Lines of best fit for the zeros and for the critical points of a polynomial

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Abstract

Combining results presented in two papers in this MONTHLY yields the following elementary result. Any line of best fit for the zeros of a polynomial is a line of best fit for its critical points.

This note gives a generalization of results on cubic polynomials presented in [1]. Our notation will follow that paper. A *line of best fit* for a set of points in the plane is defined, as in [1, p. 682], to be a line that minimizes the sum of squares of the perpendicular distances from the points to the line. (Sometimes, elsewhere, such a line is called a "least-squares perpendicular-offsets" line.)

Let $\{z_j | 1 \le j \le n\}$ be a set of $n \ge 2$ complex numbers. Define

$$z_{\mathrm{A}} = \frac{1}{n}\sum_{j=1}^{n} z_j \qquad \textit{and} \qquad Z = \sum_{j=1}^{n}(z_j - z_{\mathrm{A}})^2. \tag{1}$$

The generalization of [1, Theorem 2.4] is as follows.

Theorem 1 *Let $\{z_j | 1 \le j \le n\}$ be a set of $n \ge 3$ complex numbers . Let z_{A} and Z be as defined as in equation (1). Let $p(z)$ be the monic polynomial of degree n*

$$p(z) = z^n + \sum_{j=0}^{n-1} a_j z^j = \prod_{j=1}^{n}(z - z_j)\ ,$$

and denote the roots of its derivative by z'_k, so that $p'(z) = n\prod_{k=1}^{n-1}(z - z'_k)$. The lines of best fit for the zeros z_j of p coincide with the lines of best fit for the zeros z'_k of p', and these lines pass through z_{A}.

The zeros z'_k of p' are also called the *critical points* of p.

When $n = 3$ and $Z = 0$ the points z_j form the vertices of an equilateral triangle. When $n = 3$ and $Z \ne 0$, Theorem 1 becomes [1, Theorem 2.4], attributed there to Coolidge. (Closely related theorems are due to Grace, Bôcher, and Siebeck.)

Theorem 2 (Coolidge, $n = 3$.) *Suppose the complex numbers z_1, z_2, z_3 form the vertices of a triangle which is nonequilateral. If $p(z) = \prod_{j=1}^{3}(z - z_j)$, then the line of best fit for the three numbers is the unique line through the roots of $p'(z)$.*

The first ingredient in the proof of Theorem 1 is the following.

Theorem 3 [1, Theorem 2.3] *Suppose z_j, $1 \le j \le n$, are complex numbers, and z_{A} and Z are as in equation (1) above.*
(a) If $Z = 0$, then every line through z_{A} is a line of best fit for the points $z_1, \ldots, z_n$, and these are the only lines of best fit.
(b) If $Z \neq 0$, then the line through z_{A} that is parallel to the vector from 0 to $\sqrt{Z}$ is the unique line of best fit for $z_1, \ldots, z_n$.

The next ingredient in the proof is taken from [2]. (See also Newton's Identities.) First note that $a_{n-1} = -\sum_{j=1}^{n} z_j = -nz_{\mathrm{A}}$. Now suppose, as in [2], that $z_{\mathrm{A}} = 0$: there is no loss of generality in this translation of the points of the complex plane. Then $a_{n-1} = 0$. (It also follows that the coefficient of z^{n-2} in $p'(z)$ is also zero, so $(n-1){z'}_{\mathrm{A}} = \sum_{k=1}^{n-1} z'_k = 0$.) Squaring the equation $\sum_{j=1}^{n} z_j = 0$, we find that

$$Z = \sum_{j=1}^{n} z_j^2 = -2 \sum_{1 \le j < k \le n} z_j z_k = -2a_{n-2} \ .$$

This leads to a simple relationship between Z and Z':

Theorem 4 [2, equation (1.9)] *Denote the zeros of the polynomials p and p' as above, and suppose that their centroids are located at the origin. Then*

$$Z' = \sum_{k=1}^{n-1} (z'_k)^2 = \frac{(n-2)}{n} \sum_{j=1}^{n} z_j^2 = \frac{(n-2)}{n} Z \ . \tag{2}$$

Proof. Using $z_{\mathrm{A}} = 0$,

$$\frac{p'}{n} = z^{n-1} + \sum_{j=1}^{n-2} \frac{j a_j}{n} z^{j-1} \ ,$$

and it follows that

$$Z' = -\frac{2(n-2)}{n} a_{n-2} = \frac{(n-2)}{n} Z \ ,$$

as stated.

Finally, the proof of Theorem 1 proceeds as follows. There is no loss of generality in translating the points of the complex plane so that $z_{\mathrm{A}} = 0$. All lines of best fit, those for the zeros and those for the critical points, necessarily pass through the origin, and, by Theorem 3, are in directions $\sqrt{Z}$ associated with the zeros and $\sqrt{Z'}$ associated with the critical points. However, by Theorem 4, $\sqrt{Z'} = \sqrt{\frac{n-2}{n}}\sqrt{Z}$, and this completes the proof.

Theorem 1 can be applied repeatedly. The lines of best fit for the zeros of any derivative $p^{(k)}$, $1 \le k \le (n-2)$, coincide with the lines of best fit for the zeros of the original polynomial. If the quadratic $p^{(n-2)}$ obtained by $(n-2)$ differentiations of the original polynomial has distinct roots, the line through these roots is the line of best fit for the original polynomial. The line(s) of best fit for the zeros of p is (are) completely determined by the coefficients a_{n-1} and a_{n-2}.

While it seems highly likely that our Theorem 1 is a rediscovery, to date a search of the reasonably-accessible English-language publications has not found any prior publication. The proof by assembling results from this Monthly is original. There is a longer account at the author's website, and this will be updated with related results and further references – especially, if found, those for any prior publication.

References

[1] D. Minda and S. Phelps, Triangles, ellipses and cubic polynomials, *Amer. Math. Monthly* **115** (2008) 679-689.

[2] I. J. Schoenberg, A conjectured analogue of Rolle's theorem for polynomials with real or complex coefficients, *Amer. Math. Monthly* **93** (1986) 8-13.

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