

## Functions with constant Laplacian satisfying homogeneous Robin boundary conditions

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The authors study properties of real-valued functions  $u$  defined over  $\Omega$ , a simply-connected domain in  $\mathbb{R}^N$ , for which the Laplacian of  $u$  is constant in  $\Omega$ , and which satisfy, on the boundary of  $\Omega$ , the Robin boundary condition  $u + \beta(\partial u/\partial n) = 0$ . Here  $n$  is the outward normal and  $\beta \geq 0$ . When  $N=2$  and  $\beta=0$ , this is the classical St Venant torsion problem, but the concern in this paper is with  $N \geq 2$  and  $\beta \geq 0$ . Results concerning the magnitude  $u_m$  and location  $z_m$  of the maximum value of  $u$ , and estimates for the functional  $S_\beta = \int_\Omega u$ , and the maxima  $p_m$  and  $q_m$  of  $|\nabla u|$  and  $|\partial u/\partial n|$ , respectively, are established using comparison theorems and variational arguments.

### 1. Introduction

Qualitative properties of the solution of the Poisson problem

$$-\Delta u = 1 \quad \text{in } \Omega, \quad \beta \frac{\partial u}{\partial n} + u = 0 \quad \text{on } \partial\Omega, \quad (\text{P}(\beta))$$

in a simply connected region  $\Omega \subseteq \mathbb{R}^N$ , with closure  $\bar{\Omega}$  and boundary  $\partial\Omega$ , and of various functionals of this solution are of interest in many applied areas. Here  $\Delta$  denotes the Laplacian,  $\partial u/\partial n$  is the outward normal derivative of  $u$  on the boundary  $\partial\Omega$ , and  $\beta$ , which may depend on position, is nonnegative.

When  $N=2$ , problem P(0) is the well-known *torsion problem* of elasticity theory. If the dimension is important, we indicate it by a subscript, so that, for example,  $P_2(0)$  is the torsion problem. For this application and others involving P( $\beta$ ) (Keady & Kloeden, 1987), a functional of prime interest is

$$S_\beta = \int_\Omega u. \quad (1.1)$$

Tables of  $S_0$  and other domain functionals are given in Pólya & Szegő (1951). In particular, for problem  $P_2(0)$ ,  $S_0$  is known as the torsional rigidity. Problem  $P_2(0)$  is also associated with the steady unidirectional flow of a viscous fluid down a pipe of cross-section  $\Omega$ , sustained by a constant pressure gradient. In this context,  $S_0$  gives the volume flow rate for a given pressure gradient and the maximum fluid

velocity  $u_m$  is given by

$$u_m = \max_{\bar{\Omega}} u = u(z_m). \quad (1.2)$$

Problem  $P(0)$  also arises in combustion theory in connection with 'the time to complete combustion' (see Keady & Stakgold (1989), where equation (1.9) contains the quantity  $u_m$ , and Stakgold & McNabb (1984), especially equation (6)).

Linear diffusion and heat conduction problems concerned with transition times between constant steady states feature the functional  $S_\beta$  in expressions for the mean particle or thermal energy residence time in  $\Omega$  (McNabb & Wake 1991). The functional  $S_\beta$  is also a useful measurable parameter for identifying the trapping kinetics constants in more complex linear diffusion problems (McNabb & Keady 1993).

In certain circumstances involving heat conduction with phase transitions,  $u_m$  is related to the time for freezing or thawing of homogeneous conducting regions  $\Omega$ , and the location  $z_m$  of this maximum in  $\Omega$  is in the nature of a thermal centre, being the last point to freeze or thaw (McNabb & Wake, 1991). In the same circumstances, the smallest and greatest values of  $u$ , which occur on the boundary  $\partial\Omega$ , identify the first and last points to freeze or thaw on  $\partial\Omega$ . When  $\beta > 0$  and constant, these are also the points where  $|\partial u / \partial n|$  takes its smallest and greatest values. The quantity  $p = |\nabla u|$  also takes its greatest value in  $\bar{\Omega}$  on  $\partial\Omega$ , and we show that if  $\beta$  is small, or  $\partial\Omega$  near circular, then  $p = -\partial u / \partial n$  at the maximum value points  $z_t$ , and so  $u$  is also at a maximum on  $\partial\Omega$  at  $z_t$ . Define  $p_m$  as

$$p_m = \max_{\bar{\Omega}} |\nabla u| = p(z_t). \quad (1.3)$$

For the problem  $P_2(0)$ , the point  $z_t$  is called the principal fail point in elastic torsion theory.

The quantities  $S_\beta$ ,  $u_m$ , and  $p_m$ , may be regarded as functionals and norms of the solutions  $u$  of  $P(\beta)$  and, together with various other properties of these solutions, have significant applied interest. This paper surveys old and derives new results concerning them.

While the literature is rich in results for problem  $P(0)$  and especially  $P_2(0)$ , and functionals of elasticity interest, there are some neglected norms relevant to not-so-mainstream applications. In particular, we find it useful to add supplementary information concerning  $z_m$  and special sets which contain it.

Some useful qualitative theorems are stated, but to simplify the exposition we avoid technical problems associated with difficult regions, and so the treatment is restricted to  $\Omega$  and  $\beta$  for which the solution  $u$  of  $P(\beta)$  is at least in  $C^2(\Omega) \cap C^1(\bar{\Omega})$ . Furthermore, if  $\beta$  depends on  $z$  then  $\beta$  and  $\partial\Omega$  are  $C^\infty$ . Some results require  $\partial\Omega$  to be 'smooth' and we take this to mean at least  $C^{2,\alpha}(\bar{\Omega})$ , so that standard estimates for  $u$  and its first and second derivatives apply and  $\partial\Omega$  has bounded curvature everywhere.

## 2. Some qualitative theorems for general $\beta$ and $N$

Our starting point is a strong comparison theorem which gives a uniqueness result for the solutions of  $P(\beta)$  and shows their monotone dependence on  $\beta$ . This

is followed by some estimates relating to the asymptotic behaviour of  $u$  as  $\beta$  tends to infinity.

**THEOREM 2.1** If  $\bar{u}$  and  $u$  belong to  $C^2(\Omega) \cap C^1(\bar{\Omega})$  and satisfy the inequalities

$$-\nabla^2 \bar{u} - 1 \geq -\nabla^2 u - 1 \quad \text{in } \Omega, \quad \bar{u} + \beta \frac{\partial \bar{u}}{\partial n} \geq u + \beta \frac{\partial u}{\partial n} \quad \text{on } \partial\Omega, \quad (2.1)$$

then either  $\bar{u} > u$  in  $\Omega$  and  $\partial \bar{u} / \partial n < \partial u / \partial n$  at any point on  $\partial\Omega$  where  $\bar{u} = u$ , or  $\bar{u} \equiv u$  in  $\bar{\Omega}$ .

*Proof.* Since  $w = \bar{u} - u$  is superharmonic in  $\Omega$ , its minimum is on  $\partial\Omega$ , where

$$w + \beta \frac{\partial w}{\partial n} \geq 0. \quad (2.2)$$

At such a minimum point for  $w$  on  $\partial\Omega$ , we have  $\partial w / \partial n \leq 0$ . (Recall that  $\partial / \partial n$  denotes differentiation along the outward normal.) This in conjunction with (2.2) implies  $w \geq 0$  there and hence  $w \geq 0$  in  $\bar{\Omega}$ . The strong minimum principle (Hopf, 1952) implies  $w > 0$  in  $\Omega$  or  $w \equiv 0$  in  $\bar{\Omega}$ . Moreover, if  $\beta > 0$  on some section  $\partial\Omega_1$  of  $\partial\Omega$ , then  $w > 0$  there too or  $w \equiv 0$  in  $\bar{\Omega}$ , since  $w = 0$  at  $P^*$  on  $\partial\Omega_1$  implies  $\partial w / \partial n \geq 0$  at  $P^*$  from (2.2) and  $\partial w / \partial n \leq 0$  from the minimum requirement, so that  $\partial w / \partial n = 0$  at  $P^*$  and hence  $w \equiv 0$  in  $\bar{\Omega}$ .  $\square$

These arguments apply to more general elliptic equations (McNabb, 1961) and many of the results which follow are also readily generalized.

The problem  $P(\beta)$  for infinite  $\beta$  has no bounded solution and even the more general problem with  $\partial u / \partial n$  specified on  $\partial\Omega$  is subject to the Fredholm alternative in the sense that there is no bounded solution unless  $\int_{\partial\Omega} \partial u / \partial n + |\Omega| = 0$ , where  $|\Omega|$  is the area of  $\Omega$ . The solution of problem  $P(\beta)$  for  $\Omega$  the infinite strip  $-a < x < a$  and  $\beta$  constant is  $u = \beta a + \frac{1}{2}(a^2 - x^2)$ , and the following theorem shows that other solutions of problem  $P(\beta)$  have the same asymptotic structure for large and constant  $\beta$ .

