

**ON A BRUNN-MINKOWSKI THEOREM
FOR A GEOMETRIC DOMAIN FUNCTIONAL
CONSIDERED BY AVHADIEV**

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ABSTRACT. Suppose two bounded subsets of \mathbb{R}^n are given. Parametrise the Minkowski combination of these sets by t . The Classical Brunn-Minkowski Theorem asserts that the $1/n$ -th power of the volume of the convex combination is a concave function of t . A Brunn-Minkowski-style theorem is established for another geometric domain functional.

1. INTRODUCTION

Let Ω be a bounded domain in \mathbb{R}^n . Define

$$(1.1) \quad I(k, \partial\Omega) = \int_{\Omega} \text{dist}(z, \partial\Omega)^k d\mu_z \quad \text{for } k > 0.$$

Here $\text{dist}(z, \Omega)$ denotes the distance of the point $z \in \Omega$ to the boundary $\partial\Omega$ of Ω . The integration uses the ordinary measure in \mathbb{R}^n and is over all $z \in \Omega$. When $n = 2$ and $k = 1$ this functional was introduced, in [1] in bounds of the torsional rigidity $P(\Omega)$ of plane domains Ω . See also [10] where the inequalities

$$(1.2) \quad \frac{I(2, \partial\Omega)}{I(2, \partial B_1)} \leq \frac{P(\Omega)}{P(B_1)} \leq \frac{128}{3} \frac{I(2, \partial\Omega)}{I(2, \partial B_1)}$$

are presented. Here B_1 is the unit disk and

$$I(2, \partial B_1) = \frac{\pi}{6} = \frac{|B_1|^2}{6\pi}.$$

This inequality is one of many relating domain functionals such as these: see [9, 2, 7]. As an example, proved in [9], we instance

$$(1.3) \quad (\dot{r}(\Omega))^4 \leq \frac{P(\Omega)}{P(B_1)} \leq \left(\frac{|\Omega|}{|B_1|}\right)^2 \leq \left(\frac{|\partial\Omega|}{|\partial B_1|}\right)^4$$

giving bounds for the torsional rigidity in terms of the inner-mapping radius \dot{r} , the area $|\Omega|$ and the perimeter $|\partial\Omega|$.

We next define the Minkowski sum of domains by

$$\Omega_0 + \Omega_1 := \{z_0 + z_1 \mid z_0 \in \Omega_0, z_1 \in \Omega_1\},$$

and

$$\Omega(t) := \{(1-t)z_0 + tz_1 \mid z_0 \in \Omega_0, z_1 \in \Omega_1\}, 0 \leq t \leq 1.$$

The classical Brunn-Minkowski Theorem in the plane is that $\sqrt{|\Omega(t)|}$ is a concave function of t for $0 \leq t \leq 1$, and it also happens that $|\partial\Omega(t)|$ is, for convex Ω , a linear, hence concave, function of t . Given a nonnegative quasiconcave function $f(t)$ for which, with $\alpha > 0$, $f(t)^\alpha$ is concave, we say that f is α -concave. In [3] it was shown that, for convex domains Ω , the torsional rigidity satisfies a Brunn-Minkowski style theorem: specifically $P(\Omega(t))$ is $1/4$ -concave. Thus inequalities (1.3) show that the $1/4$ -concave function $P(\Omega(t))$ is sandwiched between the $1/4$ -concave functions $|\Omega(t)|^2$

and $|\partial\Omega(t)|^4$. In [6] it is shown that the polar moment of inertia $I_c(\Omega(t))$ about the centroid of Ω , for which

$$(1.4) \quad \left(\frac{|\Omega|}{|B_1|}\right)^2 \leq \frac{I_c(\Omega)}{I_c(B_1)} \leq \left(\frac{|\partial\Omega|}{|\partial B_1|}\right)^4,$$

holds, is also 1/4-concave. (The 1/4-concavity of $r(\Omega(t))^4$ has also been established by Borell.) In this paper we show that the same 1/4-concavity of the domain functions holds for the quantities in inequalities (1.2). Our main result will be the following.

Theorem 1. *Let \mathcal{K} denote the set of convex domains in \mathbb{R}^n . For $\Omega_0, \Omega_1 \in \mathcal{K}$, $I(k, \partial\Omega(t))$ is $1/(n+k)$ -concave in t .*

Our proof is an application of the Prekopa-Leindler inequality, Theorem 3 below.

2. PROOFS

The proof will use two little lemmas, Theorems 2 and 4, and one major Theorem, the Prekopa-Leindler Theorem 3. None of these three results is new: the new item in this paper is their use.

