Generalisation of Hadamard's inequality for convex functions to higher dimensions, and an application to the elastic torsion problem

Sever S. Dragomir,
University of Timişoara

Grant Keady,
University of Western Australia

November 25, 1998

Abstract

The (generalised) torsion function u of a domain $\Omega \subset \mathbb{R}^n$ is a function which is zero on the boundary of the domain and whose Laplacian is minus one at every point in the interior of the domain. Denote by $|\Omega|$ the measure of Ω, x_c its centroid. We establish, for convex Ω,

$$\frac{3}{2(n+1)^2} \leq \frac{1}{(n+1)^2}\frac{|\Omega|}{\int_\Omega u}\max_\Omega u \leq \frac{|\Omega| u(x_c)}{\int_\Omega u} \leq \frac{|\Omega|}{\int_\Omega u}\max_\Omega u \leq \frac{1}{2}(n+1)(n+2).$$

We generalise this to positive solutions u of the semilinear problem $-\Delta u = u^\gamma$, $0 \leq \gamma < 1$, satisfying homogeneous Dirichlet boundary conditions.

1 Foundational materials

1.1 Introduction

The main results in this paper concern positive functions u with certain concavity properties (usually that u to some power is concave) defined on a convex domain Ω and vanishing on its boundary. We find bounds for quantities, with $k > 0$, like

$$\zeta(\Omega, k, u, x_p) = \frac{|\Omega| u(x_p)^k}{\int_\Omega u^k}\,, \tag{1.1}$$

where x_p is a given point in Ω. In some bounds, notably those from the Hadamard inequality, $x_p = x_c$ the centroid of Ω. In other bounds, $x_p = x_m$ the location of the maximum of u.

There are numerous ways in which the material in this Report represents work in progress. We anticipate further results, and accordingly occasionally summarise items which we feel are likely to be useful, but which are not actually used for the main results of the report. Theorems stated, but with proofs omitted, are in this category.

1.1.1 Positive continuous functions

Without concavity properties for u or information concerning u such as one might have from being given that u solves some partial differential equation problem, results on ζ are limited. Here, however, is one easy result.

THEOREM 1.1 *For any domain Ω and positive function u defined on Ω, the functional $\zeta(\Omega, k, u, x_p)$ defined by equation (1.1) satisfies the following.*

(i) *For any $x_p \in \Omega$, $\zeta(\Omega, \cdot, u, x_p)$ is positive and logconcave on $[0, \infty)$.*

(ii) *Let x_m be a location of the maximum of u. Then $\zeta(\Omega, \cdot, u, x_m)$ is monotonic nondecreasing on $[0, \infty)$. For $k \in [0, \infty)$, $\zeta(\Omega, 0, u, x_m) = 1 \leq \zeta(\Omega, k, u, x_m)$.*

A simple application of the Hőlder inequality gives, for $0 \leq t \leq 1$ and omitting arguments which are the same in the ζ,

$$\frac{1}{\zeta((1-t)k_0 + tk_1)} \leq \frac{1}{\zeta(k_0)^{1-t}} \frac{1}{\zeta(k_1)^t},$$

which proves (i).

Write $\zeta_m(k) = \zeta(k, x_m)$. The result $1 \leq \zeta_m(k)$ is easily improved for the solutions u of our partial differential equation problems: see, for example, the leftmost inequality – on $\zeta_m(1)$ – in the abstract.

1.2 Hadamard's inequality

The Hermite-Hadamard inequality for convex functions defined on an interval of the real line is given, with some history, in [PPT]. The result immediately below is a generalisation of this to higher space dimensions.

THEOREM 1.2 *Let Ω be a convex, compact subset of $\mathbb{R}^n$ with nonempty interior. Suppose that the mapping $\Phi : \Omega \to \mathbb{R}$ is differentiable and convex on Ω. Define, for $x_p \in \Omega$,*

$$I_1(x_p) = \int_\Omega \Phi(x)dx - |\Omega|\Phi(x_p), \qquad I_2(x_p) = \int_\Omega \langle \nabla\Phi(x), x\rangle dx - \langle x_p, \int_\Omega \nabla\Phi(x)dx\rangle,$$

where $|\Omega|$ is the volume of Ω. Then one has the inequalities

$$0 \leq I_1(x_c), \tag{1.2}$$

$$I_1(x_p) \leq I_2(x_p), \tag{1.3}$$

where x_c is the centroid of Ω. If $u = \Phi$ is concave rather than convex, the inequalities are reversed, and, if additionally $u \geq 0$, $\zeta(\Omega, 1, u, x_c) \geq 1$.

Proof. As Φ is differentiable in the interior of Ω we know that

$$\Phi(x) - \Phi(y) \geq \langle (\nabla\Phi)(y), x - y\rangle \qquad \forall\ x, y \in \Omega, \tag{1.4}$$

where $\langle a, b\rangle = a \cdot b$ is the usual inner product in $\mathbb{R}^n$. Applying this with $y = x_c$ and integrating over x gives

$$\begin{aligned} I_1(x_c) &= \int_\Omega \Phi(x)dx - |\Omega|\Phi(x_c) \geq \int_\Omega \langle \nabla\Phi(x_c), x - x_c\rangle dx, \\ &= \langle \nabla\Phi(x_c), \int_\Omega (x - x_c)\, dx\rangle = 0, \end{aligned}$$

which establishes inequality (1.2).

Let x_p be any point in the interior of Ω. By inequality (1.4) we also deduce that

$$\Phi(x_p) - \Phi(x) \geq \langle (\nabla\Phi)(x), x_p - x\rangle \qquad \forall\ x \in \Omega\ .$$

Integrating this inequality with respect to x over Ω gives

$$\begin{aligned} -I_1(x_p) &= |\Omega|\Phi(x_p) - \int_\Omega \Phi(x)dx \\ &\geq \int_\Omega \langle \nabla\Phi(x), x_p\rangle dx - \int_\Omega \langle \nabla\Phi(x), x\rangle dx = -I_2(x_p), \end{aligned}$$

from which we get inequality (1.3).

The proof above actually yields the following small generalisation, which we record for possible future use. (It isn't used in this Report. We anticipate possible uses when we have further results on the concavity sets for the elastic torsion problem, $n = 2$.)

COROLLARY. *The inequalities of the preceding theorem are also true when the integrals (and measure) are taken over appropriate star-shaped subsets $\Omega_* \subseteq \Omega$ rather than the whole of Ω.*
(i) $0 \leq I_{1*}(x_{c*})$ when Ω_* is star-shaped with respect to its centroid x_{c*}.
(ii) $I_{1*}(x_{p*}) \leq I_{2*}(x_{p*})$ when x_{p*} is any point of Ω_* with respect to which Ω_* is star-shaped.

The preceding Theorem generalises. We record this for possible future use. (It isn't used in this Report. The kinds of integrals occuring in Theorem 1.3 also arise, though just incidentally, towards the end of Section 4 in connection with the Helmholtz fundamental mode problem.)

