_\epsilon)=0$, hence Ω satisfies Eq. (3) with inertia matrix A_ϵ over the interval $[s_1,s_2]$ and thus over $\mathbf{R}$ since Ω is analytic. Theorem 2 implies that there exists $\lambda>0$ such that $A_\epsilon=\lambda A$. Therefore $D=\mu A$ where $\mu=(\lambda-1)/\epsilon$ and the corollary is proved.

The quadratic form Q is determined by values of Ω and $\dot{\Omega}$ over $[s_1,s_2]$, thus Corollary 4 gives another method (Method 2) to almost-determine A from Ω. We observe that (i) Q can be approximated by sampling $(\Omega,\dot{\Omega})$ uniformly over $[s_1,s_2]$, (ii) the associations between t and $(\Omega(t),\dot{\Omega}(t))$ are not needed, (iii) when observations of $(\Omega,\dot{\Omega})$ are contaminated by noise, the effects in the formula for Q are approximately additive, and (iv) Method 2 does not require the intermediate step of solving an ODE for $C(t)$. These observations suggest that Method 2 may be more robust and practical than Method 1. However, Method 2 requires us to know $\dot{\Omega}$. It may be difficult to accurately estimate $\dot{\Omega}$ from values of Ω that are coarsely sampled or noise contaminated. These difficulties are overcome (to some degree) by using a method, based on moments of Ω (Method 3 in Sec. VIII), to almost-determine A. It turns out that moments of orders 0, 2, and 4 give most of the needed information, and it seems these quantities can be reliably estimated from uniform samples of Ω over $[s_1,s_2]$ (without regard to their time association). To simplify the discussion we assume that the moments of Ω are calculated over an entire orbit of the motion: The more general case of sampling over $[s_1,s_2]$ is a straightforward extension. Even so, the analysis is lengthy and not as clean as for Methods 1 and 2. We also note that genericity assumptions are required for Sec. VIII.

IV. MOMENTS

Let Ω be a periodic solution of Euler's equation (3), of period $T>0$.

Definition 5: Given a monomial $\mu:\mathbf{R}^3\to\mathbf{R}$ *the associated moment* Ω^μ *is defined to be*

$$\int_0^T \mu\circ\Omega(t)\,dt.$$

The moment Ω^μ *associated with the monomial* $\mu=x_{i_1}x_{i_2}\ldots x_{i_d}$ *is denoted by* $\Omega^{i_1 i_2\cdots i_d}$. *The degree* d *of the monomial* μ *is said to be the order of the moment* Ω^μ.

We shall see that the moments satisfy many interesting identities involving the inertia matrix A. As before, we frequently use conservation of energy

$$E=\tfrac{1}{2}\Omega^T A\Omega \tag{7}$$

and conservation of squared length of angular momentum

$$m^2=\|A\Omega\|^2. \tag{8}$$

Assume that the eigenspaces of A are one-dimensional, namely that the I_j are distinct where $j=1,2,3$. Then any orthogonal matrix O satisfying (4) maps eigenvectors of A to nonzero multiples of standard basis elements of $\mathbf{R}^3$. Furthermore, O is *almost-unique* in the sense that, if $O'\in O(3)$ diagonalizes A then $O'=DO$, where D is a diagonal matrix whose diagonal entries are ±1.

Let Ω be nondegenerate. Write $\alpha_j\equiv I_j-m^2/2E$. Theorem 1 says $\alpha_j\neq0$ for $j=1,2,3$ and, because the I_j are distinct, $\alpha_1<0,\alpha_2<\alpha_3$. Moments of Ω may be called *empirical*: They can be inferred from observations of the trajectory of the rigid body. Moments of $\tilde{\Omega}\equiv O\Omega$, where $O\in\mathrm{SO}(3)$ satisfies (4), are said to be *almost-canonical* (they are not quite canonical because O is not quite unique). Empirical moments are linear combinations of almost-canonical moments (and vice versa) with coefficients polynomial in entries of O. In order to obtain identities there are some simple things we can try with almost-canonical moments. First, integration of Eqs. (7) and (8) gives

$$I_1\tilde{\Omega}^{11}+I_2\tilde{\Omega}^{22}+I_3\tilde{\Omega}^{33}=2TE, \tag{9}$$

$$I_1^2\tilde{\Omega}^{11}+I_2^2\tilde{\Omega}^{22}+I_3^2\tilde{\Omega}^{33}=Tm^2. \tag{10}$$