**THEOREM 2.2** If  $\partial\Omega$  is sufficiently smooth and  $\beta$  constant, there are positive constants  $A_k$  and  $B_k$  independent of  $\beta$  for which

$$w_k + \frac{A_k}{\beta^k} \geq u \geq w_k - \frac{B_k}{\beta^k} \quad \text{in } \Omega \quad (k = 0, 1, 2, \dots), \quad (2.3)$$

where

$$w_0 = \beta \frac{|\Omega|}{|\partial\Omega|} + u_\infty, \quad w_k = \beta \frac{|\Omega|}{|\partial\Omega|} + u_\infty + \frac{u_1}{\beta} + \dots + \frac{u_k}{\beta^k} \quad (k > 0). \quad (2.4)$$

The function  $u_\infty$  is the solution of

$$\nabla^2 u_\infty + 1 = 0 \quad \text{in } \Omega, \quad \frac{\partial u_\infty}{\partial n} = -\frac{|\Omega|}{|\partial\Omega|}, \quad \int_{\partial\Omega} u_\infty = 0 \quad \text{on } \partial\Omega, \quad (P(\infty))$$

and the functions  $u_i$  ( $i = 1, 2, \dots$ ) satisfy

$$\nabla^2 u_i = 0 \quad \text{in } \Omega, \quad \frac{\partial u_1}{\partial n} = -u_\infty, \quad \frac{\partial u_i}{\partial n} = -u_{i-1} \quad (i > 1), \quad \int_{\partial\Omega} u_i = 0 \quad \text{on } \partial\Omega. \quad (2.5)$$

*Proof.* Choose  $A_k$  so that  $u_k + A_k \geq 0$  on  $\partial\Omega$ , and  $B_k$  so that  $u_k - B_k \leq 0$  on  $\partial\Omega$ . (Our smoothness conditions on  $\partial\Omega$  ensure  $u_n$  and  $u_t$  are uniformly bounded in  $\bar{\Omega}$ .) If  $\bar{u} = u_k + A_k/\beta^k$  and  $\underline{u} = u_k - B_k/\beta^k$ , then

$$\nabla^2 \bar{u} + 1 = \nabla^2 \underline{u} + 1 = 0 \quad \text{in } \Omega,$$

and

$$\bar{u} + \beta \frac{\partial \bar{u}}{\partial n} = \frac{u_k + A_k}{\beta^k} \geq 0, \quad \underline{u} + \beta \frac{\partial \underline{u}}{\partial n} = \frac{u_k - B_k}{\beta^k} \leq 0 \quad \text{on } \partial\Omega.$$

Theorem 2.1 applies and so  $\bar{u} \geq u \geq \underline{u}$  in  $\Omega$  and the result follows.  $\square$

A number of useful results follow immediately and are stated as theorems for further reference.

**THEOREM 2.3** The solutions of problem  $P(\beta)$  are unique and positive in  $\Omega$ .

*Proof.* If  $u_1$  and  $u_2$  are two solutions, Theorem 2.1 applies and  $u_1 \geq u_2 \geq u_1$  in  $\Omega$  so that  $u_1 = u_2 = u$ , say, in  $\bar{\Omega}$ . Now take  $\bar{u} = u$  and  $\underline{u} = 0$  in Theorem 2.1. Since  $u = 0$  in  $\bar{\Omega}$  is not a solution of  $P(\beta)$ , we have  $u > 0$  in  $\Omega$  and on  $\partial\Omega$  wherever  $\beta > 0$ .  $\square$

**THEOREM 2.4 (Monotone dependence on  $\beta$ )** Assume  $0 \leq \beta_1(z) \leq \beta_2(z)$  on  $\partial\Omega$ . Let  $u_i$  be the solution of problem  $P(\beta_i)$ . Then  $u_1 < u_2$  in  $\Omega$ , or  $u_1 = u_2$  on  $\bar{\Omega}$  and  $\beta_1 = \beta_2$  on  $\partial\Omega$ .

*Proof.* Let  $\bar{u} = u_2$ ,  $\underline{u} = u_1$ ,  $\beta = \beta_2$  in Theorem 2.1 and observe that since

$$u_2 + \beta_2 \frac{\partial u_2}{\partial n} = 0 \quad \text{and} \quad \frac{\partial u_1}{\partial n} < 0 \quad \text{on } \partial\Omega,$$

it follows that

$$u_1 + \beta_2 \frac{\partial u_1}{\partial n} = u_1 + \beta_1 \frac{\partial u_1}{\partial n} + (\beta_2 - \beta_1) \frac{\partial u_1}{\partial n} \leq 0.$$

The conclusions of Theorem 2.1 apply and the result follows.  $\square$

**THEOREM 2.5 (Monotone dependence of  $\partial u/\partial n$  on  $\beta$ )** Let  $u_i$  be the solution of problem  $P(\beta_i)$  in  $\Omega$ , for constant  $\beta_i$  with  $\beta_2 > \beta_1$ . Then

$$\begin{aligned} \max_{z \in \partial\Omega} \left| \frac{\partial u_2}{\partial n}(z) \right| &< \max_{z \in \partial\Omega} \left| \frac{\partial u_1}{\partial n}(z) \right|, \\ \min_{z \in \partial\Omega} \left| \frac{\partial u_2}{\partial n}(z) \right| &> \min_{z \in \partial\Omega} \left| \frac{\partial u_1}{\partial n}(z) \right|, \end{aligned} \tag{2.6}$$

and, as  $\beta_2$  tends to infinity,  $|\partial u_2/\partial n|$  tends uniformly to  $|\Omega|/|\partial\Omega|$  on  $\partial\Omega$ .

*Proof.* If  $u$  is the solution of problem  $P(\beta)$  in  $\Omega$ , there is a constant  $K$  independent of  $\beta$  such that

$$\beta \frac{|\Omega|}{|\partial\Omega|} - K < u = \beta \left| \frac{\partial u}{\partial n} \right| < \beta \frac{|\Omega|}{|\partial\Omega|} + K \quad \text{on } \partial\Omega,$$

and so  $|\partial u/\partial n|$  tends uniformly to  $|\Omega|/|\partial\Omega|$  on  $\partial\Omega$  as  $\beta$  goes to infinity.

Let  $w = u_2 - u_1$  so that

$$\nabla^2 w = 0 \text{ in } \Omega, \quad w + \beta_2 \frac{\partial w}{\partial n} = (\beta_2 - \beta_1) \left( -\frac{\partial u_2}{\partial n} \right) > 0 \text{ on } \partial\Omega. \quad (2.7)$$

Suppose  $-\partial u_2/\partial n$  takes its greatest value  $q_m$  at  $Q$  on  $\partial\Omega$  and  $\bar{w} = (\beta_2 - \beta_1)q_m$ . Theorem 2.1 implies  $w < \bar{w}$  on  $\partial\Omega$  or  $w \equiv \bar{w} = (\beta_2 - \beta_1)(-\partial u_2/\partial n)$  on  $\partial\Omega$ . But the latter implies, firstly  $u_2 = \beta_2 q_m$  on  $\partial\Omega$ , and secondly  $\partial u_1/\partial n = -q_m$ , with  $u_1 = \beta_1 q_m$  on  $\partial\Omega$ . Since  $u_2 = u_1 + \bar{w}$  in  $\Omega$ , we have  $\partial u_2/\partial n = \partial u_1/\partial n$  on  $\partial\Omega$  and hence  $\beta_1 = \beta_2$  contrary to assumptions. This establishes  $w < \bar{w}$  on  $\partial\Omega$  and in particular at  $Q$ , where  $\partial w/\partial n > 0$ . Hence

$$q_m = \left| \frac{\partial u_2}{\partial n} \right|_Q < \left| \frac{\partial u_1}{\partial n} \right|_Q \leq \max_{z \in \partial\Omega} \left| \frac{\partial u_1}{\partial n} (z) \right|. \quad (2.8)$$

The complementary inequality is provided in the same way.  $\square$

In a private communication, Kosmodem'yanskii pointed out that the constant  $K$  above is bounded by an inequality of isoperimetric character, derivable from estimates given in his 1989 paper and given by  $K \leq (\frac{1}{2} |\Omega| - |\Omega|^2/I)^{1/2}$ , where  $I = \int_{\partial\Omega} \kappa^{-1} ds$  and  $\kappa$  is the curvature of  $\partial\Omega$ .

We now establish some results which compare solutions on differing domains of problem  $P(\beta)$  for constant  $\beta$ .