THEOREM 1.3 *Let Φ, Ω, x_c be as in Theorem 1.2. Define, for $x_p \in \Omega$, and $0 \leq t \leq 1$,*

$$\begin{aligned} H(x_p, t) &= \frac{1}{|\Omega|}\int_\Omega \Phi(tx + (1-t)x_p)dx, \\ I_2(x_p, t) &= \frac{1}{|\Omega|}(\int_\Omega \langle \nabla\Phi(tx + (1-t)x_p), x\rangle dx - \langle x_p, \int_\Omega \nabla\Phi(tx + (1-t)x_p)dx\rangle). \end{aligned}$$

Then

(i) *$H(x_p, \cdot)$ is convex on $[0, 1]$.*

(ii)

$$\sup_{t\in[0,1]} H(x_c, t) = H(x_c, 1) = \frac{1}{|\Omega|}\int_\Omega \Phi,$$

and

$$\inf_{t\in[0,1]} H(x_c, t) = H(x_c, 0) = \Phi(x_c).$$

(iii) $H(x_c,\cdot)$ *is monotonic nondecreasing on* $[0,1]$.

(iv)

$$0 \le H(x_c,t) - \Phi(x_c) \le tI_2(x_c,t), \qquad \forall t \in [0,1],$$

and

$$0 \le (1-t)I_2(x_c,t) \le \frac{1}{|\Omega|}\int_\Omega \Phi - H(x_c,t), \qquad \forall t \in [0,1].$$

2 Positive power-concave functions vanishing on $\partial\Omega$

2.1 Notation

We apologise for the switch from convex functions to concave ones. However, this seems to be forced on us because of applications to partial differential equations coming in Sections 3 and 4.

In this section we are concerned with classes of positive functions defined on the convex set Ω. Specifically, for $1 \le \alpha$, define

$$\begin{aligned} \mathcal{U}_\alpha &= \{u \ge 0 \mid \ u^{1/\alpha} \ \text{ is concave}\}, \\ \mathcal{U}_{\alpha,0} &= \{u \mid \ u \in \mathcal{U}_\alpha \ \text{ which are zero on } \partial\Omega\}. \end{aligned}$$

The centroid x_c is introduced in Section 1.2. Define also

$$u_m = \max_\Omega u, \qquad u(x_m) = u_m.$$

The results stated in the rest of this subsection, subsection 2.1, are for $u \ge 0$ and $1 \le \xi$. We have

$$(\frac{1}{|\Omega|}\int_\Omega u^{1/\xi}) \le (\frac{1}{|\Omega|}\int_\Omega u)^{1/\xi}, \qquad \text{or} \qquad \zeta(1)^{1/\xi} \le \zeta(\frac{1}{\xi}). \tag{2.1}$$

This inequality is simply Hőlder's inequality, and is recorded here because of our concern – in our main application, the torsion problem – with $\int_\Omega u$. (It is also follows from the logconcavity of ζ using $\zeta(0) = 1$.) Another trivial inequality is

$$\frac{1}{|\Omega|}\int_\Omega u \le u_m^{(\xi-1)/\xi}\frac{1}{|\Omega|}\int_\Omega u^{1/\xi}, \qquad \text{or} \qquad \zeta_m(\frac{1}{\xi}) \le \zeta_m(1), \tag{2.2}$$

consistent with Theorem 1.1 (ii).

If $u \ge 0$ and $1 \le \xi$, and $u = 0$ on the boundary of Ω, an application of the divergence theorem gives

$$-\int_\Omega x\cdot\nabla(u^{1/\xi}) = -\int_\Omega \operatorname{div}(xu^{1/\xi}) + n\int_\Omega u^{1/\xi} = n\int_\Omega u^{1/\xi}. \tag{2.3}$$

2.2 Concave function results

THEOREM 2.1 *Let* $\alpha \geq 1$. $\forall u \in \mathcal{U}_{\alpha,0}$

$$\frac{1}{n+1} u_m^{1/\alpha} \leq \frac{1}{|\Omega|} \int_\Omega u^{1/\alpha} \leq u(x_c)^{1/\alpha}, \tag{2.4}$$

or

$$\frac{1}{(n+1)} \zeta_m(\frac{1}{\alpha}) \leq 1 \leq \zeta(\frac{1}{\alpha}, x_c) \leq \zeta_m(\frac{1}{\alpha}) \leq (n+1).$$

From this:

$$u_m \leq c_m(n,\alpha) u_c \qquad \textit{where} \qquad c_m(n,\alpha) = (n+1)^\alpha; \tag{2.5}$$

$$\zeta_m(1) = \frac{|\Omega| u_m}{\int_\Omega u} \leq c_{r,0}(n,\alpha) \qquad \textit{where} \qquad c_{r,0}(n,\alpha) = (n+1)^\alpha. \tag{2.6}$$

Proof. The right-hand part of the first inequalities (2.4-r) follows on applying the generalisation of Hadamard's inequality (1.2) given in Section 1.2 to the function $\Phi = -u^{1/\alpha}$. The left-hand part (2.4-l) follows from the fact that $u^{1/\alpha}$ lies above a 'cone' base Ω and height $u_m^{1/\alpha}$. The next inequality is a combination of both parts of inequalities (2.4).

Inequality (2.6) follows from inequalities (2.1), with $\xi = \alpha$, and the left-hand part (2.4-l).

Here is an alternative proof of the left-hand inequality (2.4-l). We start with applying inequality (1.3) at any point $x_p \in \Omega$:

$$-\int_\Omega u^{1/\alpha} + |\Omega|(u(x_p))^{1/\alpha} \leq -\int_\Omega x \cdot \nabla(u^{1/\alpha}).$$

Next apply (2.3), with $\xi = \alpha$. This gives

$$|\Omega| u(x_p)^{1/\alpha} \leq (n+1) \int_\Omega u^{1/\alpha}.$$

This inequality is best when $x_p = x_m$ which gives the left-hand inequality (2.4-l) as stated in the theorem.

Combining the next two Theorems improves on inequality (2.6).

THEOREM 2.2 *Let* $\Omega^* = \{x \in \mathbb{R}^n \mid |x| < \rho\}$. *Let* $U = U_m(1 - |x|/\rho)$ *define a conical graph over* Ω^*. *Let* $\xi \geq 0$.

$$\frac{|\Omega^*| U_m^\xi}{\int_\Omega U^\xi} = c_r(n,\xi), \qquad \textit{where} \qquad c_r(n,\xi) = \frac{\Gamma(n+\xi+1)}{\Gamma(n+1)\Gamma(\xi+1)}. \tag{2.7}$$

Proof. Using spherical polar coordinates, we find

$$\frac{\int_\Omega U^\xi}{\int_\Omega 1} = U_m^\xi \frac{\int_0^1 (1-t)^\xi t^{n-1}\, dt}{\int_0^1 t^{n-1}\, dt} = U_m^\xi \frac{\Gamma(n+1)\Gamma(\xi+1)}{\Gamma(n+\xi+1)}.$$

Refer to [PS, Ka1] for the definitions and properties of Schwarz symmetrisation. The following result is in the standard books on convexity for Steiner symmetrisation, and is also true for Schwarz symmetrisation. If v is concave over Ω, its Schwarz symmetrisation v^* is concave over the symmetrised domain Ω^*. (The result for Schwarz symmetrisation is proved in some books by combining the Blaschke Selection Principle with the result for Steiner symmetrisation.) This fact is used in the following proof.