Solving Eqs. (9) and (10) gives

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$$\tilde{\Omega}^{22} = \frac{2\alpha_3 TE - I_1(I_3 - I_1)\tilde{\Omega}^{11}}{I_2(I_3 - I_2)}, \tag{11}$$

$$\tilde{\Omega}^{33} = \frac{-2\alpha_2 TE + I_1(I_2 - I_1)\tilde{\Omega}^{11}}{I_3(I_3 - I_2)}. \tag{12}$$

From Eqs. (11) and (12) we obtain the following bounds on $\tilde{\Omega}^{11}$ in terms of E, m^2, T and the I_j:

$$\frac{2\alpha_2 TE}{I_1(I_2 - I_1)} < \tilde{\Omega}^{11} < \frac{2\alpha_3 TE}{I_1(I_3 - I_1)}. \tag{13}$$

This begs the question of how to calculate $\tilde{\Omega}^{11}$. This can be done by quadrature, as in Secs. V and VI, where we also consider third-order moments. Before doing that, we say some things about fourth-order moments. Multiplying Eqs. (7) and (8) by $\tilde{\Omega}_i^2$ and integrating gives

$$I_1\tilde{\Omega}^{1111} + I_2\tilde{\Omega}^{1122} + I_3\tilde{\Omega}^{1133} = 2E\tilde{\Omega}^{11}, \tag{14}$$

$$I_1\tilde{\Omega}^{1122} + I_2\tilde{\Omega}^{2222} + I_3\tilde{\Omega}^{2233} = 2E\tilde{\Omega}^{22}, \tag{15}$$

$$I_1\tilde{\Omega}^{1133} + I_2\tilde{\Omega}^{2233} + I_3\tilde{\Omega}^{3333} = 2E\tilde{\Omega}^{33}, \tag{16}$$

$$I_1^2\tilde{\Omega}^{1111} + I_2^2\tilde{\Omega}^{1122} + I_3^2\tilde{\Omega}^{1133} = m^2\tilde{\Omega}^{11}, \tag{17}$$

$$I_1^2\tilde{\Omega}^{1122} + I_2^2\tilde{\Omega}^{2222} + I_3^2\tilde{\Omega}^{2233} = m^2\tilde{\Omega}^{22}, \tag{18}$$

$$I_1^2\tilde{\Omega}^{1133} + I_2^2\tilde{\Omega}^{2233} + I_3^2\tilde{\Omega}^{3333} = m^2\tilde{\Omega}^{33}. \tag{19}$$

The system of equations (14)–(18) has rank 5. We now need to consider the cases $\alpha_2 < 0$ and $\alpha_2 > 0$ separately.

V. THE CASE $\alpha_2 < 0$

Suppose $\alpha_2 < 0$. Then $x_3 \neq 0$ for $x \in \epsilon_E \cap \epsilon_m$. Solving Eqs. (7) and (8), x lies on at least one of eight arcs $x_{\sigma_1,\sigma_2,\sigma_3}$ given parametrically by

$$x_{\sigma_1,\sigma_2,\sigma_3}(v) = \begin{bmatrix} \sigma_1\sqrt{\dfrac{I_3 v^2(I_3 - I_2) + 2\alpha_2 E}{I_1(I_2 - I_1)}} \\ \sigma_2\sqrt{\dfrac{-2\alpha_1 E - I_3 v^2(I_3 - I_1)}{I_2(I_2 - I_1)}} \\ \sigma_3 v \end{bmatrix}, \quad v \in [\beta_L, \beta_U], \tag{20}$$

where $\sigma_j = \pm$, $j = 1,2,3$, $\beta_L = \sqrt{-2\alpha_2 E/(I_3(I_3 - I_2))}$, and $\beta_U = \sqrt{-2\alpha_1 E/(I_3(I_3 - I_1))}$. Then $0 < \beta_L < \beta_U$, and

$$x_{\sigma_1,\sigma_2,\sigma_3 1}(v) \neq 0 \quad \text{for} \quad v \in (\beta_L, \beta_U], \quad x_{\sigma_1,\sigma_2,\sigma_3 1}(\beta_L) = 0, \tag{21}$$

$$x_{\sigma_1,\sigma_2,\sigma_3 2}(v) \neq 0 \quad \text{for} \quad v \in [\beta_L, \beta_U), \quad x_{\sigma_1,\sigma_2,\sigma_3 2}(\beta_U) = 0, \tag{22}$$

$$x_{\sigma_1,\sigma_2,\sigma_3 3}(v) \neq 0 \quad \text{for} \quad v \in [\beta_L, \beta_U]. \tag{23}$$

