On triangle-free graphs of order 10 with prescribed 1-defective chromatic number

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Abstract

A graph is (m, k)-colourable if its vertices can be coloured with m colours such that the maximum degree of any subgraph induced on vertices receiving the same colour is at most k. The k-defective chromatic number for a graph is the least positive integer m for which the graph is (m, k)-colourable. All triangle-free graphs on 8 or fewer vertices are $(2, 1)$-colourable. There are exactly four triangle-free graphs of order 9 which have 1-defective chromatic number 3. We show that these four graphs appear as a subgraph in almost all triangle-free graphs of order 10 with 1-defective chromatic number equal to 3. In fact there is a unique triangle-free $(3, 1)$-critical graph on 10 vertices and we exhibit this graph.
Key Words: k-defective chromatic number; k-independence; triangle-free graph; $(3, 1)-$critical graph.

1 Introduction

We consider in this paper undirected graphs with no loops or multiple edges. For all undefined concepts and terminology we refer to [4].

Given a graph G, $d_G(u)$, $N_G(u)$ and $N_G[u]$ denote respectively the degree, the neighbourhood, and the closed neighbourhood of a vertex u in G. The union of graphs G_1 and G_2 is denoted by $G_1 \cup G_2$. For convenience we write $2G$ in place of $G \cup G$.

Let k be a nonnegative integer. A subset U of the vertex set $V(G)$ is *k-independent* if $\Delta(G[U]) \leq k$. A 0-independent set is an independent set in the usual sense. A graph G is *(m, k)-colourable* if it is possible to assign m colours, say $1, 2, \dots, m$ to the vertices of G, one colour to each vertex, such that the set of all vertices receiving the same colour is k-independent. The smallest integer m for which G is (m, k)-colourable is called the *k-defective chromatic number* of G and is denoted by $\chi_k(G)$. A graph G is said to be

(m,k)-*critical* if $\chi_k(G) = m$ and $\chi_k(G-u) < m$ for every u in $V(G)$. A graph G is said to be (m,k)-*edge-critical* if $\chi_k(G) = m$ and $\chi_k(G-e) < m$ for every e in $E(G)$.

It is easy to see that the following statements are equivalent.

(i) G is (m,k)-colourable.

(ii) There exists a partition of $V(G)$ into m sets each of which is k-independent.

(iii) $\chi_k(G) \leq m$.

Note that $\chi_0(G)$ is the usual chromatic number. It is easy to see that $\chi_k(G) \leq \lceil \frac{|V(G)|}{k+1} \rceil$. The concept of k-defective chromatic number has been extensively studied in the literature (see [2, 6, 7, 8, 11, 13, 14]). Given a positive integer m, it is well known that there exists a triangle-free graph with G with $\chi_k(G) = m$. A natural question that arises is: what is the smallest order of a triangle-free graph G with $\chi_k(G) = m$? We denote this smallest order by $f(m,k)$. The parameter $f(m,0)$ has been studied by several authors (see [3, 5, 9, 10]) and $f(m,0)$ is determined for $m \leq 5$. It has also been shown that $f(3,1) = 9$ and $f(3,2) = 13$. Furthermore the corresponding extremal graphs have been characterized (see [13, 2]).

In this paper we characterize triangle-free graphs of order 10 with $\chi_1(G) = 3$. In a subsequent paper [1] we build from the results of this paper to determine the smallest order of a triangle-free planar graph which has 1-defective chromatic number 3.

In all the figures in this paper a double line between sets X and Y means that every vertex of X is adjacent to every vertex of Y.

2 Preliminary results

We need the following results, proofs of the theorems being in the papers cited.

Theorem 1 *([11, 12]) Let G be a graph with maximum degree Δ. Then*

$$\chi_k(G) \leq \lceil \frac{\Delta+1}{k+1} \rceil = 1 + \lfloor \frac{\Delta}{k+1} \rfloor.$$

Theorem 2 *([13]) The smallest order of a triangle-free graph with $\chi_1(G) = 3$ is 9, that is, $f(3,1) = 9$. Moreover, G is a triangle-free graph of order 9 with $\chi_1(G) = 3$ if and only if it is isomorphic to one of the graphs G_i, $1 \leq i \leq 4$ given in Figure 1.*