THEOREM 2.3 *Let $\alpha \geq 1$, $\xi \geq 0$. $\forall u \in \mathcal{U}_{\alpha,0}$*

$$\zeta_m(\xi) = \frac{|\Omega| u_m^{\xi}}{\int_\Omega u^{\xi}} \leq c_r(n, \alpha\xi), \tag{2.8}$$

where c_r is defined in the preceding lemma.

Proof. Translate the origin so that it is at x_m, $x_m = 0$. Define, for $x_{\partial\Omega} \in \partial\Omega$, the cone

$$U_c(t x_{\partial\Omega}) = u_m^{1/\alpha}(1-t).$$

$U_c \in \mathcal{U}_{1,0}$ and $u \geq U_c^{\alpha}$.

Next, consider the Schwarz symmetrisation U_c^* of U_c.

$$\int_\Omega u^{\xi} \geq \int_\Omega U_c^{\alpha\xi} = \int_{\Omega^*} (U_c^*)^{\alpha\xi}.$$

The level curve $U_c = u_m^{1/\alpha}(1-t)$ is geometrically similar to $\partial\Omega$: it is $\{t x_{\partial\Omega} \ \mid \ x_{\partial\Omega} \in \partial\Omega\}$.

$$|\{x \mid \ U_c(x) > u_m^{1/\alpha}(1-t)\}| = t^n |\Omega|.$$

From this U_c^* is a 'circular cone', so the preceding lemma can be applied and gives the result.

When $\xi = 1$ and $\alpha > 1$ inequality (2.8) improves on inequality (2.6): in the case $\alpha = 1$ they are equal. For the application to the torsion problem, we note $c_r(n, 2) = (n+1)(n+2)/2$, the right-most expression in the inequality given in the Abstract.

2.3 Weighted 'centroids'

Items in this subsection are not used elsewhere in this report.

Define, for $\rho > 0$,

$$x_c(\rho) = \frac{\int_\Omega x\rho}{\int_\Omega \rho}.$$

With $u \in \mathcal{U}_{0,\alpha}$, $\xi > 0$ and $\rho = u^{\xi}$, write

$$x_c(\xi) = \frac{\int_\Omega x u^{\xi}}{\int_\Omega u^{\xi}}.$$

We have $x_c(0) = x_c$ – the usual centroid – and we expect that $x_c(\infty) = x_m$. The proof of the Hadamard inequality also gives

$$\int_\Omega u^{\xi} u^{1/\alpha} \leq u(x_c(\xi))^{1/\alpha} \int_\Omega u^{\xi}, \qquad \text{or} \qquad \zeta(\Omega, \xi, u, x_c(\xi)) \leq \zeta(\Omega, \xi + \frac{1}{\alpha}, u, x_c(\xi)).$$

We remark that, as in inequality (2.4), for $u \in \mathcal{U}_{0,\alpha}$ and $0 \leq \xi \leq 1/\alpha$,

$$\frac{1}{n+1} u_m^{1/\alpha} \leq \frac{1}{|\Omega|} \int_\Omega u^{1/\alpha} = \frac{1}{|\Omega|} \int_\Omega u^{\xi} u^{1/\alpha - \xi} \leq u(x_c(\xi))^{1/\alpha - \xi} \frac{1}{|\Omega|} \int_\Omega u^{\xi},$$

the extreme parts of which may be rewritten

$$(\frac{\zeta_m(\theta)}{\zeta_m(\theta, x_c(\xi))})^{1/(\alpha\theta)} \leq (\frac{u_m}{u(x_c(\xi))})^{1/\alpha} \leq \frac{n+1}{\zeta(\xi, x_c(\xi))}$$

Because our interest, to date, has been in the torsion problem of Section 3, and the usual $\xi = 0$ centroid, we have not yet explored possible inequalities involving weighted centroids, or used them in any remaining parts of this report. Our lack of progress with the Helmholtz problem, $\gamma = 1$, at the end of Section 4, and the usual $\xi = 0$ centroid, makes it at least possible that something more – such as some form of generalisation of the centroid – may be needed to generalise the results of the earlier parts of Section 4 obtained for $0 \le \gamma < 1$.

2.4 $n = 2$: further identities

The items recorded in this subsection are not used in this report, for which the main focus is general n. It is, however, the intention of one of the authors to attempt to get better bounds for the torsion problem when $n = 2$.

In [KM, Kea2a] the notation is as follows. The quantities $Q(k)$ are defined

$$Q(k) = T + kuH,$$

where

$$T = -u_y^2 u_{xx} + 2u_x u_y u_{xy} - u_x^2 u_{yy} = Q(0),$$
$$H = u_{xx}u_{yy} - u_{xy}^2.$$

Define also, for $0 < \beta < 1$,

$$I(\beta) = \frac{u^2 \det(\text{hessian}(u^\beta))}{\beta^2(1-\beta)}.$$

Thus a superharmonic function $u \in \mathcal{U}_\alpha$ if $I(1/\alpha) \ge 0$. Note

$$I(\beta) = u^{2\beta-1}(T + \frac{uH}{(1-\beta)}).$$

Define

$$w = \frac{1}{2}(u_x u_{yy} - u_y u_{xy}, u_y u_{xx} - u_x u_{xy}).$$

Then

$$H = \text{div}(w) \qquad \text{and} \qquad T = 2w \cdot \nabla u.$$

An easy identity is

$$-|\nabla u|^2 H - (\Delta u)T = 4|w|^2.$$

For the torsion problem, and the generalisation to u^γ, $0 \le \gamma \le 1$, w is the gradient of $P_2/4$ where P_2 is defined in the Theorems 3.2 and 4.2. For further identities, see [KM].

3 The elastic torsion problem

3.1 Preliminaries

In their 1951 book [PS], and subsequent papers, Pólya and Szegő were concerned with bounding various 'physical' domain functionals in terms of 'geometrical' ones. The geometric functionals include (when $n = 2$) the area of Ω (or the measure of Ω for general n), $|\Omega|$, its centroid x_c, its polar moment of inertia I_c about the centroid, etc.. The physical domain functionals arise from various partial differential equation problems. For ease of exposition we consider, in this Section, just one problem, the elastic torsion problem. In the actual physical problem concerning elastic torsion, $n = 2$: see [PS].

Given a domain Ω, the problem of finding a u, twice continuously differentiable in Ω and continuous on the closure of Ω satisfying, for some given positive constant μ,

$$\begin{aligned} -\Delta u &= \mu && \text{in} \quad \Omega, \\ u &= 0 && \text{on the boundary of} \quad \Omega, \end{aligned}$$

is called the (St Venant elastic) torsion problem. There is no loss of generality in taking $\mu = 1$.

A functional of interest in some applications, e.g. [MK], is the maximum of the torsion function, u_m, and its location x_m, $u(x_m) = u_m = \max_\Omega u$. Other functionals of significance in elasticity include the following. The *torsional rigidity* is

$$S = \int_\Omega u = \int_\Omega |\nabla u|^2 = -\frac{1}{n} \int_\Omega x \cdot \nabla u.$$

Another functional studied here is $u_c = u(x_c)$, the torsion function evaluated at the centroid, x_c.