Since the image C of $\tilde{\Omega}$ is diffeomorphic to a circle, C is a union of arcs. By Eq. (23) and because C is connected, σ_3 is fixed. Given σ_3, there are at most four such arcs in C, corresponding to choices of σ_1, σ_2. Inspection of Eqs. (21) and (22) show that, in order for C to be connected, all four arcs are needed: C is the track sum of

$$x_{+,+,\sigma_3}(v), \quad x_{+,-,\sigma_3}(\beta_U-\beta_L-v), \quad x_{-,-,\sigma_3}(v), \quad x_{-,+,\sigma_3}(\beta_U-\beta_L-v),$$

where $v\in[\beta_L,\beta_U]$. By Eq. (3) $I_3\dot{\tilde{\Omega}}_3(v)=-(I_2-I_1)\tilde{\Omega}_1(v)\tilde{\Omega}_2(v)\neq 0$ for $v\in(\beta_L,\beta_U)$, and so $\tilde{\Omega}^{\mu}=[I_3/(I_2-I_1)]\int_{\beta_L}^{\beta_U}P(v)dv$, where

$$P=\frac{\mu(x_1,x_2,x_3)+\mu(x_1,-x_2,x_3)+\mu(-x_1,-x_2,x_3)+\mu(-x_1,x_2,x_3)}{x_1x_2}$$

and $x\equiv x_{+,+,\sigma_3}(v)$. From this formula the moments $\tilde{\Omega}^{\mu}$ can be evaluated by quadrature. The following result follows directly from the form of P.

Proposition 6: Write $\mu=x_1^{a_1}x_2^{a_2}x_3^{a_3}$. *If* a_1 *or* a_2 *is odd then* $\tilde{\Omega}^{\mu}=0$. *Otherwise*

$$\tilde{\Omega}^{\mu}=\frac{4I_3}{I_2-I_1}\int_{\beta_L}^{\beta_U}x_1(v)^{a_1-1}x_2(v)^{a_2-1}x_3(v)^{a_3}\,dv\neq 0.$$

This result provides the following examples of zeroth-, first-, and second-order almost-canonical moment identities.

Example 7: If μ *is the monomial of degree* 0 *whose value is* 1 *everywhere then the corresponding moment is the period*

$$T=4I_3\sqrt{I_1I_2}\int_{\sqrt{-2\alpha_2E/I_3(I_3-I_2)}}^{\sqrt{-2\alpha_1E/I_3(I_3-I_1)}}\times\frac{1}{\sqrt{(2\alpha_2E+I_3v^2(I_3-I_2))(-2\alpha_1E-I_3v^2(I_3-I_1))}}\,dv.$$

Example 8: $\tilde{\Omega}^1=\tilde{\Omega}^2=0$, *and*

$$\tilde{\Omega}^3=4I_3\sqrt{I_1I_2}\,\mathrm{sign}(\tilde{\Omega}_3)\int_{\sqrt{-2\alpha_2E/I_3(I_3-I_2)}}^{\sqrt{-2\alpha_1E/I_3(I_3-I_1)}}\times\frac{v}{\sqrt{(2\alpha_2E+I_3v^2(I_3-I_2))(-2\alpha_1E-I_3v^2(I_3-I_1))}}\,dv\neq 0.$$

Example 9: $\tilde{\Omega}^{12}=\tilde{\Omega}^{23}=\tilde{\Omega}^{31}=0$, *and the* $\tilde{\Omega}^{ii}$ *are nonzero. In particular*

$$\tilde{\Omega}^{11}=\frac{4I_3}{I_2-I_1}\sqrt{\frac{I_2}{I_1}}\int_{\sqrt{-2\alpha_2E/I_3(I_3-I_2)}}^{\sqrt{-2\alpha_1E/I_3(I_3-I_1)}}\sqrt{\frac{2\alpha_2E+I_3v^2(I_3-I_2)}{-2\alpha_1E-I_3v^2(I_3-I_1)}}\,dv>0,$$

and the two remaining nonzero second-order almost-canonical moments can be calculated algebraically in terms of $\tilde{\Omega}^{11}$ *as in Sec. IV. Alternatively, we can calculate either* $\tilde{\Omega}^{22}$ *or* $\tilde{\Omega}^{33}$ *by quadrature and solve algebraically for the other nonzero second-order moments.*