**THEOREM 2.6** (Domain monotonicity when  $\beta = 0$ ) If  $\Omega_1 \subset \Omega_2$  and  $u_j$  is the solution of  $P(0)$  in  $\Omega_j$ , then  $u_1 < u_2$  in  $\Omega_1$  or  $u_1 \equiv u_2$  and  $\Omega_1$  coincides with  $\Omega_2$ .

*Proof.* When  $\Omega_1 \subset \Omega_2$ ,  $u_2 \geq 0$  on  $\partial\Omega_1$ , and so  $u_1 \leq u_2$  on  $\partial\Omega_1$ . Therefore Theorem 2.1 applies and gives the result above.  $\square$

The analogue of this theorem is not true for  $\beta > 0$ , even if  $\Omega_1$  and  $\Omega_2$  are both convex. Counterexamples can be given using the asymptotic behaviour of  $u$  for large  $\beta$ .

Suppose  $\Omega_1 \subset \Omega_2$ ,  $u_j$  is the solution of problem  $P(\beta)$  in  $\Omega_j$ , and a weaker domain monotonicity result than Theorem 2.6 were true. Suppose for any  $\beta$  no matter how large, there are always greater values of  $\beta$  for which  $u_1 < u_2$ . If such a result were true, it would imply  $|\Omega_1|/|\partial\Omega_1| \leq |\Omega_2|/|\partial\Omega_2|$ . This follows from (2.3), which implies that for large  $\beta$  there is a constant  $K$  independent of  $\beta$  such that

$$\beta \frac{|\Omega_i|}{|\partial\Omega_i|} - K < u_i < \beta \frac{|\Omega_i|}{|\partial\Omega_i|} + K \text{ in } \Omega_i,$$

and so, arbitrarily large  $\beta$  values exist for which

$$0 \leq \frac{u_2 - u_1}{\beta} \leq \frac{|\Omega_2|}{|\partial\Omega_2|} - \frac{|\Omega_1|}{|\partial\Omega_1|} + \frac{2K}{\beta}.$$

It is not hard to find regions  $\Omega_1$  and  $\Omega_2$  with  $\Omega_1 \subset \Omega_2$  for which  $|\Omega_2|/|\partial\Omega_2| < |\Omega_1|/|\partial\Omega_1|$  even if  $\Omega_1$  and  $\Omega_2$  are both convex. For example, let  $\Omega_1$  be a rectangle  $ABCD$  with an isosceles triangle  $AEB$  forming a 'roof' on top of height  $EF$  above  $AB$  where  $F$  is the midpoint of  $AB$  and  $EF$  is small. The region  $\Omega_2$  is

the same rectangle but with a slightly different roof. The line  $AE$  is extended to meet the extension of  $CB$  at  $G$ . The region  $\Omega_2$  is the quadrilateral  $AGCD$  containing the five-sided figure  $AEBGD$ . When  $EF$  is small, and  $AD$  is greater than  $AB$ ,  $|\Omega_2|/|\partial\Omega_2| \leq |\Omega_1|/|\partial\Omega_1|$ .

Theorem 2.6 is a useful source of lower bounds for solutions of problem  $P(0)$ . For example, if  $B(O, \rho)$ , a ball of radius  $\rho$  and centre  $O$ , is contained in  $\Omega$  and  $w$  is the solution of  $P(0)$  in  $B(O, \rho)$ , while  $u$  is that in  $\Omega$ , then in  $B(O, \rho)$

$$u \geq w \quad \text{and} \quad u_{\min} \geq \rho^2/2N. \quad (2.9)$$

The following theorem provides much more limited but nevertheless useful analogous bounds for problem  $P(\beta)$  when  $\beta > 0$ .

**THEOREM 2.7** Suppose for finite  $a > 0$ , and points  $P$  in  $\Omega$ , the ball  $B(P, a)$  of radius  $a$  centred at  $P$  is in  $\Omega$ . If  $w_{P,a}$  solves  $P(\beta)$  in  $B(P, a)$  and  $\Omega$  is the union of all such  $B(P, a)$ , then  $u > w_{P,a}$  in  $B(P, a) \subset \Omega$ , or  $u \equiv w_{P,a}$  and  $\Omega$  and  $B(P, a)$  coincide.

*Proof.* Let  $Q$  be a point on  $\partial\Omega$  where  $u$ , the solution of  $P(\beta)$  in  $\Omega$ , takes its smallest value. By assumption, there is a  $P$  such that  $B(P, a) \subset \Omega$  and  $Q$  is on the surface of  $\bar{B}(P, a)$ . If  $w_{P,a} \geq u$  at  $Q$ , there is a constant  $K \geq 0$  such that  $v = w_{P,a} - K = u$  at  $Q$ , and

$$v + \beta \frac{\partial v}{\partial n} = -K \leq 0, \quad v \leq u \quad \text{on} \quad \partial B(P, a). \quad (2.10)$$

But at  $Q$ ,

$$u + \beta \frac{\partial u}{\partial n} = 0, \quad u = v, \quad \text{and so} \quad \frac{\partial u}{\partial n} \geq \frac{\partial v}{\partial n}.$$

Theorem 2.1 now implies  $u \equiv v$  in  $B(P, a)$  and from (2.11) that  $K = 0$ . Thus  $w_{P,a} < u$  at  $Q$  and hence also in  $\bar{B}(P, a)$ , otherwise  $u \equiv w_{P,a}$  and  $\Omega$  and  $B(P, a)$  coincide.  $\square$

The constant  $a$  of Theorem 2.7 is bounded above by the smallest radius of curvature of the boundary  $\partial\Omega$  of  $\Omega$ . If  $\Omega$  is convex and  $\beta$  constant, there is a stronger result involving a less restrictive bound on  $a$ .

**THEOREM 2.8** If  $\Omega$  is convex with  $B(O, a) \subset \Omega$ , and if  $u$  and  $w_a$  solve problem  $P(\beta)$  in  $\Omega$  and  $B(O, a)$  respectively, then  $u > w_a$  in  $B(O, a)$  or  $\Omega$  and  $B(O, a)$  coincide.

*Proof.* The solution  $w_a$  is given by

$$w_a = \frac{a^2 - r^2}{2N} + \frac{\beta a}{N}, \quad (2.11)$$

and at any point  $P$  on  $\partial\Omega$

$$w_a + \beta \frac{\partial w_a}{\partial n} = \frac{a^2 - r_p^2}{2N} + \frac{\beta a}{N} - \frac{\beta}{N} r_p \cos \theta, \quad (2.12)$$

where  $\theta$  is the angle  $r_p$  makes with the normal at  $P$ . But  $r_p \cos \theta$  is the distance from  $O$  to the tangent to  $\partial\Omega$  at  $P$ , and since this tangent plane does not intersect  $\Omega$  (convexity condition),  $r_p \cos \theta \geq a$ . Hence

$$w_a + \beta \frac{\partial w_a}{\partial n} \leq n \quad \text{on } \partial\Omega, \tag{2.13}$$

so that by Theorem 2.1,  $w_a < u$  or  $w_a = u$  in  $B(O, a)$  and the result follows  $\square$

An implication of Theorem 2.8 is that  $|\Omega|/\rho |\partial\Omega|$  has a minimum value  $1/N$  for convex regions. This minimum value is attained by many convex shapes other than  $B(O, a)$ . For example, when  $N=2$ , convex polygons whose boundaries consist of arcs of the circle  $r = a$  and tangent lines to it. Such figures include the circle  $r = a$ , the circumscribing square, or any other circumscribing polygon. For these regions

$$u_\infty = c_0 - \frac{1}{4}r^2 \tag{2.14}$$

is the solution of  $P(\infty)$  when  $c_0$  is given by

$$c_0 |\partial\Omega| = \int_{\partial\Omega} \frac{1}{4}r^2. \tag{2.15}$$

There is also a simple result, analogous to Theorems 2.7 and 2.8, for spheres circumscribing the region  $\Omega$ .

**THEOREM 2.9** If  $B(O, R)$  is a ball of radius  $R$  with centre at  $O$  containing  $\Omega$ ,  $u$  is the solution of  $P(\beta)$  with  $\beta$  constant in  $\Omega$ , and  $w$  the solution of the same problem in  $B(O, R)$ , then  $u < w$  in  $\Omega$ , or  $u = w$  in  $\bar{\Omega}$  and  $\Omega$  and  $B(O, R)$  coincide.