In [PS] there is some concern with nondimensional combinations of domain functionals. Quantities like $S|\Omega|^{-2}$, $SI_c|\Omega|^{-4}$ – appropriate when $n = 2$ – and similar appear in tables at the back of the book, and others are scattered throughout the text and elsewhere in the literature. Amongst various ways of unifying the bounds on domain functionals, say for some non-dimensional combination Q is to find positive lower bounds Q_{LB} and finite upper bounds Q_{UB} so that the bounds have the form

$$Q_{LB} \le Q(\Omega) \le Q_{UB} \qquad \text{for } \Omega \text{ in some class of domains.}$$

A favorite class of domains is the bounded convex domains.

For a survey of the elastic torsion problem in convex domains, for the case $n = 2$, see [KM]. For an application of the 'generalised torsion problem' with $n > 2$, see [MK].

THEOREM 3.1 *Let u be the torsion function of a convex domain Ω. Then the square root of u, $\sqrt{u}$ is concave.*

For $n = 2$ this was proved in Makar-Limanov, [ML]. The result for higher space dimensions was first proved in [Ken].

THEOREM 3.2 (Sperb [Sp]). *Let u be the torsion function of a bounded convex domain Ω. Then,*

$$\frac{\beta+2}{2} \int_\Omega u^{1/\beta} \le u_m \int_\Omega u^{(1-\beta)/\beta}, \qquad \beta > 0, \tag{3.1}$$

$$\frac{3}{2} \le \frac{|\Omega| u_m}{\int_\Omega u} = \zeta_m(1). \tag{3.2}$$

PROOF. Define the quantities P_k,

$$P_k = |\nabla u|^2 + k\mu u.$$

Henceforth $\mu = 1$. In this paragraph we repeat proofs by Payne and by Sperb [Sp] that, for convex Ω,

$$P_2 \le 2u_m. \tag{3.3}$$

P_2 satisfies an elliptic differential inequality (or equation when $n = 2$),

$$\Delta P_2 + \frac{\ell \cdot \nabla P_2}{|\nabla u|^2} = R(n) \ge 0, \qquad \ell = 2\nabla u - \frac{1}{2}\nabla P_2,$$

and $R(2) = |\nabla P_2|^2/(2|\nabla u|^2)$. The coefficients in this differential inequality become singular at points where $|\nabla u| = 0$, and only at these points. An application of the maximum principle establishes that the maximum of P_2 occurs either at a point where $|\nabla u| = 0$ or on the boundary of Ω. A calculation given on p.76 of [Sp] shows that at any point on $\partial\Omega$,

$$\frac{\partial P_2}{\partial n} = (n-1)|\nabla u|^2 M,$$

where M is the mean curvature of $\partial\Omega$. When Ω is convex, $M \ge 0$, and the Hopf form of the maximum principle shows that it is impossible for the maximum of P_2 to be attained on $\partial\Omega$. The details are given at pp.76-77 of [Sp]. Thus inequality (3.3) is established. (The result was first established, for $n = 2$, by Payne.)

An application of the divergence theorem (to $\mathrm{div}(u^{1/\beta}\nabla u)$) shows that

$$\int_\Omega u^{1/\beta} = \frac{1}{\beta}\int_\Omega u^{(1-\beta)/\beta}|\nabla u|^2.$$

Applying inequality (3.3), $P_2 \le 2u_m$, in the preceding gives inequality (3.1) which at $\beta = 1$ is

$$3\int_\Omega u = \int_\Omega P_2 \le 2u_m|\Omega|,$$

hence inequality (3.2).

Open Problem 3.1. Can inequality (3.3) be improved for torsion functions which are in $\mathcal{U}_{\alpha,0}$, $\alpha \ge 1$, in such a way as to allow useful improvements to inequality (3.1)?

3.2 The new results

THEOREM 3.3 *The torsion function u of a bounded convex domain Ω satisfies*

$$\frac{3}{2(n+1)^2} \le \frac{1}{(n+1)^2}\frac{|\Omega|u_m}{\int_\Omega u} \le \frac{|\Omega|u_c}{\int_\Omega u}\ , \tag{3.4}$$

$$c_r(n,2)^{-1} \le \frac{|\Omega|u_c}{\int_\Omega u} \le \frac{|\Omega|u_m}{\int_\Omega u} \le c_r(n,2). \tag{3.5}$$

If, in addition, $u \in \mathcal{U}_{0,\alpha}$ for $1 \le \alpha \le 2$,

$$\frac{3}{2(n+1)^\alpha} \le \frac{1}{(n+1)^\alpha}\frac{|\Omega|u_m}{\int_\Omega u} \le \frac{|\Omega|u_c}{\int_\Omega u}, \tag{3.6}$$

$$c_r(n,\alpha)^{1-\alpha} \le \frac{|\Omega|u_c}{\int_\Omega u} \le \frac{|\Omega|u_m}{\int_\Omega u} \le c_r(n,\alpha). \tag{3.7}$$

Proof. Inequalities (3.4) and (3.5) follow on using Theorem 3.1 and inequalities (3.6) and (3.7).

The right-most parts of inequalities (3.7) follow from (2.6). From inequality (2.2) with $\xi = \alpha$ and the Hadamard inequality (2.4)

$$\frac{1}{|\Omega|}\int_\Omega u \le u_m^{(\alpha-1)/\alpha}\frac{1}{|\Omega|}\int_\Omega u^{1/\alpha} \le u_m^{(\alpha-1)/\alpha}u_c^{1/\alpha}\ ,$$

from which

$$1 \le (\frac{|\Omega|u_m}{\int_\Omega u})^{(}\alpha-1)\ (\frac{|\Omega|u_c}{\int_\Omega u})\ .$$

Using the rightmost inequality of (3.7) to control the first term, the u_m-term, of the last inequality yields the leftmost inequality of (3.7).

From Theorem 3.2 and inequalities (2.5) (for $u \in \mathcal{U}_{0,\alpha}$, $1 \le \alpha \le 2$), we have

$$\frac{3}{2} \le \frac{|\Omega|u_m}{\int_\Omega u} \qquad \text{and} \qquad u_m \le (n+1)^\alpha u_c.$$

These yield inequalities (3.6). This completes the proof.

Open Problem 3.2. Find neater, improved inequalities of the form

$$f_1(n,\alpha) \le \frac{|\Omega|u_c}{\int_\Omega u},$$

where f_1 is a simple, explicitly-given function with $f_1(n,1) \ge 1$.

Open Problem 3.3. Find improvements to our inequalities

$$u_c \le u_m \le c_m(n,\alpha)u_c,$$

i.e. c_m, specifying the extent to which u_c differs from u_m. (In connection with c_m, at present, the partial differential equation is used only to get Kennington's concavity result. After that, we merely use (2.4).)

3.3 Exact solutions, possible best constants, related items

3.3.1 $n = 1$.

The case $n = 1$ is trivial: $|\Omega|u_c/\int_\Omega u = \frac{3}{2}$.

3.3.2 $n = 2$ and some closed-form torsion functions in some simple domains.

Simple exact solutions are convenient for checking against inequalities, and helping in the formulation of conjectures. The solutions given in this section are polynomials. (Numerous other exact solutions are available. There is some interest in these when the domain Ω is easy to describe, and the solutions are elementary transcendental functions. See references in [KM, PS].)

Considering quadratic and cubic polynomials for u one can solve the elastic torsion problem in an ellipse and in an equilateral triangle. One finds the following.