Theorem 10: *There are ten third-order almost-canonical moments, all except three are zero, and these are given by* $(\tilde{\Omega}^{113},\tilde{\Omega}^{223},\tilde{\Omega}^{333})=\tilde{\Omega}^3Q/((I_3-I_1)(I_3-I_2))$ *where* $Q=((I_3-I_2)\alpha_3/I_1,(I_3-I_1)\alpha_3/I_2,(-I_3(\alpha_1+\alpha_2)+I_1\alpha_2+I_2\alpha_1)/I_3)$.

Proof: We multiply both sides of Eq. (7) and (8) by $\tilde{\Omega}_3$ and integrate to obtain

$$I_1\tilde{\Omega}^{113}+I_2\tilde{\Omega}^{223}+I_3\tilde{\Omega}^{333}=2E\tilde{\Omega}^3, \tag{24}$$

$$I_1^2\tilde{\Omega}^{113}+I_2^2\tilde{\Omega}^{223}+I_3^2\tilde{\Omega}^{333}=m^2\tilde{\Omega}^3. \tag{25}$$

We integrate

$$\tilde{\Omega}_1\dot{\tilde{\Omega}}_2+\dot{\tilde{\Omega}}_1\Omega_2=\frac{I_2-I_3}{I_1}\tilde{\Omega}_2^2\tilde{\Omega}_3+\frac{I_3-I_1}{I_2}\tilde{\Omega}_1^2\tilde{\Omega}_3$$

(from Euler's Equation) and use $\tilde{\Omega}(0)=\tilde{\Omega}(T)$ to obtain

$$\frac{I_2-I_3}{I_1}\tilde{\Omega}^{223}+\frac{I_3-I_1}{I_2}\tilde{\Omega}^{113}=0, \tag{26}$$

and then solve the system (24), (25), (26) to complete the proof.

Theorem 11: *There are fifteen almost-canonical fourth-order moments. All of these are zero except for* $\tilde{\Omega}^{1111},\tilde{\Omega}^{1122},\tilde{\Omega}^{1133},\tilde{\Omega}^{2222},\tilde{\Omega}^{2233},\tilde{\Omega}^{3333}$. *These are positive and they satisfy a nondegenerate system of six linear equations whose matrix coefficients are rational functions of the* I_j *and whose right-hand sides are linear separately in* E,m^2, *and the second-order almost-canonical moments* $\tilde{\Omega}^{jj},j=1,2,3$.

Proof: We integrate

$$\tilde{\Omega}_2\tilde{\Omega}_3\dot{\tilde{\Omega}}_1+\tilde{\Omega}_3\tilde{\Omega}_1\dot{\tilde{\Omega}}_2+\tilde{\Omega}_1\tilde{\Omega}_2\dot{\tilde{\Omega}}_3=-\frac{I_3-I_2}{I_1}\tilde{\Omega}_2^2\tilde{\Omega}_3^2+\frac{I_3-I_1}{I_2}\tilde{\Omega}_1^2\tilde{\Omega}_3^2-\frac{I_2-I_1}{I_3}\tilde{\Omega}_1^2\tilde{\Omega}_2^2,$$

[from Eq. (3)] to obtain

$$-\frac{I_3-I_2}{I_1}\tilde{\Omega}^{2233}+\frac{I_3-I_1}{I_2}\tilde{\Omega}^{1133}-\frac{I_2-I_1}{I_3}\tilde{\Omega}^{1122}=0. \tag{27}$$

The proof is complete since the determinant of the coefficient matrix of the system (14), (15), (16), (17), (18), (27) equals $3I_1I_2I_3(I_2-I_1)(I_3-I_1)(I_3-I_2)$, which is positive.

Comparing Theorem 11 with Example 9 we obtain

Corollary 12: The second- and fourth-order almost-canonical moments are independent of the choice of O. *They are determined algebraically by* E,m^2,T, *the* I_j, $j=1,2,3$, *and any one of* $\tilde{\Omega}^{ii}$, $j=1,2,3$.