*Proof.* The solution  $w$  of  $P_N(\beta)$  in  $B(O, R)$  is

$$w = \frac{R^2 - r^2}{2N} + \frac{\beta R}{N}. \tag{2.16}$$

At a point  $P$  on  $\partial\Omega$  distance  $r_p \leq R$  from  $O$ ,

$$w + \beta \frac{\partial w}{\partial n} = \frac{R^2 - r_p^2}{2N} + \beta \frac{R}{N} - \frac{\beta}{N} r_p \cos \theta \geq 0, \tag{2.17}$$

where  $\theta$  is the angle  $OP$  makes with the normal to  $\partial\Omega$  at  $P$ . The result now follows from Theorem 2.1.  $\square$

This theorem is still true if  $B(O, R)$  is an infinite cylinder in  $M$ -space ( $M < N$ ) containing  $\Omega$  in  $\mathbb{R}^N$ , and in the extreme case  $M = 1$  with  $\Omega$  contained in the strip  $|x| \leq b$  we have

$$w_b = \frac{1}{2}(b^2 - x^2) + \beta b \geq u \quad \text{in } \Omega. \tag{2.18}$$

This result leads to a generalized version of an inequality given by Sperb (1981).

**THEOREM 2.10** Let  $\Omega^*$  be the smallest convex region containing  $\Omega$ , with in-radius  $\rho^*$ , and let  $a^*$  be the distance from  $z_m$  to  $\partial\Omega^*$ . Then

$$u_m \leq \frac{1}{2}a^{*2} + \beta a^* \leq \frac{1}{2}\rho^{*2} + \beta\rho^*. \tag{2.19}$$

*Proof.* If  $\Omega$  is contained in  $|x| \leq b$ , then from (2.18)

$$u < w_b \text{ in } \Omega \quad \text{or} \quad u \equiv w_b \text{ in } \bar{\Omega}.$$

We may assume the plane  $x=0$  contains  $z_m$  and allow  $b$  to decrease (even though  $x+b=0$  may intersect  $\Omega$ ) till  $x=b$  is first tangent to  $\bar{\Omega}$  or there is a point  $P^*$  in  $\Omega \cap \{x \geq 0\}$  where  $u = w_b$ , while  $u < w_b$  elsewhere in this region. The point  $P^*$  must lie on the boundary  $\partial\Omega \cap \{x \geq 0\}$  or on  $\Omega \cap \{x=0\}$ . But on  $\partial\Omega \cap \{x \geq 0\}$ ,

$$w_b + \beta \frac{\partial w_b}{\partial n} \geq 0,$$

and on  $\Omega \cap \{x=0\}$ ,  $w_b \geq u_m \geq u$ . The point  $P^*$  cannot be on  $\partial\Omega \cap \{x=0\}$  since  $u_m$  is in the interior of  $\Omega$ , and the arguments of Theorem 2.1 infer  $P^*$  can only exist where  $\Omega$  coincides with the strip. Hence

$$u < w_b \text{ in } \Omega \cap \{x \geq 0\} \quad \text{or} \quad u \equiv w_b,$$

even when  $x=b$  is tangent to  $\Omega$ . Consider all possible tangent planes corresponding to the different orientations of the  $x$ -axis through  $z_m$ , and let  $a^*$  be the smallest attainable value of  $b$ . If  $\Omega^*$  is the smallest convex region containing  $\Omega$ , then  $a^*$  is the distance from  $z_m$  to  $\partial\Omega^*$  and is less than or equal to  $\rho^*$  the in-radius of  $\Omega^*$ .

Thus the inequalities (2.19) must be satisfied and equality is attained only when  $\Omega$  is a strip.  $\square$

The combination of Theorems 2.8 and 2.9 show that if  $\Omega$  is convex, while  $B(O, a)$  is a ball contained in  $\Omega$  and containing a region  $\Omega^*$ , then the solutions  $u$  and  $u^*$  of problem  $P(\beta)$  in  $\Omega$  and  $\Omega^*$  are such that  $u < u^*$  in  $\Omega$ , or  $u \equiv u^*$  in  $\bar{\Omega}$ , and  $\Omega$ ,  $\Omega^*$ , and  $B(O, a)$  coincide. A more general domain monotonicity result can be stated in terms of a region  $\Omega_\beta$ , instead of  $B(O, a)$ , contained in  $\Omega$  and defined as follows. For any given region  $\Omega$ , let  $u$  be the solution of  $P(\beta)$  in  $\Omega$ . Then  $u$  will have a maximum  $u_m$  in  $\Omega$  at  $x_m$  where  $p = |\nabla u| = 0$  and hence

$$(u - \beta p)(x_m) = u_m > 0. \quad (2.20)$$

Let  $\Omega_\beta$  be the region in  $\Omega$  where

$$\phi = u - \beta p > 0. \quad (2.21)$$

This region  $\Omega_\beta$  is not empty since it contains a neighbourhood of  $x_m$ . It also touches  $\partial\Omega$  at least at two points, since the points on  $\partial\Omega$  where  $u$  takes its greatest and least values are points where  $p = -\partial u / \partial n$  and hence where  $\phi = 0$ .

The function  $\phi$  is superharmonic since  $u$  is, and since

$$p^2 = \sum_i \left( \frac{\partial u}{\partial x_i} \right)^2, \quad p^3 \nabla^2 p = \frac{1}{2} \sum_{i,j,k} \left( \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_j \partial x_k} - \frac{\partial u}{\partial x_k} \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2, \quad (2.22)$$

so that any point inside a closed surface on which  $\phi = 0$  is in  $\Omega_\beta$ . In the special circumstance that  $u$  is constant on  $\partial\Omega$ , the function  $\phi$  vanishes on  $\partial\Omega$ , so that  $\Omega$  and  $\Omega_\beta$  coincide for all  $\beta$ .

**THEOREM 2.11** If  $\underline{\Omega}$  is any region in  $\Omega_\beta$  and  $\underline{u}$  is the solution of  $P(\beta)$  in  $\underline{\Omega}$ , for  $\beta$  constant, then  $u > \underline{u}$  in  $\underline{\Omega}$ , or  $u \equiv \underline{u}$  and  $\Omega$ ,  $\Omega_\beta$ , and  $\underline{\Omega}$  coincide.

*Proof.* At any point  $P$  on  $\partial\Omega$ ,

$$u + \beta \frac{\partial u}{\partial n} \geq u - \beta |\nabla u| \geq 0,$$

and hence by Theorem 2.1 the result follows.  $\square$

Note that, for the balls and cylinders of Theorems 2.9 and 2.10,

$$w + \beta \frac{\partial w}{\partial n} = w - \beta p = 0 \quad \text{on } \partial\Omega,$$

since these are geometries for which  $w$  the solution of  $P(\beta)$  in  $\Omega$  is constant on  $\partial\Omega$  and  $\Omega$  and  $\Omega_\beta$  coincide.

When  $\Omega$  is a pseudo-ellipsoid (McNabb *et al.*, 1990) with axes  $X_i$  for which

$$u = A \left( 1 - \sum_i \frac{x_i^2}{X_i^2 + 2\beta X_i} \right), \quad \frac{1}{2A} = \sum_i \frac{1}{X_i^2 + 2\beta X_i}, \quad (2.23)$$

the region  $\Omega_\beta$  in which (2.21) holds is bounded by the quartic polynomial surface  $\partial\Omega_\beta$  given by

$$u^2 = \beta^2 \left[ \sum_i \left( \frac{\partial u}{\partial x_i} \right)^2 \right]. \quad (2.24)$$

This lies inside the pseudo-ellipsoid but touches it on the  $x_i$ -axes at  $x_i = X_i$ , and, as  $\beta$  tends to infinity,  $\Omega_\beta$  tends to the ellipsoid  $\Omega_0$

$$\sum_i (x_i^2/X_i^2) = 1. \quad (2.25)$$

This means there is an upper bound for  $|\Omega|/|\partial\Omega|$  for regions in this ellipsoid given by

$$\frac{|\Omega|}{|\partial\Omega|} \leq 1 / \sum_i (1/X_i). \quad (2.26)$$

This is the value of  $|\Omega|/|\partial\Omega|$  for the pseudo-ellipsoid and the rectangle with sides  $2X_i$ . One wonders whether other polygons composed of tangent planes to the pseudo-ellipsoid have the same ratio.

### 3. Bounds on $u_m$

Theorem 2.9 can be restated thus: amongst all domains with a given circumradius  $R$ , the ball maximizes the quantity  $u_m$ . In the case  $\beta = 0$ , this can be improved to: amongst all domains with a given volume  $|\Omega|$ , the ball maximizes the quantity  $u_m$ . Thus, for the disc of area  $A$ ,

$$u_m \leq A/4\pi. \quad (3.1)$$

(See Payne, 1967; Bandle, 1980; and Pólya & Szegő, 1951.) These references give numerous inequalities on other functionals, inequalities that are *isoperimetric*, that is, exact for a disc. (Further work associated with the centre-on-

circumference lenses of Keady & McNabb (1989) which, when  $\zeta_1$  tends to  $\pi$ , tend to a disc, may be of use in suggesting possible improvement to some of these isoperimetric inequalities.) We do not know of any improvement on this result from Theorem 2.9 for  $u_m$  when  $\beta$  is nonzero, using symmetrization techniques.