Ω	$\{\frac{x^2}{a^2}+\frac{y^2}{b^2}<1\}$	$\{y+\frac{a}{\sqrt{3}}>0, \sqrt{3}\lvert x\rvert<y-\frac{2a}{\sqrt{3}}\}$
$u(x,y)$	$\frac{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}{2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$	$\frac{1}{4\sqrt{3}a}(y+\frac{a}{\sqrt{3}})((y-\frac{2a}{\sqrt{3}})^2-3x^2)$
$S=\int_\Omega u$	$\frac{\pi}{4}\frac{ab}{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$	$\frac{\sqrt{3}a^4}{20}$
u_m	$\frac{1}{2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$	$\frac{a^2}{9}$
Ω_1	Ω	$\{x^2+y^2\le\frac{a^2}{3}\}$

The concavity set Ω_1 is defined in subsection 3.3.4 below. For both the ellipse and the equilateral triangle, x_m coincides with the centroid.

For the ellipse, u is concave. For the equilateral triangle, u is not concave (and, in fact, $u^{1/\alpha}$ is not concave for any $\alpha<2$). For the ellipse, Ω_e,

$$\frac{|\Omega_e|u_c}{S}=2.$$

We expect that for a family of rectangles tending to an infinite strip,

$$\frac{|\Omega_s|u_c}{S}\to\frac{3}{2}.$$

To date the extreme values we have found are
$|\Omega|u_c/S=\frac{20}{9}$ for an equilateral triangle, and
$|\Omega|u_c/S\sim\frac{4}{3}$ for a thin isosceles triangle or a thin sector.
We have no numerical evidence to contradict a conjecture that, at least amongst circular-arc triangles, these are the extreme values.

3.3.3 $n=2$: other known inequalities

The following isoperimetric inequalities do not depend on Ω being convex. The left-hand of the following is given in [Pa1]:

$$f_l(\frac{8\pi S}{|\Omega|^2})\equiv\frac{|\Omega|^2}{4\pi S}(1-(1-\frac{8\pi S}{|\Omega|^2})^{1/2})\le\frac{u_m|\Omega|}{S}\le 2(\frac{|\Omega|^2}{8\pi S})^{1/2}\le\frac{|\Omega|^2}{4\pi S}. \tag{3.8}$$

Equality holds when Ω is a disk.

Let $r=8\pi S/|\Omega|^2$. The function $f_l(r)$ increases over its range $0\le r\le 1$, from $f_l(0)=1$ to $f_l(1)=2$.

Inequality (3.8) is an example of a more elaborate inequality involving nondimensional combinations (here r as defined above) rather than simple fixed numeric constants. We have yet to explore whether the inequalities in Theorem 3.3 can be developed in this direction. We remark that with pure numeric constants in

$$c_l\le\frac{u_m|\Omega|}{S}\le c_r,$$

no useful combination with inequality (3.8) seems possible, as, for convex domains,

$$f_l(r)\le f_l(1)\le 2 \qquad \text{and} \qquad c_r\ge\frac{20}{9}, \qquad\qquad 2/\sqrt{r}\ge 2 \qquad \text{and} \qquad c_l=\frac{3}{2}.$$

3.3.4 $n = 2$: concavity sets

Define

$$\Omega_\alpha = \{x \mid D^2 u^{1/\alpha} \text{ is negative semidefinite } \}.$$

and $x_{c,\alpha}$ as the centroid of Ω_α. Makar-Limanov's result is $\Omega = \Omega_2$ when Ω is convex.

We are concerned with $1 \le \alpha \le 2$. The sets Ω_α increase as α increases: see [Ken]. Some properties of Ω_1 are established in [KM]: its components are simply connected.

If star-shaped properties were established for the Ω_α it might be possible to use the Corollary stated in Section 1.2.

3.3.5 $n = 2$: motivation and further problems

In a fluid mechanics problem, [Kea1], part of the problem involves the torsion problem (with $n = 2$) in (convex) domains with given area and centroid.

Define, with a different use of α than elsewhere in this paper,

$$\mathcal{C}_\alpha = \{\Omega \ \mid \ \Omega \text{ convex, centroid at origin, } |\Omega| = \alpha\}.$$

Nearly-circular convex domains occur in the application of [Kea1]. Let $B_\alpha \in \mathcal{C}_\alpha$ be a disk. Of possible use in the applications related to [Kea1] are inequalities involving $\mu = |\Omega \setminus B_\alpha|/\alpha$. This leads to the following.

Open Problem 3.4. With the preceding notation, find $c_m(\mu)$ so that

$$0 \le 1 - \frac{u_c}{u_m} \le c_m(\mu),$$

and also $c_m(\mu)$ tends to zero as μ tends to zero.

4 A semilinear problem

4.1 Preliminaries

Semilinear equations of kind broadly similar to those treated in this section arise in applications: see [Sp]. For one where concavity properties are investigated and which arose in one of the authors' own researches, see [KS].

Consider now positive solutions u of the semilinear problem

$$-\Delta u = u^\gamma, \quad \text{in } \Omega, \qquad\qquad u = 0 \quad \text{on } \partial\Omega. \tag{4.1}$$

We consider mostly $0 \le \gamma < 1$, but sometimes also allow $\gamma = 1$.

THEOREM 4.1 (Kennington [Ken]). *Let $0 \le \gamma < 1$. Let u be a positive solution of problem (4.1) in a convex domain Ω. Then $u^{(1-\gamma)/2}$ is concave.*

See [Ken]. A generalisation of Makar-Limanov's proof techniques to this semilinear elliptic equation is given in [Kea2b]. A survey of some results related to this is given in [Ka1].

For this paragraph, we use this concavity result, but no other properties from the p.d.e.. An application of inequality (2.4) gives

$$u_m \leq (n+1)^{\frac{2}{1-\gamma}} u_c. \tag{4.2}$$

We now record some other inequalities. Here is Holder's inequality in a form we can use:

$$(\frac{1}{|\Omega|}\int_\Omega u^{(1+\gamma)/\xi}) \leq (\frac{1}{|\Omega|}\int_\Omega u^{1+\gamma})^{1/\xi}. \tag{4.3}$$

For functions $u \in \mathcal{U}_{(1-\gamma)/2,0}$

$$\frac{|\Omega| u_m^{1+\gamma}}{\int_\Omega u^{1+\gamma}} \leq c_{r,\gamma}(n). \tag{4.4}$$

For $c_{r,\gamma}(n) = N(n,\gamma)$, where N is defined in Theorem 4.3, this is established as follows. Use (4.3) with $\xi = 2(1+\gamma)/(1-\gamma)$,

$$(\frac{1}{|\Omega|}\int_\Omega u^{(1-\gamma)/2}) \leq (\frac{1}{|\Omega|}\int_\Omega u^{1+\gamma})^{(1-\gamma)/(2(1+\gamma))}.$$

and (2.4-l) with $\alpha = 2/(1-\gamma)$,

$$\frac{1}{n+1} u_m^{(1-\gamma)/2} \leq \frac{1}{|\Omega|}\int_\Omega u^{(1-\gamma)/2}.$$