VI. THE CASE $\alpha_2>0$

When $\alpha_2>0$, $x_1\neq 0$ for $x\in\epsilon_E\cap\epsilon_m$, and x lies on at least one of eight arcs $x_{\sigma_1,\sigma_2,\sigma_3}$ given by

$$x_{\sigma_1,\sigma_2,\sigma_3}(v)=\begin{bmatrix}\sigma_1 v\\ \sigma_2\sqrt{\dfrac{2\alpha_3E-I_1v^2(I_3-I_1)}{I_2(I_3-I_2)}}\\ \sigma_3\sqrt{\dfrac{I_1v^2(I_2-I_1)-2\alpha_2E}{I_3(I_3-I_2)}}\end{bmatrix}, \tag{28}$$

where the $\sigma_j=\pm$. The arguments of Sec. V then adapt as follows: $v\in[\beta_L,\beta_U]$ where

$$\beta_L = \sqrt{\frac{2\alpha_2 E}{I_1(I_2 - I_1)}}, \quad \beta_U = \sqrt{\frac{2\alpha_3 E}{I_1(I_3 - I_1)}}.$$

In this case

$$x_{\sigma_1,\sigma_2,\sigma_3 1}(v) \neq 0 \quad \text{for} \quad v \in [\beta_L, \beta_U], \tag{29}$$

$$x_{\sigma_1,\sigma_2,\sigma_3 2}(v) \neq 0 \quad \text{for} \quad v \in [\beta_L, \beta_U), \quad x_{\sigma_1,\sigma_2,\sigma_3 2}(\beta_U) = 0, \tag{30}$$

$$x_{\sigma_1,\sigma_2,\sigma_3 3}(v) \neq 0 \quad \text{for} \quad v \in (\beta_L, \beta_U], \quad \text{and} \quad x_{\sigma_1,\sigma_2,\sigma_3 3}(\beta_L) = 0. \tag{31}$$

As before, the image C of $\tilde{\Omega}$ is a union of arcs, but this time σ_1 is fixed. Given σ_1, we argue as before, but using Eqs. (30) and (31), that C is parametrized by four arcs, namely the track sum of

$$x_{\sigma_1,+,+}(v), \quad x_{\sigma_1,-,+}(\beta_U - \beta_L - v), \quad x_{\sigma_1,-,-}(v), \quad x_{\sigma_1,-,+}(\beta_U - \beta_L - v),$$

where $v \in [\beta_L, \beta_U]$. By Eq. (3) $I_1\tilde{\Omega}_1(v) = (I_3 - I_2)\tilde{\Omega}_2(v)\tilde{\Omega}_3(v) \neq 0$ for $v \in (\beta_L, \beta_L)$, and so

$$\tilde{\Omega}^{\mu} = \frac{I_1}{I_3 - I_2} \int_{\beta_L}^{\beta_U} P(v)\, dv,$$

where

$$P = \frac{\mu(x_1, x_2, x_3) + \mu(x_1, -x_2, x_3) + \mu(-x_1, -x_2, x_3) + \mu(-x_1, x_2, x_3)}{x_1 x_2}$$

and $x \equiv x_{\sigma_1,+,+}(v)$. The following result is analogous to Proposition (6).

Proposition 13: Write $\mu = x_1^{a_1} x_2^{a_2} x_3^{a_3}$. *If* a_2 or a_3 *is odd then* $\tilde{\Omega}^{\mu} = 0$. *Otherwise*

$$\tilde{\Omega}^{\mu} = \frac{4I_1}{I_3 - I_2} \int_{\beta_L}^{\beta_U} x_1(v)^{a_1} x_2(v)^{a_2 - 1} x_3(v)^{a_3 - 1}\, dv \neq 0.$$

Example 14: The moment that corresponds to the monomial of degree 0 $(which = 1)$ *is the period*

$$T = 4I_1\sqrt{I_2 I_3} \int_{\sqrt{2\alpha_2 E/I_1(I_2 - I_1)}}^{\sqrt{2\alpha_3 E/I_1(I_3 - I_1)}} \times \frac{1}{\sqrt{(-2\alpha_2 E + I_1 v^2 (I_2 - I_1))(2\alpha_3 E - I_1 v^2 (I_3 - I_1))}}\, dv.$$

Example 15: Let $\tilde{\Omega}^j$ *be the first-order moments corresponding to* x_j *where* $j = 1,2,3$. *Then* $\tilde{\Omega}^2 = \tilde{\Omega}^3 = 0$, *and* $\tilde{\Omega}^1 \neq 0$.