For problem  $P_2(0)$  and for convex domains, Pólya & Szegő (1951: p.113) suggest approximating the torsional rigidity by a formula which is exact for an arbitrary ellipse. The functional  $u_m$  might be estimated in various similar ways, one of which is as follows. For problem  $P_2(0)$  in an ellipse,  $u_m$  is equal to  $u_R(R, \rho)$ , where

$$u_R(R, \rho) = \frac{1}{2} \left( \frac{1}{R^2} + \frac{1}{\rho^2} \right)^{-1} \quad \text{when } \beta = 0. \quad (3.2)$$

The corresponding 'rule-of-thumb' when  $\beta$  is a nonzero constant and  $\Omega$  is a convex domain,

$$u_R(R, \rho, \beta) = \frac{1}{2} \left( \frac{1}{R^2 + 2\beta R} + \frac{1}{\rho^2 + 2\beta\rho} \right)^{-1}, \quad (3.3)$$

is described by McNabb *et al.* (1991). Of course, this is exact when  $\Omega$  is a pseudo-ellipse and in particular a disc (so that  $R = \rho$ ). Moreover, if  $u_m$  is the maximum value of the solution of problem  $P(\beta)$  in any region  $\Omega$  contained in the region  $\Omega_\rho$  defined by the quartic surface (2.23, 2.24),

$$u_m < u_R(R, \rho, \beta).$$

It is easy to construct other 'ellipse approximation' formulae, analogous to (3.1), for  $p_m$  and other functionals. For example, in table 1 of Keady & McNabb (1989) an 'ellipse approximation' formula estimating  $u_{xx}(z_m)/u_{yy}(z_m)$  when  $\beta = 0$ , namely  $(\rho/R)^2$ .

**THEOREM 3.1** Let  $\Omega$  be convex, and either (i)  $\beta = 0$  or (ii)  $N = 2$  and  $\beta$  constant. Then

$$P_2 = |\nabla u|^2 + 2u \leq 2u_m \quad \text{in } \Omega. \quad (3.4)$$

This is proved in Sperb (1981), where the inequality is integrated to give for convex  $\Omega$  and  $\beta = 0$ ,

$$u_m \leq \frac{1}{2} \text{distance}(z_m, \partial\Omega)^2 \leq \frac{1}{2}\rho^2, \quad (3.5)$$

where  $\rho$  is the in-radius of  $\Omega$ . Theorem 2.10 generalizes inequality (3.5) to arbitrary regions for  $\beta \geq 0$  and  $N \geq 2$ .

Let  $\kappa$  denote the curvature of the boundary of  $\Omega$  and let  $\kappa_{\max}$  denote the maximum, and  $\kappa_{\min}$  the minimum values of  $\kappa$ . Payne & Philippin (1983) prove for problem  $P_2(0)$  that

$$\frac{1}{4\kappa_{\max}^2} \leq u_m \leq \frac{1}{4\kappa_{\min}^2}. \quad (3.6)$$

(See also Kosodem'yanskii, 1987.) The inequalities become identities for a disc.

(For a lens, though, inequalities (3.6) do not tell us any more than monotonicity under domain inclusion. Lens domains are treated by Keady & McNabb (1989).)

Theorems 2.9 and 2.7 extend this result to  $P_N(\beta)$ , for which

$$\frac{1}{N} \left( \frac{1}{2\kappa_{\max}^2} + \frac{\beta}{\kappa_{\max}} \right) \leq u_m \leq \frac{1}{N} \left( \frac{1}{2\kappa_{\min}^2} + \frac{\beta}{\kappa_{\min}} \right), \tag{3.7}$$

where  $\kappa_{\max}$  and  $\kappa_{\min}$  relate to radii of spheres inside and outside  $\Omega$  and touching  $\partial\Omega$ . For convex  $\Omega$  in  $\mathbb{R}^2$  we note the following.

**THEOREM 3.2** (Sakaguchi, 1990) Let  $\Omega \subset \mathbb{R}^2$  be bounded and convex with  $\beta \geq 0$  constant. The solution  $u$  of  $P_2(\beta)$  has a unique critical point in  $\Omega$ . This critical point is  $z_m$ .

For large  $\beta$ , we see from equations (2.3, 2.4) that  $u$  is asymptotic to  $w_m$  when  $\partial\Omega$  is sufficiently smooth, and inequalities for  $u_m$  may be obtained from equation (2.4).

**4. Inequalities on  $S_\beta$**

Useful estimates for  $S_\beta$  when  $\beta$  is large can be derived from a variational formulation for the general Poisson problem  $P(\beta)$ . The solution  $u_\beta$  of  $P(\beta)$  can be seen to maximize

$$J(\varphi) = \int_{\Omega} [2\varphi - (\nabla\varphi)^2] - \frac{1}{\beta} \int_{\partial\Omega} \varphi^2 \tag{4.1}$$

by writing  $\varphi = u_\beta + v$  and using Gauss's theorem to express the functional in the form

$$J(u_\beta + v) = \int_{\Omega} u_\beta - \int_{\Omega} (\nabla v)^2 \, dv - \frac{1}{\beta} \int_{\partial\Omega} v^2. \tag{4.2}$$

Evidently  $S_\beta = \int_{\Omega} u_\beta$  is the greatest value attained by  $J(\varphi)$  and the expression (4.1) may be used to obtain approximate formula and some useful inequalities for  $S_\beta$ . For example, if  $u_0$  is the solution of  $P(0)$  and we choose constants  $a_1$  and  $a_2$  so that  $J(\varphi)$  is maximized for

$$\varphi_0 = a_1 + a_2 u_0,$$

then we find  $a_1 = \beta |\Omega|/|\partial\Omega|$  and  $a_2 = 1$ . Now

$$J(u_\beta) \geq J(\varphi_0) = \frac{\beta |\Omega|^2}{|\partial\Omega|} + \int_{\Omega} u_0. \tag{4.3}$$

so that

$$S_\beta \geq \frac{\beta |\Omega|^2}{|\partial\Omega|} + S_0. \tag{4.4}$$

It is interesting to note that, when  $\beta$  is small and  $\partial\Omega$  is sufficiently smooth,

$$S_\beta = S_0 + \beta \int_{\partial\Omega} \left( \frac{\partial u_0}{\partial n} \right)^2 + O(\beta^2), \tag{4.5}$$

and so (4.4) and (4.5) imply

$$\int_{\partial\Omega} \left( \frac{\partial u_0}{\partial n} \right)^2 \geq \frac{|\Omega|^2}{|\partial\Omega|}. \quad (4.6)$$

The solution  $u = (a^2 - r^2 + 2\beta a)/2N$  of problem  $P(\beta)$  shows that equality holds when  $\Omega$  is a sphere in  $N$  dimensions.

In like fashion, if  $u_\infty$  is the solutions of problem  $P(\infty)$ , and we write

$$u_\beta = \frac{\beta |\Omega|}{|\partial\Omega|} + u_\infty + v, \quad (4.7)$$

then, by Gauss's theorem,

$$S_\beta = J(u_\beta) = \frac{\beta |\Omega|^2}{|\partial\Omega|} + \int_\Omega u_\infty - \int_\Omega (\nabla v)^2 - \frac{1}{\beta} \int_{\partial\Omega} (u_\infty + v)^2 \quad (4.8)$$

From this we conclude that

$$S_\beta \leq \frac{\beta |\Omega|^2}{|\partial\Omega|} + \Sigma_\infty \quad \text{and} \quad \Sigma_\infty \geq S_0, \quad \text{where} \quad \Sigma_\infty = \int_\Omega u_\infty. \quad (4.9)$$

On the other hand, if we set  $v$  to zero in (4.6) instead of choosing it to maximize the value of  $J(\varphi)$ , we obtain the inequality

$$S_\beta \geq \frac{\beta |\Omega|^2}{|\partial\Omega|} + \Sigma_\infty - \frac{1}{\beta} \int_{\partial\Omega} u_\infty^2, \quad (4.10)$$

and since

$$\Sigma_1 = \int_\Omega u_1 = - \int_{\partial\Omega} u_\infty^2, \quad (4.11)$$

the inequalities (4.9) and (4.10) give

$$\frac{\beta |\Omega|^2}{|\partial\Omega|} + \Sigma_\infty + \frac{1}{\beta} \Sigma_1 \leq S_\beta \leq \frac{\beta |\Omega|^2}{|\partial\Omega|} + \Sigma_\infty. \quad (4.12)$$

It is interesting that both the functions

$$w_0 = \frac{\beta |\Omega|}{|\partial\Omega|} + u_0 \quad \text{and} \quad w_\infty = \frac{\beta |\Omega|}{|\partial\Omega|} + u_\infty \quad (4.13)$$

satisfy the Poisson equation, but on  $\partial\Omega$ , while

$$\int_{\partial\Omega} \left( w_0 + \beta \frac{\partial w_0}{\partial n} \right) = \int_{\partial\Omega} \left( w_\infty + \beta \frac{\partial w_\infty}{\partial n} \right) = 0, \quad (4.14)$$

the boundary conditions of problem  $P(\beta)$  are only satisfied pointwise if  $\Sigma_\infty = S_0$ .