As in the case $\gamma = 0$, we can improve on the preceding inequality using Theorem 2.3, this time with $\xi = 1+\gamma$ and $\alpha = 2/(1-\gamma)$. This gives inequality (4.4) with

$$c_{r,\gamma}(n) = c_r(n, \frac{2(1+\gamma)}{1-\gamma}). \tag{4.5}$$

THEOREM 4.2 (Sperb [Sp]). *Let $0 \leq \gamma \leq 1$. Let u be a positive solution of problem (4.1) in a bounded convex domain Ω. Then*

$$(1 + \frac{\beta(\gamma+1)}{2})\int_\Omega u^{\gamma+\frac{1}{\beta}} \quad \leq \quad u_m^{1+\gamma}\int_\Omega u^{(1-\beta)/\beta}, \qquad \beta > 0, \tag{4.6}$$

$$\frac{\gamma+3}{2}\int_\Omega u^{1+\gamma} \quad \leq \quad u_m^{1+\gamma}|\Omega| \tag{4.7}$$

Proof. The proof techniques are similar to those of Theorem 3.2. Define the quantities P_k,

$$P_k = |\nabla u|^2 + \frac{k u^{1+\gamma}}{1+\gamma}. \tag{4.8}$$

Sperb shows that, for convex Ω,

$$P_2 \leq \frac{2u_m^{1+\gamma}}{1+\gamma}. \tag{4.9}$$

P_2 satisfies an elliptic differential inequality (or equation when $n = 2$): for the details see [Sp].

An application of the divergence theorem (to $\text{div}(u^{1/\beta}\nabla u)$) combined with the p.d.e. shows that

$$\int_\Omega u^{(1/\beta)+\gamma} = \frac{1}{\beta}\int_\Omega u^{(1-\beta)/\beta}|\nabla u|^2.$$

Applying inequality (4.9), in the preceding gives inequality (4.6). At $\beta = 1$ this is inequality (4.7), an inequality which is already in the literature. (See [Sp], p102, inequality (6.66).)

Some observations that inequalities (4.4) and (4.5) give no information when γ tends up to 1 are in order here. For example,

$$c_{r,\gamma}(2) = \frac{(\frac{2(1+\gamma)}{1-\gamma} + 2)(\frac{2(1+\gamma)}{1-\gamma} + 1)}{2},$$

which is unbounded when γ tends up to 1.

4.2 The new results, $0 \le \gamma < 1$

The results (3.4) and (3.5) of Theorem 3.3 are the $\gamma = 0$ case of the following.

THEOREM 4.3 *Let $0 \le \gamma < 1$. Let u be a positive solution of problem (4.1) in a bounded convex domain Ω. Define*

$$N(n, \gamma) = (n+1)^{\frac{2(1+\gamma)}{1-\gamma}}.$$

Then

$$\begin{aligned} \frac{\gamma+3}{2} &\le \frac{|\Omega| u_m^{1+\gamma}}{\int_\Omega u^{1+\gamma}} \le \frac{|\Omega| u_c^{1+\gamma}}{\int_\Omega u^{1+\gamma}} N(n, \gamma), && (4.10)\\ \frac{|\Omega| u_c^{1+\gamma}}{\int_\Omega u^{1+\gamma}} &\le \frac{|\Omega| u_m^{1+\gamma}}{\int_\Omega u^{1+\gamma}} \le c_{r,\gamma}(n). && (4.11)\end{aligned}$$

Proof. The left-most inequality of (4.10) is inequality (4.7). The starting point for the remaining inequalities of the theorem is the result of of Kennington that $u^{(1-\gamma)/2}$ is concave. The right-most inequality of (4.10) is inequality (4.2).

The left-most inequality of (4.11) is just $u_c \le u_m$. The right-most inequality of (4.11) is just inequality (4.4).

Some of the above inequalities give

$$\frac{\gamma+3}{2} \le \zeta_m(1+\gamma) \le c_{r,\gamma}(n), \tag{4.12}$$

where, as we have remarked earlier, our best bounds so far have $c_{r,\gamma}(n)$ tending to infinity as γ tends up to 1.

4.2.1 $0 \le \gamma \le 1$, $n = 1$

A short calculation gives

$$\frac{|\Omega| u_c^{1+\gamma}}{\int_\Omega u^{1+\gamma}} = \frac{\gamma+3}{2}.$$

4.3 $\gamma = 1$

The methods we have used with $\gamma < 1$ have not, so far, yielded significant results when $\gamma = 1$. Power-concavity can no longer be the sole ingredient. While Brascamp and Lieb's result – the next theorem we state – is valid all the way up to the boundary of Ω while no power-concavity is, it seems possible, that in this $\gamma = 1$ case, the better concavity away from the boundary may be important to get bounds generalising our $\gamma < 1$ ones.

The following result, due to Brascamp and Lieb, is proved in [Ken].

THEOREM 4.1.1 *Let u be a positive solution of*

$$-\Delta u = \lambda_1 u \text{ in } \Omega, \qquad\qquad u = 0 \text{ on } \partial\Omega.$$

Then $\log(u)$ *is concave.*

The function u is called the *fundamental mode.*

Consider the problem of finding some inequality, like inequality (4.12), of the form

$$2 \le \zeta_m(2) \le c_{r,1}(n), \tag{4.13}$$

with $c_{r,1}(n)$ finite. The left-hand inequality is that of Sperb, proved earlier in this report. If we succeeeded in finding a finite $c_{r,1}$ we would have answered Question 9 in Payne, [Pa2]:

Payne, [Pa2], Question 9. Is it possible to obtain, for (plane) convex Ω, an explicit bound

$$u_m \le f(|\Omega|, \int_\Omega u^2),$$

for the fundamental mode? f must be independent of λ_1 and all other geometric quantities.

Numerous other problems related to generalising Theorem 4.3 suggest themselves.

Open Problem 4.($\gamma = 1$).0. Find inequalities $c_{m,1}(n)$,

$$u_c \le u_m \le c_{m,1}(n) u_c.$$

There are various lower bounds already available for $\zeta_m(k)$, where ζ_m is defined, as before, by setting $x_p = x_m$ in equation (1.1). A joint paper of Payne and Stakgold (treated in [Sp] p104, and in [Pa2] p151), gave $\pi/2 \le \zeta_m(1)$. See [Sp] p207 gives, for the more general semilinear problem $-\Delta u = u^\gamma$,

$$2^{-2\gamma/(1+\gamma)} \frac{\Gamma(\frac{1}{1+\gamma})^2}{\Gamma(\frac{2}{1+\gamma})} \le \zeta_m(\gamma) \qquad\qquad \text{for } n = 2. \tag{4.14}$$

The point x_m might be replaced in some inequalities with x_c.

Open Problem 4.($\gamma = 1$).1. What inequalities can be found for $\zeta(k, x_c) = |\Omega| u_c^k / \int_\Omega u^k$, where u is the fundamental mode, e.g. for $k = 2$?

The starting point for developments in earlier sections was Hadamard's inequality. With apologies in advance that, so far, this has not proved useful in the $\gamma = 1$ case, here are items related to it. The Hadamard inequality result (1.2) is

$$\int_\Omega \log(\frac{u}{u_c}) \le 0,$$

or equivalently, for $k \ge 0$,

$$|\Omega| \exp(\frac{k}{|\Omega|} \int_\Omega \log(u)) \le u_c^k |\Omega|. \tag{4.15}$$

Inequality (4.15) is similar in appearance to the following consequence of Jensen's inequality

$$|\Omega| \exp(\frac{k}{|\Omega|} \int_\Omega \log(u)) \le \int_\Omega u^k. \tag{4.16}$$

(The other inequality (1.3) from our section 1.2 states:

$$0 \le |\Omega| \log(u_m) - \int_\Omega \log(u) \le - \int_\Omega (x - x_m) \cdot \frac{\nabla u}{u},$$

and we haven't yet seen a use for this.)