Example 16: Let $\tilde{\Omega}^{ij}$ *be the second-order moments corresponding to* $x_1^i x_2^j$. *Then* $\tilde{\Omega}^{12} = \tilde{\Omega}^{23} = \tilde{\Omega}^{31} = 0$, *and the* $\tilde{\Omega}^{ii}$ *are nonzero. As in Example 9, it suffices to calculate a single* $\tilde{\Omega}^{ii}$ *by quadrature. The other nonzero second-order moments are then determined algebraically.*

Example 17: Of the ten almost-canonical third-order moments, the nonzero ones are $\tilde{\Omega}^{111}, \tilde{\Omega}^{122}, \tilde{\Omega}^{133}$, *which can be calculated algebraically from* $\tilde{\Omega}^1$, E, m^2, *and the* I_j, *by imitating the proof of Theorem 10.*

Example 18: Of the fifteen almost-canonical moments of order 4, the nonvanishing ones are the same as in Theorem 11. The proofs of Theorem 11 and Corollary 12 are valid in the present case where $\alpha_2 > 0$.

VII. *G* AND NULL VECTORS: THE GENERIC CASE

Given $\overline{O} \in O(3)$ write $\overline{A} = \overline{O} A \overline{O}^T$, and $\overline{\Omega}(t) = \overline{O}\Omega(t)$. The transformations $A \mapsto \overline{A}$, $\Omega \mapsto \overline{\Omega}$ leave Eq. (3) invariant and, using $\overline{O}$, moments $\overline{\Omega}^{\mu}$ of $\overline{\Omega}$ can be calculated from empirical moments. Suppose $\overline{O} \in O(3)$ diagonalizes the symmetric matrix

$$E^{[2]} \equiv \begin{bmatrix} \Omega^{11} & \Omega^{12} & \Omega^{13} \\ \Omega^{12} & \Omega^{22} & \Omega^{23} \\ \Omega^{13} & \Omega^{23} & \Omega^{33} \end{bmatrix}$$

of empirical second-order moments, namely $\overline{O} E^{[2]} \overline{O}^T$ is diagonal. Suppose that the eigenvalues of $E^{[2]}$ are distinct, and let $\mathbf{e}_j$ denote the jth standard basis vector of $\mathbf{R}^3$. Examples 9 and 16 imply that $\overline{O} = \Pi O$ where $\Pi(\mathbf{e}_j) = \pm \mathbf{e}_{\pi(j)}$, $j = 1,2,3$, and π is a permutation of $\{1,2,3\}$. The eigenvalues of $E^{[2]}$ are the nonvanishing second-order almost-canonical moments $\tilde{\Omega}^{jj}$, and $\overline{A} = \mathrm{diag}(\overline{I}_1, \overline{I}_2, \overline{I}_3)$. where $\overline{I}_{\pi(j)} \equiv I_j$. Write $\overline{I} \equiv [\overline{I}_1\ \overline{I}_2\ \overline{I}_3]^T$. Equations (9), (14), (15), (16) are invariant with respect to permutations of coordinates in monomials. Substitution of Eq. (9) in the right-hand sides of (14), (15), (16) shows that $\overline{I}$ is a null-vector of $G \equiv T F^{[4]} - F^{[2]} F^{[2]T}$, where

$$F^{[4]} \equiv \begin{bmatrix} \overline{\Omega}^{1111} & \overline{\Omega}^{1122} & \overline{\Omega}^{1133} \\ \overline{\Omega}^{1122} & \overline{\Omega}^{2222} & \overline{\Omega}^{2233} \\ \overline{\Omega}^{1133} & \overline{\Omega}^{2233} & \overline{\Omega}^{3333} \end{bmatrix}, \quad F^{[2]} \equiv \begin{bmatrix} \overline{\Omega}^{11} \\ \overline{\Omega}^{22} \\ \overline{\Omega}^{33} \end{bmatrix}.$$

In the same way, with (10), (17), (18), (19) in place of (9), (14), (15), (16), $\overline{I^{(2)}} \equiv [\overline{I}_1^2\ \overline{I}_2^2\ \overline{I}_3^2]^T$ is also a null-vector of the symmetric matrix G. Because $\{\overline{I}, \overline{I^{(2)}}\}$ is linearly independent, G is either trivial or rank 1.