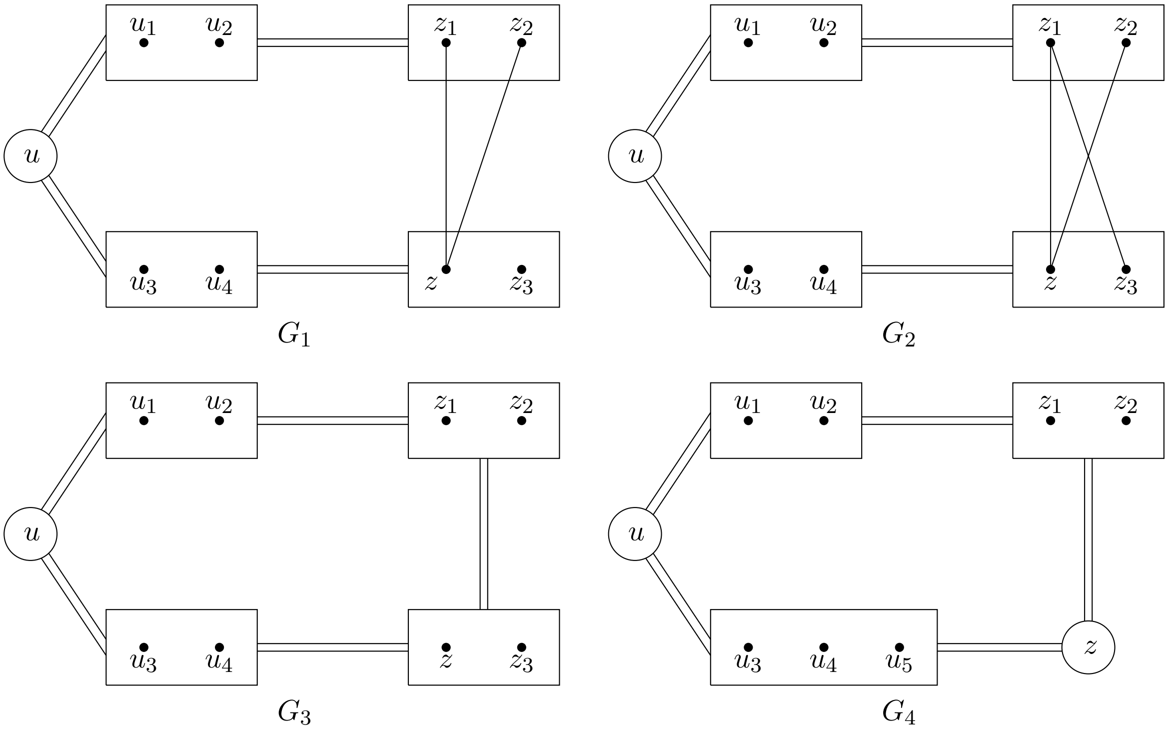


Figure 1: The critical graphs of order 9 with $\chi_1(G) = 3$: G_1 to G_4 of [13].

3 Main results

Consider a graph G of order n. The following notation is used repeatedly in the paper:

$$u \text{ is a vertex degree } \Delta(G), \qquad A = N_G(u), \qquad B = V(G) - N_G[u], \tag{1}$$

$$H = G[B] \quad \text{and} \quad z \in B \text{ with } d_H(z) = \Delta(H). \tag{2}$$

We henceforth denote the vertex set $V(G)$ by V and the edge set $E(G)$ by E.

Lemma 1 *Let G be a triangle-free graph. In the notation described above, suppose that $\Delta(H) = |B| - 1$ and $|A \cap N_G(z)| \leq 2k$, where k is a nonnegative integer. Then $\chi_k(G) \leq 2$.*

Proof. Consider a partition of $A \cap N_G(z)$ into two sets A_{11} and A_{12} such that $|A_{1i}| \leq k$ for $i = 1$ and 2. Since G is triangle-free, the sets $N_H(z) \cup \{u\} \cup A_{11}$ and $(A - A_{11}) \cup \{z\}$ are both k-independent. Hence $\chi_k(G) \leq 2$. □

Lemma 2 *Let G be a triangle-free graph of order 10 with $\chi_1(G) \geq 3$. Then (i) $\Delta(H) \geq 2$ and (ii) $4 \leq \Delta(G) \leq 6$.*

Proof. The lower bound for $\Delta(G)$ follows from Theorem 1. Let $u \in V$ with $d_G(u) = \Delta(G)$. If $\Delta(H) \leq 1$, then $\{u\} \cup B$ is 1-independent. Since A is also 1-independent, this implies $\chi_1(G) \leq 2$. Thus $\Delta(H) \geq 2$ and hence $|B| \geq 3$ implying that $\Delta(G) = |A| \leq 6$. □

Lemma 3 *Let G be a triangle-free graph of order 10 with $\Delta(G) = 6$. If $\chi_1(G) = 3$ then there exists a vertex u^* in G such that $G - u^* \cong G_4$.*

Proof. Assume that $\chi_1(G) = 3$. Using the notation described before we have $|B| = 3$. From (i) of Lemma 2 we have $\Delta(H) \geq 2$. Thus $\Delta(H) = 2$.