The logconcavity of the fundamental mode (or its k-th power for $k > 0$) is (for $k = 1$)

$$u(tx + (1-t)x_p) \ge u(x)^t u(x_p)^{1-t} \qquad t \in [0,1].$$

Define

$$\Omega_t(x_p) = \{x_p + t(y - x_p) \ \mid \ y \in \Omega\}.$$

Integrating gives

$$\frac{1}{|\Omega_t(x_p)|} \int_{\Omega_t(x_p)} u \ge u(x_p)^{1-t} \frac{1}{|\Omega|} \int_\Omega u^t,$$

which can be written

$$\zeta(t(\Omega - x_p), k, u, x_p) \le \zeta(\Omega, tk, u, x_p), \qquad t \in [0,1],\ k > 0\ .$$

The presence of sets $\Omega_t(x_p)$ in both this and in Theorem 1.3 leads to a wild hope that it might be possible to combine them somehow.

4.4 $0 \le \gamma \le 1$. Further items for $n = 2$

See [Kea2a].

5 Speculation: possible further extensions

5.1 Elliptic p.d.e.

Suppose that $\rho \in C(\bar{\Omega})$ and $g \in C(\mathbb{R})$ are given, with $\rho \ge 0$ and g nondecreasing. The differential equation might be generalised to

$$-\Delta u = g(u)\rho.$$

[Ken] has results on the power-concavity of u when, for example $g(u) = u^\gamma$ with $0 \le \gamma < 1$, and $\rho > 0$ has suitable power-concavity.

5.1.1 $-\Delta u = \rho(x)$

After being told of our $\rho = 1$, $\gamma = 0$ results, L. Ragoub
`ragoub@mat.ulaval.ca` or `F40T011%SAKSU00.BITNET@VTBIT.CC.VT.EDU`
considered this $\gamma = 0$ problem with more general ρ. (We remark that Sperb's inequality appears to be available only for $\rho = 1$.) Ragoub considered

$$S = \int_\Omega u\rho = \int_\Omega |\nabla u|^2,$$

and bounded quantities like $|\Omega|u_c/S$. (Ragoub makes greater use of the right-hand form of S, and uses inequalities like the Rodemich and Poincaré inequalities. See Gilbarg and Trudinger, 2nd ed..)

5.1.2 $-\Delta u = g(u)$

When $k > 0$ and $g(u) = H(k - u)$ and H is the Heaviside step function, the problem is called the 'dead-core problem'. (Similar problems, also with decreasing g, are studied in [KS].) When k is very large and positive this tends to the torsion problem. For the dead-core problem, $\sqrt{u}$ is concave.

5.2 Parabolic p.d.e.

The elliptic problem of Section 4 arises in connection with the large-time asymptotics of the 'porous medium equation', $\mu > 1$ in

$$u_t = -\Delta u^\mu \text{ in } \Omega, \qquad\qquad u = 0 \text{ on } \partial\Omega,$$

with positive initial data $u(.,0)$ with suitable power-concavity. Again we might investigate inequalities concerning $u(x_c, t)$.

Acknowledgement

Sever Dragomir acknowledges support from the Mathematics Department of the University of Western Australia which funded his visits during August 1995, and May 1996.

References

[BC] R. Banuelos and T. Carroll, Brownian motion and the fundamental frequency of a drum. *Duke Math. Jnl* **75** (1994), 575-602.

[Be] T.B. Benjamin, The alliance of practical and analytical insights into the nonlinear problems of fluid mechanics, in *Applications of methods of functional analysis to problems in mechanics, Symposium IUTAM/IMU, Marseilles, Sept 1975* (eds P. Germain and B. Nayroles) Lecture Notes in Maths **503**, (Springer-Verlag, Berlin : 1976).

[BF] M.S. Berger and L.E. Fraenkel, Nonlinear desingularization in certain free-boundary problems. *Comm. Math. Phys.* **77** (1980), 149-172.

[Bu] G.R. Burton, Steady symmetric vortex pairs and rearrangements. *Proc. Roy. Soc. Edinburgh* **108A** (1988), 269-290.

[Du] Y.G. Dutkevich, Deviation of the centroid of a convex figure from the centre of the circumscribed circle. *Math. Notes USSR* **19** (1976), 97-99.

[Ka1] B. Kawohl, *Rearrangements and Convexity of Level Sets in PDE*, Lecture Notes in Maths **1150**, (Springer-Verlag, Berlin : 1986).

[Kea1] G. Keady, Asymptotic estimates for symmetric vortex streets. *J. Australian Math. Soc.* **26B** (1985), 487-502.

[Kea2a] G. Keady, The concavity of solutions of $-\Delta u = u^\gamma$. *Research Report, CMA, Australian National University*, (1984).

[Kea2b] G. Keady, The power concavity of solutions of some semilinear elliptic boundary-value problems. *Bull. Australian Math. Soc.* **31** (1985), 181-184.

[KA] G. Keady and P. Abbott, Tables of stress and strain – the elastic torsion problem – in a CAS. To appear in *Proceedings of the Australian Engineering Mathematics Conference, Sydney, July 1996.*

[KM] G.Keady and A.McNabb, The elastic torsion problem: solutions in convex domains. *N.Z. Journal of Mathematics* **22** (1993), 43-64.

[KS] G. Keady and I. Stakgold, Some geometric properties of solids in combustion. Pp. 137-151 in *Proceedings of the Cortona Conference on Geometry of solutions of PDE*, G. Talenti (ed.), Academic Press, London: 1989).

[Ken] A.U. Kennington, Power concavity and boundary-value problems. *Indiana J. Math. Anal.*, **34** (1985), 687-704.

[ML] L.G. Makar-Limanov, Solution of Dirichlet's problem for the equation $\Delta u = -1$ in a convex region. *Math. Notes of Acad. Sci. of U.S.S.R.* **9** (1981), 52-53.

[MK] A.McNabb and G.Keady, Diffusion and the torsion parameter. *J. Australian Math. Soc.*, **35B** (1994), 289-301.

[Pa1] L.E. Payne, Isoperimetric inequalities and their application. *S.I.A.M. Review* **9** (1967), 453-488.

[Pa2] L.E. Payne, *Some comments on the past fifty years of isoperimetric inequalities*, in W.N. Everitt (ed.) *Inequalities: fifty years on from Hardy, Littlewood and Polya*, Lecture Notes in Pure and Applied Mathematics **129**, Dekker, New York, 1991.

[PPT] J.E. Pecarić, F.E. Proschan, Y.L. Tong, *Convex Functions, Partial Orderings and Statistical Applications* (Academic Press: 1992).

[PS] G. Polya and G. Szego, *Isoperimetric Inequalities of Mathematical Physics* (Princeton U.P.: 1951).

[Sc] P.R. Scott, Centre of gravity and circumcentre of a convex body in the plane. *Quart. J. Math. Oxford* **40** (1989), 111-117.