This variational characterization of solutions is also useful for proving the following results for (Steiner) 'symmetrized' domains. (Convex domains which are symmetric about an axis are particular instances of Steiner symmetrized domains.) Our notation is that of Pólya & Szegő (1951) and Kawohl (1986).

**THEOREM 4.1** Let  $\Omega$  be Steiner symmetrized about some axis, and let  $\beta$  be symmetric about the same axis. Then the solution  $u$  of problem  $P(\beta)$  is Steiner symmetrized, and, in particular, any maximum occurs on the axis of symmetry.

*Proof.* Let  $u^*$  be the Steiner symmetrization of the function  $u$  which maximizes  $E_\beta(u)$ . Consider the formula (4.1) defining  $E(u)$  and  $E(u^*)$ . The integral  $\int_\Omega v$  is preserved under Steiner symmetrization, while the second term increases and hence decreases the value of  $E$ . Using the symmetry of the function  $\beta$  about the axis, we see the boundary integral is unchanged so that  $E(u^*) \geq E(u)$ . If  $u$  is a maximizer of  $E$ , then so is  $u^*$ . Thus both solve problem  $P(\beta)$  and, as this problem has a unique solution,  $u$  must be symmetrized. A similar argument establishes the same result for  $u_{\infty}$ .

**5. First derivative bounds**

The function  $P_0$  defined as

$$P_0 = |\nabla u|^2 = p^2, \tag{5.1}$$

and the functions  $p$  are subharmonic in  $\Omega$  (see (2.22)), so that  $p_m$  the maximum value attained by  $p$  in  $\bar{\Omega}$  is attained on  $\partial\Omega$ . Moreover, if  $\partial\Omega$  is sufficiently smooth (of bounded curvature), then  $p$ , as a function on  $\partial\Omega$ , has a critical point wherever  $u$  has one on  $\partial\Omega$ . In general, however, it also has critical points on  $\partial\Omega$  other than these and, when  $\beta$  is sufficiently large,  $p_m$  is not located at a critical point of  $u$  unless  $\Omega$  is a ball.

**THEOREM 5.1** If  $u$  is a solution of  $P(\beta)$  for  $\beta$  constant in  $\Omega$ ,  $\partial\Omega$  has bounded curvature, and  $u \in C^2$  on  $\partial\Omega$ , then  $p$  has critical points at the critical points of  $u$ . Furthermore,  $u$  and  $p$  have their smallest value on  $\partial\Omega$  at the same points.

*Proof.* At each point  $P$  on  $\partial\Omega$ , we assume  $\partial\Omega$  has a local representation  $x_N = f(x_1, x_2, \dots, x_{N-1})$ , where  $f \in C^2$ , and  $P$  is the origin so that  $f(0, 0, \dots, 0) = 0$ . Since  $u + \beta(\partial u / \partial n) = 0$  at  $P$ ,

$$u(f_1^2 + \dots + f_{N-1}^2 + 1)^{1/2} + \beta \left( \frac{\partial u}{\partial x_1} f_1 + \dots + \frac{\partial u}{\partial x_{N-1}} f_{N-1} - \frac{\partial u}{\partial x_N} \right) = 0 \quad \text{at } P, \tag{5.2}$$

where  $f_i = \partial f / \partial x_i$  for  $i = 1, 2, \dots, N - 1$ , and  $f_i = 0$  at  $P$ .

If  $u$  has a critical point at  $P$ ,

$$\frac{\partial u}{\partial x_i} = 0 \quad (i = 1, 2, \dots, N - 1), \quad u = \beta \frac{\partial u}{\partial x_N}, \quad p = \frac{\partial u}{\partial x_N} \quad \text{at } P, \tag{5.3}$$

and so if  $u$  takes its smallest value on  $\partial\Omega$  at  $P$ , then  $\partial u / \partial n$  is smallest and  $p$  takes the least possible value it could assume on  $\partial\Omega$ .

We may differentiate (5.2) with respect to  $x_i$  ( $i = 1, 2, \dots, N - 1$ ), so that

$$\frac{\partial u}{\partial x_i} + \beta \sum_{\alpha=1}^{N-1} \left( \frac{\partial u}{\partial x_\alpha} f_{\alpha i} - \frac{\partial^2 u}{\partial x_N \partial x_i} \right) = 0 \quad \text{at } P, \tag{5.4}$$

and so  $\partial^2 u / \partial x_N \partial x_i = 0$  for  $i = 1, 2, \dots, N - 1$  at critical points for  $u$  on  $\partial\Omega$ .

Since

$$p \frac{\partial p}{\partial x_i} = p p_i = \sum_{\alpha=1}^n \frac{\partial u}{\partial x_\alpha} \frac{\partial^2 u}{\partial x_\alpha \partial x_i}, \quad (5.5)$$

we see  $p_i$  vanishes at the critical points for  $u$  on  $\partial\Omega$ , and Theorem 5.1 is established.  $\square$

Although the points where the least value of  $p$  is assumed on  $\partial\Omega$  are also critical points where  $u$  assumes its least value on  $\partial\Omega$ , there is no clear-cut general association between other critical points of  $p$  (e.g.  $p_m$ ) and those of  $u$ . We will pursue this association in more detail for the case  $N=2$ .

At each point  $P$  of  $\partial\Omega$ , we have a representation  $y=f(x)$  of  $\partial\Omega$ , such that

$$u = \beta \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial x} + \beta \left( \frac{\partial u}{\partial x} f_{xx} - \frac{\partial^2 u}{\partial x \partial y} \right) = 0, \quad (5.6)$$

and if  $P$  is a critical point for  $p$  on  $\partial\Omega$  then

$$\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} = 0 \quad \text{at } P. \quad (5.7)$$

Define

$$G(\beta) = \beta \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} (1 + \beta f_{xx}) \quad \text{at } P,$$

so that equations (5.6) and (5.7) give

$$\frac{\partial u}{\partial x} G(\beta) = 0 \quad \text{at } P. \quad (5.8)$$

If, for a given  $\beta$  and  $\Omega$ ,  $G$  is nonzero at  $P$ , then  $\partial u / \partial x = 0$  there, and  $P$  is a critical point for  $u$  on  $\partial\Omega$ .

When  $\beta$  is near zero,  $G$  is positive, since  $\partial u / \partial y$  is positive when  $\beta$  is zero and therefore, by continuity, when  $\beta$  is close enough to zero. Also, if  $\Omega$  is a circle of radius  $a$ , then

$$y - f(x) = y - a + (a^2 - x^2)^{\frac{1}{2}} = 0, \quad u = \frac{1}{4}[a^2 - x^2 - (y - a)^2] + \beta a,$$

and so

$$G = \frac{1}{2}a. \quad (5.9)$$

Hence  $G$  is positive for all  $\beta$  when  $\Omega$  is a circle and so, for each finite  $\beta$ , is positive for  $\Omega$  nearly circular.

In another approach for the case when  $\beta$  is small,  $\partial\Omega$  smooth, and  $N$  unrestricted, we assume  $u$  can be written

$$u = u_0 + \beta u_1 + \beta^2 u_2 + \dots, \quad (5.10)$$

where  $u_i$  satisfy the boundary value problems

$$\left. \begin{aligned} \nabla^2 u_0 + 1 = 0, \quad \nabla^2 u_i = 0 \quad \text{in } \Omega; \quad u_0 = 0, \quad u_1 = -\frac{\partial u_0}{\partial n}, \quad u_2 = -\frac{\partial u_1}{\partial n}, \dots \quad \text{on } \partial\Omega. \\ p^2 = u_1^2 + 2\beta u_1 u_2 + \dots, \quad u = \beta u_1 + \beta^2 u_2 + \dots \quad \text{on } \partial\Omega. \end{aligned} \right\} \quad (5.11)$$

For small  $\beta$ , then, we expect to find the greatest and least values of  $p$  and  $u$  on  $\partial\Omega$  at the same points, confirming our earlier conclusions. The picture when  $\beta$  is very large is interesting and different. The asymptotic expressions (2.3, 2.4) for large  $\beta$  and smooth  $\partial\Omega$  give

$$u \sim \frac{\beta |\Omega|}{|\partial\Omega|} + u_\infty + \frac{u_1}{\beta} + \dots, \quad \frac{\partial u}{\partial y} \sim \frac{|\Omega|}{|\partial\Omega|} + \frac{u_\infty}{\beta} + \dots \tag{5.12}$$

Naturally, the maxima and minima of  $u$  on  $\partial\Omega$  are at the same points as those of  $\partial u/\partial y$  on  $\partial\Omega$ ; but  $\partial u/\partial y$  is almost constant and we need to study  $u_\infty$  to see the association between the critical points of  $p$  and  $u$  on  $\partial\Omega$  for large  $\beta$ .