Theorem 2. (Knothe). *Let $0 < t < 1$. Let $\Omega_0, \Omega_1 \in \mathcal{K}$. With*

$$z_t = (1-t)z_0 + tz_1,$$

we have

$$(2.1) \quad \text{dist}(z_t, \partial\Omega(t)) \geq (1-t)\text{dist}(z_0, \partial\Omega_0) + t\text{dist}(z_1, \partial\Omega_1).$$

Proof. Let $z_t \in \Omega(t)$ be as above. Denote the usual Euclidean norm with $|\cdot|$. Let $w_t \in \partial\Omega(t)$ be a point such that

$$|z_t - w_t| = \text{dist}(z_t, \partial\Omega(t)).$$

Define the direction u by

$$u = \frac{z_t - w_t}{|z_t - w_t|}.$$

Define $v_0 \in \Omega_0$, and $v_1 \in \Omega_1$ as the points on these boundaries which are on the rays, in direction u , from z_0 and z_1 respectively. Thus

$$v_0 = z_0 + |z_0 - v_0|u, \quad v_1 = z_1 + |z_1 - v_1|u.$$

Now let p be any unit vector perpendicular to u . The preceding definitions give that

$$\langle w_t - ((1-t)v_0 + tv_1), p \rangle = 0,$$

from which, on defining

$$v_t = (1-t)v_0 + tv_1 \text{ we have } w_t = v_t + \eta u.$$

for some number η . Now, we do not know (or care) if v_t is on the boundary of $\Omega(t)$, but we do know that v_t is in the closed set $\overline{\Omega(t)}$. Using the convexity of $D(t)$ we have that v_t is on the ray joining z_t with w_t , and between z_t and w_t . From this

$$\begin{aligned} \text{dist}(z_t, \partial\Omega(t)) &= |z_t - w_t| \geq |z_t - v_t|, \\ &= (1-t)|z_0 - v_0| + t|z_1 - v_1|, \\ &\geq (1-t)\text{dist}(z_0, \partial\Omega_0) + t\text{dist}(z_1, \partial\Omega_1), \end{aligned}$$

as required.

Theorem 3. (Prekopa-Leindler). *Let $0 < t < 1$ and let f_0 , f_1 , and h be nonnegative integrable functions on \mathbb{R}^n satisfying*

$$(2.2) \quad h((1-t)x + ty) \geq f_0(x)^{1-t} f_1(y)^t,$$

for all $x, y \in \mathbb{R}^n$. Then

$$(2.3) \quad \int_{\mathbb{R}^n} h(x) dx \geq \left(\int_{\mathbb{R}^n} f_0(x) dx \right)^{1-t} \left(\int_{\mathbb{R}^n} f_1(x) dx \right)^t.$$

For references to proofs, see [5].

Theorem 4. (Homogeneity Lemma). *If F is positive and homogeneous of degree 1*

$$F(s\Omega) = sF(\Omega) \quad \forall s > 0, \Omega,$$

and quasiconcave

$$(2.4) \quad F(\Omega(t)) \geq \min(F(\Omega(0)), F(\Omega(1))) \quad \forall 0 \leq t \leq 1 \quad \forall \Omega_0, \Omega_1 \in \mathcal{K},$$

then it is concave:

$$F(\Omega(t)) \geq (1-t)F(\Omega(0)) + tF(\Omega(1)) \quad \forall 0 \leq t \leq 1.$$

Proof. See [5]. Replace Ω_0 by $\Omega_0/F(\Omega_0)$, Ω_1 by $\Omega_1/F(\Omega_1)$. Using the homogeneity of degree 1, and applying (2.4) we have

$$F\left((1-t)\frac{\Omega_0}{F(\Omega_0)} + t\frac{\Omega_1}{F(\Omega_1)}\right) \geq 1.$$

With

$$t = \frac{F(\Omega_1)}{F(\Omega_0) + F(\Omega_1)}, \quad \text{so } (1-t) = \frac{F(\Omega_0)}{F(\Omega_0) + F(\Omega_1)},$$

the last inequality on F becomes

$$F\left(\frac{\Omega_0 + \Omega_1}{F(\Omega_0) + F(\Omega_1)}\right) \geq 1.$$

Finally, using the homogeneity we have

$$F(\Omega_0 + \Omega_1) \geq F(\Omega_0) + F(\Omega_1),$$

and using homogeneity again,

$$F((1-t)\Omega_0 + t\Omega_1) \geq (1-t)F(\Omega_0) + tF(\Omega_1),$$

as required.

Proof of the Main Theorem 1. Knothe's Lemma 2 and the AGM inequality give

$$(2.5) \quad \text{dist}(z_t, \partial\Omega(t)) \geq \text{dist}(z_0, \partial\Omega_0)^{(1-t)} \text{dist}(z_1, \partial\Omega_1)^t,$$

and similarly for any positive k -th power of the distance. Denote the characteristic function of Ω by χ_Ω . A standard argument, as given in [5] for example, establishes that

$$\chi_{\Omega(t)}((1-t)z_0 + tz_1) \geq \chi_{\Omega_0}(z_0)^{1-t} \chi_{\Omega_1}(z_1)^t.$$

So, with

$$\begin{aligned} h(z) &= \text{dist}(z, \partial\Omega(t)) \chi_{\Omega(t)}(z), \\ f_0(z) &= \text{dist}(z, \partial\Omega_0) \chi_{\Omega_0}(z), \\ f_1(z) &= \text{dist}(z, \partial\Omega_1) \chi_{\Omega_1}(z), \end{aligned}$$

the conditions of the Prekopa-Leindler Theorem are satisfied. This gives that $I(k, \partial\Omega(t))$ is log-concave in t . Now define $F(\Omega(t)) := I(k, \partial\Omega(t))^{1/(n+k)}$. The function F is quasiconcave in t (as it inherits the stronger property of logconcavity in t from $I(k, \partial\Omega(t))$). Since $I(k, \partial\Omega(t))$ is homogeneous of

degree $n + k$, F is homogeneous of degree 1. The Homogeneity Lemma applied to F yields that $I(k, \partial\Omega(t))$ is $1/(n + k)$ -concave.

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