Theorem 19: *G has rank 1, with non-null eigenvector*

$$w \equiv \left[\frac{\overline{I}_2 - \overline{I}_3}{\overline{I}_1}\ \ \frac{\overline{I}_3 - \overline{I}_1}{\overline{I}_2}\ \ \frac{\overline{I}_1 - \overline{I}_2}{\overline{I}_3} \right]^T.$$

Proof: Given $v \in \mathbf{R}^3$ define $f(t) = v_1 \overline{\Omega}_1(t)^2 + v_2 \overline{\Omega}_2(t)^2 + v_3 \overline{\Omega}_3(t)^2$, where $t \in [0,T]$. Let $g:[0,T] \to \mathbf{R}$ be identically 1. Cauchy–Schwarz for $f,g \in L^2[0,T]$ says

$$(F^{[2]T} v)^2 < T v^T F^{[4]} v \tag{32}$$

unless f is a scalar multiple of g. Let v be a nonzero null-vector of G. By Eq. (32) f is constant, and by Eq. (3)

$$0 = v^T \begin{bmatrix} \overline{\Omega}_1(t) \dot{\overline{\Omega}}_1(t) \\ \overline{\Omega}_2(t) \dot{\overline{\Omega}}_2(t) \\ \overline{\Omega}_3(t) \dot{\overline{\Omega}}_3(t) \end{bmatrix} = 2 \overline{\Omega}_1(t) \overline{\Omega}_2(t) \overline{\Omega}_3(t) v^T w$$

for $t\in[0,T]$. By Eq. (20), and the corresponding equation (28) for the other case, $\overline{\Omega}_1\overline{\Omega}_2\overline{\Omega}_3 = \tilde{\Omega}_1\tilde{\Omega}_2\tilde{\Omega}_3$ vanishes for only finitely many parameter values. Therefore v is orthogonal to w. Because w is nonzero it is not a null-vector of G. However it is an eigenvector, because it is orthogonal to the null-vectors $\overline{I,I^{(2)}}$, and G is symmetric.

Remark 20: The eigenvalue of w is

$$-R\frac{\overline{I}_3^2\overline{I}_1^2(\overline{I}_3-\overline{I}_1)^2+\overline{I}_2^2\overline{I}_3^2(\overline{I}_2-\overline{I}_3)^2+\overline{I}_1^2\overline{I}_2^2(\overline{I}_1-\overline{I}_2)^2}{3\overline{I}_1^2\overline{I}_2^2\overline{I}_3^2(\overline{I}_1-\overline{I}_2)(\overline{I}_2-\overline{I}_3)(\overline{I}_3-\overline{I}_1)}$$

where $R\equiv\overline{I}_2\overline{I}_3(\overline{I}_2-\overline{I}_3)\overline{\Omega}^{22}\overline{\Omega}^{33}+\overline{I}_3\overline{I}_1(\overline{I}_3-\overline{I}_1)\overline{\Omega}^{33}\overline{\Omega}^{11}+\overline{I}_1\overline{I}_2(\overline{I}_1-\overline{I}_2)\overline{\Omega}^{11}\overline{\Omega}^{22}$.

VIII. METHOD 3: *A* FROM MOMENTS

Let Ω be a nondegenerate solution of Eq. (3) of period $T>0$. In Sec. III A is almost-determined by the restriction of Ω to an interval $[s_1,s_2]$ of positive length. It turns out that in generic cases we can obtain a similar result using moments of Ω of order $\leqslant 4$, together with a small amount of auxiliary information. First we explain what is meant by *generic*.

Definition 21: The symmetric positive-definite matrix A is generic *when its eigenvalues I_j are distinct for $j=1,2,3$. When A is generic a solution Ω of Eq. (3) is* generic *provided the matrix $E^{[2]}$ of Sec. VII has distinct eigenvalues.*

Let Ω be generic. In Secs. VI and V we see that either Ω_3 or Ω_1 has constant sign on $[0,T]$, but not both: The *index* of Ω is defined to be 3 or 1 accordingly.

Theorem 22: *For Ω generic, A is almost-determined by the index of Ω and the moments of Ω of orders* 0,1,2,4.