If  $p_m$ , the maximum of  $|\nabla u_\infty|$  on  $\partial\Omega$ , coincides with  $u_\infty^*$  the maximum of  $u_\infty$  on  $\partial\Omega$ , then  $p_m = \partial u_\infty/\partial y = |\Omega|/|\partial\Omega|$  at the critical point. This implies  $p$  is constant (the greatest and least values being  $|\Omega|/|\partial\Omega|$ ) on  $\partial\Omega$  and equal to  $\partial u_\infty/\partial y$ , so that  $\partial u_\infty/\partial x = 0$  on  $\partial\Omega$  and so  $u_\infty = 0$  on  $\partial\Omega$ . But then  $u_\infty = u_0$ , the solution of  $P(0)$  on  $\Omega$ , and a symmetrization argument shows  $\partial\Omega$  can only be a circle.

**THEOREM 5.2** If  $\Omega$  is bounded and  $\partial\Omega \in C^2$  and if  $u_\infty \equiv u_0$ , then  $\partial\Omega$  is a circle.

*Proof.* Let  $O$  be the contact point of a tangent line  $y = 0$ , with  $\partial\Omega$  chosen so that  $\Omega$  lies in  $y > 0$ . Let  $\Omega_\alpha$  and  $u_\alpha$  denote the image of  $\Omega$  and  $u$  reflected in the line  $y = \alpha$ . As  $\alpha$  increases from 0, there will be a first number  $\alpha^* > 0$  such that  $\Omega_\alpha \cap \{y \geq \alpha^*\} \subset \Omega \cap \{y \geq \alpha^*\}$ , and  $\partial\Omega_\alpha \cap \{y \geq \alpha^*\}$  and  $\partial\Omega \cap \{y \geq \alpha^*\}$  either have a point  $P$  in common not on  $y = \alpha^*$  or they have a common tangent  $x = \beta$  at  $(\beta, \alpha^*)$  on  $\partial\Omega$ .

In the first situation, we have  $u_\alpha \leq u$  in  $\Omega_\alpha \cap \{y \geq \alpha^*\}$  (see McNabb, 1967) and at  $P$  we have  $u_\alpha = u = 0$  and  $\partial u_\alpha/\partial n = \partial u/\partial n = -|\Omega|/|\partial\Omega|$ , and this by the Hopf lemma (Hopf, 1952) implies  $u_\alpha \equiv u$  and  $\Omega_\alpha \cap \{y \geq \alpha^*\}$  is identical to  $\Omega \cap \{y \geq \alpha^*\}$ .

In the second case,

$$\frac{\partial u}{\partial y} = \frac{\partial u_\alpha}{\partial y} = u = u_\alpha = 0 \quad \text{at } (\beta, \alpha^*)$$

and

$$\frac{\partial u_\alpha}{\partial x} = \frac{\partial u}{\partial n} = \frac{|\Omega|}{|\partial\Omega|} \quad \text{at } (\beta, \alpha^*).$$

Moreover

$$\frac{\partial^2 u_\alpha}{\partial x \partial y} = \frac{\partial^2 u}{\partial x \partial y} = 0 \quad \text{and} \quad \nabla^2(u - u_\alpha) = 0 \quad \text{at } (\beta, \alpha^*),$$

and so  $w = u - u_\alpha = O(t^3)$  along the line  $(\beta + t, \alpha^* + t)$ . But  $u_\alpha \leq u$  in  $\Omega_\alpha \cap \{y \geq \alpha^*\}$  requires, by an argument analogous to that of Hopf (1952),  $u_\alpha - u$  to be of order  $t^2$  along this line as  $t \rightarrow 0$  or else  $u_\alpha \equiv u$  and  $\Omega_\alpha \cap \{y \geq \alpha^*\}$  is identical with  $\Omega \cap \{y \geq \alpha^*\}$ . This means that  $\partial\Omega$  must be a circle if  $u_\infty \equiv u_0$ , and the result follows.  $\square$

If  $u_\infty$  is not constant on  $\partial\Omega$ , then  $(\partial u_\infty/\partial x)^2$  is greatest on  $\partial\Omega$  for some choices of axes, and at this point  $p^2 = (\partial u_\infty/\partial x)^2 + |\Omega|^2/|\partial\Omega|^2$  takes its greatest value.

Hence, for any  $\Omega$  not a disc, and  $\beta$  large enough, the location of  $p_m$  cannot coincide with that of the greatest value of  $u$  on  $\partial\Omega$ . Analogous arguments may be carried through in  $\mathbb{R}^N$ .

There is a simple exact solution for problem  $P_2(\beta)$  when  $\Omega$  is an equilateral triangle (McNabb & Keady, 1993). Although this case for which  $\partial\Omega$  has corners is strictly outside the scope of the treatment in this paper, it is interesting to see how the location of  $p_m$  changes with increasing  $\beta$  from an association with  $u^*$  at the midpoints of the sides to an association with the least value of  $u$  on  $\partial\Omega$  at the corners.

We see from Theorem 5.1 that if  $u^*$  is the greatest value of  $u$  assumed at  $Q$  on  $\partial\Omega$ , then  $p$  has a critical value at  $Q$  which will be  $p_m$  if  $\beta$  is small enough, but in general

$$p_m \geq u^*/\beta.$$

If the point  $z_t$  can be touched by a ball of radius  $\rho^*$  lying in  $\Omega$ , then, by Theorem 2.7,

$$p_m \geq u^*/\beta \geq \rho^*/N.$$

Sperb (1981: p.85) gives numerous inequalities for  $p_m$  using his  $P_n$  functions

$$P_n = |\nabla u|^2 + nu. \quad (5.13)$$

For  $N=2$ ,  $P_1$  takes its maximum value on  $\partial\Omega$ , and, when  $\beta=0$ ,

$$p_m^2 \geq u_m. \quad (5.14)$$

The analogue of this in higher dimensions is that Sperb's function  $P_{2/N}$  takes its maximum on  $\partial\Omega$ , and so, for problem  $P_N(0)$ ,

$$p_m^2 \geq 2u_m/N. \quad (5.15)$$

For convex  $\Omega$ , St Venant and Filon have speculated about the location of the fail points  $z_f$  (see Kawohl, 1986; Sweers, 1989; Ramaswamy, 1990). For probably all of the domains for which St Venant and Filon had explicit solutions of the torsion problem, they noticed that the fail points occurred at a point of intersection of an in-circle, a largest inscribed circle, with the boundary of the domain. They speculated whether it might be more generally true for convex domains. It is not true in general. The first counterexample we found was for a slender isosceles triangle. (The semi-infinite strip is easier because a Fourier series expansion for the torsion function is available. However, St Venant and Filon were almost certainly considering bounded domains.) Kawohl (1986) suggests that the speculations of St Venant and of Filon might have been intended for convex domains with two orthogonal axes of symmetry. Again there is a counterexample, another convex curvilinear polygon (see Sweers, 1989). We remark that, for any domain in two dimensions, the NAG routine D03EAF is a particularly appropriate tool for locating fail points and for investigating other aspects of the behaviour of  $\nabla u$  on the boundary.

Estimates for extreme values of  $p$  on  $\partial\Omega$  should be tested against those obtained from the requirement that  $p_m$  must be greater than or equal to the

average value of  $u/\beta$  on  $\partial\Omega$ . We have, from Gauss's theorem,

$$p_m \geq \frac{u^*}{\beta} \geq \frac{1}{\beta |\partial\Omega|} \int_{\partial\Omega} u = -\frac{1}{|\partial\Omega|} \int_{\partial\Omega} \frac{\partial u}{\partial n} = -\frac{1}{|\partial\Omega|} \int_{\Omega} \nabla^2 u = \frac{|\Omega|}{|\partial\Omega|}, \quad (5.16)$$

and this result is exact for spheres in  $\mathbb{R}^N$ .