[Sp] R. Sperb, *Maximum Principles and Applications* (Academic, London: 1981).

[T] B. Turkington, On steady vortex flow in two dimensions. *Comm. Partial Diff. Eq.* **8** (1983), 999-1071.

A Appendix. Vortex pairs: a possible area of application

In this application $n = 2$ and the domain on which the torsion problem is solved is denoted A, while Ω has another use, e.g. $\Omega = (-\infty, \infty) \times (0, b)$ for $b > 0$. The vortex pair problem is to solve, given $\lambda > 0$, for (ψ, V, k) satisfying

$$\begin{aligned} -\Delta\psi &= \lambda \mathrm{Heaviside}(\psi - Vy - k) && \text{in } \Omega, \\ \psi &= 0 && \text{on } \partial\Omega, \\ \psi &\to 0 && \text{as } x \to \pm\infty, \\ \psi_x &= 0 && \text{on } x = 0. \end{aligned}$$

There are faimilies of solutions. We restrict our attention to those which are symmetric about $x = 0$ and which have $\psi_x < 0$ for $x > 0$. There are various methods of parametrising the problem. Define the vortex *core*

$$A = \{(x, y) \mid \psi(x, y) > Vy + k\}$$

and denote the centroid of A by $(0, y_c)$. Benjamin [Be] popularised a rearrangement variational formulation in which the area and centroids of the sets allowed were fixed. This variational formulation was later used in [T, Kea1, Bu]. Denote the area of A by α.

The goal in [Kea1] was to build on the approach of [BF] to show that, with the centroids fixed throughout all solutions, considering variational solutions with connected cores,

$$\alpha = \mathrm{area}(A_\alpha) \to 0 \qquad \Longrightarrow \qquad \mathrm{diameter}(A_\alpha) \to 0 \ .$$

A key step in getting to this is establishing

$$\alpha = \mathrm{area}(A_\alpha) \to 0 \qquad \Longrightarrow \qquad \mathrm{cap}(A_\alpha, \Omega) \to 0 \ ,$$

where $\mathrm{cap}(A_\alpha, \Omega)$ denotes the electrostatic capacity of A_α relative to Ω. For details see [Kea1]. A goal, as yet not achieved by this approach is to show that as α becomes ever smaller the cores A_α are asymptotically circular. To make this more precise, define B_α to be the disk, centroid $(0, y_c)$ the same as A_α, and with area α also the same as A_α. Then the goal can be expressed as

$$\alpha = \to 0 \qquad \Longrightarrow \qquad \frac{|A_\alpha \setminus B_\alpha|}{\alpha} \to 0 \ ,$$

A preliminary goal is to establish

$$\alpha = \to 0 \qquad \Longrightarrow \qquad \frac{|\mathrm{cap}(A_\alpha, \Omega) - \mathrm{cap}(A_\alpha, \Omega)|}{\alpha} \to 0 \ .$$

One can image the program continuing: having established the asymptotic circularity, eastablish that A_α is asymptotically elliptical with the small eccentricity having the right rate of approach to zero with α. In establishing these, as begun in [Kea1], one requires good estimates for k_α, V_α, etc..

Here is a simple illustration of how inequalities of the kind in the main part of this paper might find use. In some of the account we simplify to the case $b = \infty$, the case of *vortex pairs*. The notation is that of [Kea1] where Ψ denotes the elastic torsion function for A, and subscript c denotes evaluation at the centroid.

$$\begin{aligned}
\psi_c &= k + Vy_c + \Psi_c, \\
&= \lambda \int_A G(x, y; 0, y_c)\, dxdy, \\
&\geq \lambda \int_{B((0,y_c),\rho_c)} G(x, y; 0, y_c)\, dxdy \qquad \text{where } \rho_c = \text{distance}((0, y_c), \partial A), \\
&= \frac{\lambda\rho_c^2}{4}(1 + \log(\frac{4y_c^2}{\rho_c^2})\ ) \qquad \text{when } b = \infty.
\end{aligned}$$

ρ_c is the radius of the largest disk, centre $(0, y_c)$, contained in A. Also define R_c to be radius of the smallest disk, centre $(0, y_c)$, containing A. Having shown that diameter$(A_\alpha) \to 0$ as $\alpha \to 0$, we have that, for sufficiently small α, $B((0, y_c), R_c) \subset \Omega$. An argument similar to that above establishing the left-hand-side of the inequality below establishes the right-hand-side:

$$\frac{\lambda\rho_c^2}{4}(1 + \log(\frac{4y_c^2}{\rho_c^2})\ ) \leq k + Vy_c + \Psi_c \leq \frac{\lambda R_c^2}{4}(1 + \log(\frac{4y_c^2}{R_c^2})\ ) \qquad \text{when } b = \infty.$$

The identity

$$Vy_c = \frac{1}{\alpha}\int_A \Psi \qquad \text{when } b = \infty,$$

and the inequality from the main part of this paper

$$\frac{1}{6\alpha}\int_A \Psi \leq \Psi_c \leq \frac{4}{\alpha}\int_A \Psi,$$

show that the last two terms of $k + Vy_c + \Psi_c$ are, asymptotically for α small, of the same order. Both are of smaller order that k. However, there is a major weakness in this program. Though there is numerical evidence that A is convex (for all α), there is not yet any simple proof (except via [T] and avoiding the whole strategy outlined above) that A is convex even for α small. For this reason we have, as yet, not explored many of the combinations of the numerous inequalities, e.g. those in [KM] as well as in this paper, concerning convex A.

B Appendix. An improved upper bound for the $n = 2$ torsion problem

Using a pair of well-known inequalities for the $n = 2$ torsion problem, the inequalities (3.5) can be improved replacing $c(2, 2) = 6$ by 4. Let ρ denote the inradius of Ω. The first of these inequalities, proved in [PS], is

$$\frac{1}{8}\rho^2|\Omega| \leq S,$$

an inequality which becomes an equality when Ω is a disk. The second, proved in [Sp] is

$$u_m \le \frac{1}{2}\rho^2, \qquad \text{for convex } \Omega,$$

an inequality which becomes an equality when Ω is a strip. Combining these last two inequalities gives

$$\frac{|\Omega| u_m}{S} \le 4,$$

as previously asserted. (See [BC] for an inequality which can be used to establish an upper bound for $|\Omega|u_m/S$ for any simply connected $\Omega \subset \mathbb{R}^2$.)

C Appendix. Centroid and circumcentre

THEOREM C.1 ([Du, Sc]) *The centroid x_c of a plane convex domain Ω having a circumscribed disk of radius R_{circum} is at a distance at most $2zR_{circum}/3$ from the centre x_{circum} of the circumscribed disk, where z is the positive root of the equation*

$$2z\arccos(z) + z^2 - 2\sqrt{1-z^2} = 0.$$

The figure for which $|x_c - x_{circum}| = 2zR_{circum}/3$, to within rigid-body movements, is unique and is a truncated half-disk Ω_D. (Ω_D is constructed from the half-disk, radius R_{circum}, centre O in $x > 0$, by truncating it removing two segments, one with an end points at $(0, R_{circum})$ and the other with an end-point at $(0, -R_{circum})$. The Ω_D is symmetric about the x-axis.)