Proof: As in Sec. VII, the matrix $E^{[2]}$ of second-order moments determines O up to conjugation with a permutation matrix Π. As in Sec. VII, the corresponding permutation of $\{1,2,3\}$ is denoted by π. Examining first-order moments with Examples 15, 8, choose π so that $\pi(\iota)=\iota$ where ι is the index of Ω. Second- and fourth-order moments determine a rank 1 symmetric matrix G. By Theorem 19 any non-null eigenvector $\hat{w}$ of G has all coordinates nonzero. Choose $\hat{w}$ so that $\hat{w}_\iota<0$. Furthermore, the permutation π of $\{1,2,3\}\setminus\{\iota\}$ is uniquely determined by the condition

$$\hat{w}_{\pi(2)}>0. \tag{33}$$

Knowing π, return to Sec. VII and rechoose $\overline{O}$ so that Π becomes a diagonal matrix with entries ± 1. Define w and choose $\hat{w}$ as before, but with the new $\overline{O}$. Then $\overline{I}_j=I_j$ for $j=1,2,3$, and w becomes

$$\left[\frac{I_2-I_3}{I_1}\ \ \frac{I_3-I_1}{I_2}\ \ \frac{I_1-I_2}{I_3}\right]^T=\sigma\hat{w}, \tag{34}$$

where $\sigma>0$. For $j=1,2$, set $v_j=\hat{w}_j/\hat{w}_3$. The v_j are determined by the index and the moments. Since $I_1<I_2<I_3$,

$$v_1>0,\ \ v_2<0. \tag{35}$$

Proposition 23: Suppose that $J\in\mathbf{R}^3$ satisfies

$$\left[\frac{J_2-J_3}{J_1}\ \ \frac{J_3-J_1}{J_2}\ \ \frac{J_1-J_2}{J_3}\right]^T=\sigma\hat{w}, \tag{36}$$

where $\sigma>0$ is unknown, and $J_j>0$ for $j=1,2,3$. Then J is a positive scalar multiple of I.

Proof: Since $\hat{w}_1<0,\hat{w}_2>0,\hat{w}_3<0$, we have

$$0<J_1<J_2<J_3 .$$

Define $\Delta_1,\Delta_2>0$ by $J_2=(1-\Delta_2)J_3$ and $J_1=(1-\Delta_1-\Delta_2)J_3$. Then $\Delta_1+\Delta_2<1$. Substituting for the J_j in Eq. (36), and eliminating σ,

$$\frac{\Delta_2}{\Delta_1(1-\Delta_1-\Delta_2)}=v_1, \tag{37}$$

$$-\frac{\Delta_1+\Delta_2}{\Delta_1(1-\Delta_2)}=v_2. \tag{38}$$

Consider the possible solutions for $\Delta\equiv(\Delta_1,\Delta_2)$ of the simultaneous equations (37) and (38). We consider two cases.

Case 1: When $v_1+v_2\neq 0$ there are at most two possible solutions, namely

$$\Delta=\left(\frac{\sqrt{-v_1v_2(1+v_1+v_2)}}{v_1v_2},\frac{(1+v_1+v_2)v_2-\sqrt{-v_1v_2(1+v_1+v_2)}}{(v_1+v_2)v_2}\right), \tag{39}$$

and

$$\Delta=\left(-\frac{\sqrt{-v_1v_2(1+v_1+v_2)}}{v_1v_2},\frac{(1+v_1+v_2)v_2+\sqrt{-v_1v_2(1+v_1+v_2)}}{(v_1+v_2)v_2}\right). \tag{40}$$

Since Eqs. (35) and (39) imply that $\Delta_1<0$, hence Eq. (40) is the only possible solution.

Case 2: When $v_1+v_2=0$ there is at most one solution, namely

$$\Delta=\left(\frac{1}{v_1},\frac{v_1-1}{2v_1}\right).$$

In both cases there is at most one solution Δ (corresponding to $J=I$). This proves the proposition.

Proposition 23 implies that Eq. (34) determines $[I_1\, I_2\, I_3]$ up to positive scalar multiple. Since O is determined up to multiplication by a scalar matrix with entries ± 1, A is almost-determined. This completes the proof of Theorem 22 and provides Method 3, for almost-determining A from qualitative information about Ω (the index) and moments of Ω of orders 0,1,2,4.

ACKNOWLEDGMENT

Research supported in part by the NUS Wavelets Program funded by the National Science and Technology Board and the Ministry of Education, Republic of Singapore.

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