If  $w$  is the solution of  $P(\beta)$  for  $\beta$  constant in the in-sphere  $B(\rho)$  touching  $\partial\Omega$  at  $Q$ , say, then there is a constant  $c$  for which  $w + c \leq u$  in  $B(\rho)$  and  $w + c = u$  at a point  $Q^*$  on  $\partial B(\rho)$ . By Theorem 2.1,  $\partial u/\partial r \leq \partial w/\partial r = -\rho/N$  at  $Q^*$ , and so

$$p_m \geq \rho/N. \quad (5.17)$$

This result can be better than (5.16) for nonconvex regions with an in-sphere for which  $\rho/N > |\Omega|/|\partial\Omega|$ . If the boundary  $\partial\Omega$  is sufficiently smooth, the asymptotic expressions (2.3, 2.4) give useful estimates for  $p_m$ . For large  $\beta$ , and sufficiently smooth  $\partial\Omega$ ,

$$p_m \geq -\frac{\partial u}{\partial n}(z_i) \sim \frac{|\Omega|}{|\partial\Omega|} + \left( \frac{u_{\infty}}{\beta} + \dots + \frac{u_{i-1}}{\beta^i} \right)(z_i). \quad (5.18)$$

For diffusion and conduction problems, bounds on  $\partial u/\partial n$  give bounds on the local fluxes at the boundary. Theorem 2.4 shows that bounds on  $\partial u/\partial n$  for problem  $P(\beta)$  for  $\beta$  constant are provided by bounds on  $\partial u/\partial n$  for problem  $P(0)$ .

If  $u$  is the solution of problem  $P(\beta)$  for  $\beta$  constant in  $\Omega$ , then, unlike  $p_m$ , the greatest value  $q_m$  of  $|\partial u/\partial n|$  on  $\partial\Omega$  occurs at the same points on  $\partial\Omega$  where  $u$  assumes its maximum value  $u^*$  on  $\partial\Omega$ , and, when  $\beta > 0$ ,  $q_m = u^*/\beta$  at these points. Theorems 2.10 and 3.1 give a useful upper limit for  $q_m$ .

**THEOREM 5.3** If  $\Omega$  is convex and  $u$  is the solution of problem  $P(\beta)$  for  $\beta$  constant in  $\Omega$ , with a maximum value  $u_m$  at  $z_m$  in  $\Omega$  and a maximum value  $q_m$  for  $|\partial u/\partial n|$  on  $\partial\Omega$ , then

$$q_m \leq a = \text{minimum distance from } z_m \text{ to } \partial\Omega. \quad (5.19)$$

*Proof.* Suppose  $u = u^*$  at  $Q$  on  $\partial\Omega$  and let  $w$  be the solution of problem  $P(0)$  in  $\Omega$ , so that  $u \leq u^* + w$  on  $\partial\Omega$  and hence in  $\Omega$ , and  $q_m \leq |\partial w/\partial n|_Q$ . From Theorem 3.1

$$\left| \frac{\partial w}{\partial n} \right|_Q^2 \leq |\nabla w|_Q^2 \leq 2w_m, \quad (5.20)$$

and from Theorem 2.10

$$w_m \leq \frac{1}{2}a^{*2} = \frac{1}{2}a^2. \quad (5.21)$$

Hence,

$$q_m^2 \leq a^2 \quad \text{and} \quad q_m \leq a. \quad (5.22)$$

□

Fu & Wheeler (1973) showed that, for problem  $P_2(0)$  and convex  $\Omega$ ,

$$P_m^2 \leq \rho^2(1 - \frac{1}{2}\kappa_t\rho)^2 \leq \rho^2(1 - \frac{1}{2}\kappa_{\min}\rho)^2, \quad (5.23)$$

where  $\kappa_f$  is the curvature at the principal fail point. This result is exact both when  $\Omega$  is a strip and when it is a disc. It points to possible extensions of Theorem 2.5 by replacing the comparison functions (2.5) with the solution of the problem  $P(\beta)$  for  $\beta$  constant in a ball instead of a strip, so that  $w(r)$  satisfies

$$\nabla^2 w + 1 = 0 \text{ in } R_0 < r < R_1, \quad \frac{\partial w}{\partial r} = 0 \text{ on } r = R_0, \quad w + \beta \frac{\partial w}{\partial r} = 0 \text{ on } r = R_1. \quad (5.24)$$

The solution  $w$  takes its greatest value  $w_m$  on  $r = R_0$ . If  $N = 2$ ,

$$w_m = \frac{R_1^2 - R_0^2}{4R_1} (R_1 + 2\beta) - \frac{1}{2} R_0^2 \ln \frac{R_1}{R_0}, \quad (5.25)$$

and, if  $N > 2$ ,

$$w_m = \frac{R_1^2 - R_0^2}{2NR_1} (R_1 + N\beta) + \frac{R_0^N}{N(N-2)} \left( \frac{1}{R_1^{N-2}} - \frac{1}{R_0^{N-2}} \right). \quad (5.26)$$

**THEOREM 5.4** Let  $\Omega^*$  be the smallest region containing  $\Omega$  such that every point  $P$  on  $\partial\Omega^*$  lies on the surface of a sphere of radius  $R_1$  touching  $\Omega^*$  at  $P$ , and containing  $\Omega^*$ . Let  $a^*$  be the minimum distance from  $z_m$  to  $\partial\Omega^*$ , where  $z_m$  is a point in  $\Omega$  where  $u$ , the solution of  $P(\beta)$  for  $\beta$  constant in  $\Omega$ , takes its greatest value  $u_m$ . Then

$$u_m \leq w_m \text{ in } \Omega, \quad (5.27)$$

where  $w_m$  is given by (5.25) or (5.26), and  $R_0 = R_1 - a^*$ . If  $\rho^*$  is the radius of the in-sphere of  $\Omega^*$ , we may set  $R_0 = R_1 - \rho^*$ . If  $\Omega$  and  $\Omega^*$  coincide, and  $q_m$  is the maximum of  $|\partial u / \partial n|$  on  $\partial\Omega$ , then

$$q_m^2 \leq 2w_m \text{ and } q_m \leq \left[ -\frac{\partial w}{\partial r} \right]_{R_1} = \frac{a^*}{N} \left[ 1 + \left( 1 - \frac{a^*}{R_1} \right) + \dots + \left( 1 - \frac{a^*}{R_1} \right)^{N-1} \right] \quad (5.28)$$

*Proof.* The first inequality (5.27) is derived by following the proof plan of Theorem 2.10 using the new functions defined by the boundary value problem (5.24). Suppose  $\Omega$  is convex, so that  $\Omega$  and  $\Omega^*$  coincide. The first inequality (5.28) then follows from Theorem 3.1. Let  $Q$  be a point on  $\partial\Omega$  at the minimum distance  $a$  from  $z_m$  and  $w(r)$  the solution of problem (5.24) in the sphere which contains  $\Omega$  and touches it at  $Q$ . For this sphere and  $R_1 - R_0 = a$ ,  $u < w$  in  $\Omega \cap \{r: R_0 \leq r \leq R_1\}$ , or  $u \equiv w$  and  $\Omega$  is the ball of radius  $R_1$ . In the former case  $u < w$ , we have  $u_Q \leq w(R_1)$  and  $u_m < w(R_0)$ .

Let  $w_c = w - c$  for  $c$  a positive constant and compare  $w_c$  with  $u$  for increasing values of  $c$ . When  $c = 0$ ,  $w_c = w > u$  in  $\Omega \cap \{r: R_0 \leq r \leq R_1\}$  and as  $c$  increases there is a  $c^* > 0$  and a first point in the region where  $u = w$ . By the arguments of Theorem 2.10, we find this contact point must be at  $Q$  if  $\Omega$  is not a ball. Now let  $Q^*$  be a point on  $\partial\Omega$  where  $u$  takes its greatest value  $u^*$  and hence where  $q = |\partial u / \partial n|$  takes its greatest value  $q_m$ . Consider  $w_c$  for  $c = w(R_1) - u^* \leq w(R_1) - u_Q \leq w(R_0) - u_m$ .

This function  $w_c$  associated with the sphere containing  $\Omega$  and touching  $\partial\Omega$  at  $Q^*$  is such that  $u \leq w_c$  in  $\Omega \cap \{r: R_0 \leq r \leq R_1\}$  and  $u = u^* = w_c$  at  $Q^*$ . Then

$$q_m < \left[ -\frac{\partial w_c}{\partial r} \right]_{R_1} = \left[ -\frac{\partial w}{\partial r} \right]_{R_1} = \frac{R_1^N - R_0^N}{NR_1^{N-1}}, \quad (5.29)$$

and the second inequality (5.27) is obtained.  $\square$

For  $N = 2$  and  $a = \rho$  inequality (5.22) is obtained, and for  $N = 3$

$$q_m \leq \rho \left( 1 - \frac{\rho}{R_1} + \frac{\rho^2}{3R_1^2} \right). \quad (5.30)$$

When  $R_1$  tends to infinity, the inequality (5.19) is re-established.

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