Let $z \in B$ with $d_H(z) = 2$. Using Lemma 1, we conclude that $|A \cap N_G(z)| \geq 3$. Also, as $d_G(z) \leq 6$, $|A \cap N_G(z)| \leq 4$.

Let $A_1 = A \cap N_G(z)$, $A_2 = A - A_1$ and $N_H(z) = \{z_1, z_2\}$. Since G is K_3-free, the set $A_1 \cup \{z_1, z_2\}$ is 0-independent. If z_1 is adjacent to at most one vertex of A_2, then

$$A \cup \{z_1\} \text{is 1-independent. So is } V - (A \cup \{z_1\}) = \{u, z, z_2\}.$$

It follows that $\chi_1(G) \leq 2$, a contradiction. Hence z_1 (similarly z_2) has at least two neighbours in A_2. Since $|A_2| \leq 3$, z_1 and z_2 have at least one common neighbour in A_2.

Suppose that there is exactly one common neighbour, say x, of z_1 and z_2 in the set A_2. This implies that $|A_2| = 3$ and

$$X = (A - \{x\}) \cup \{z_1, z_2\} \text{ is 1-independent. Since } V - X = \{u, x, z\} \text{ is also 1-independent}$$

we have $\chi_1(G) \leq 2$, a contradiction. Thus A_2 has at least two common neighbours, say x and y, of z_1 and z_2.

Now select a vertex u^* from A as follows. If $|A_1| = 4$ then u^* is any vertex of A_1. Otherwise, that is, if $|A_1| = 3$ then u^* is a vertex in A_2 (note tht $|A_2| = 3$) different from x and y. Now it is easy to verify that $G - u^* \cong G_4$. Hence the result. □

Lemma 4 *Let G be a triangle-free graph of order 10 with $\Delta(G) = 5$. If $\chi_1(G) = 3$ then either there exists a vertex u^* with $G - u^* \cong G_i$ for $1 \leq i \leq 4$ or $G \cong G_5$ illustrated in Figure 2.*

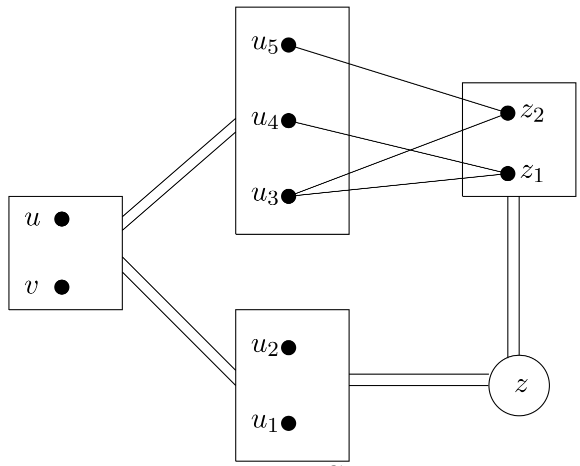


Figure 2: G_5

Proof. Suppose that $\chi_1(G) = 3$. Using the notation described before, it follows that $|B| = 4$. Now using Lemma 1 and Lemma 2(i), we have $\Delta(H) = 2$. Let $v \in B$ such that $(z, v) \notin E$, $N_H(z) = \{z_1, z_2\}$ and $A_1 = A \cap N_G(z)$. Note that $|A_1| \leq 3$.
Case i. $|A_1| = 3$.
Let $A - A_1 = \{x_1, x_2\}$. Suppose that $(z_1, x_1) \notin E$.
Claim 4.1. $(v, z_2) \in E$.
Since $\chi_1(G) = 3$ and

$$A \cup \{z_1\} \text{ is 1-independent}, V - (A \cup \{z_1\}) = \{u, v, z, z_2\} \text{ is not 1-independent.}$$

This proves Claim 4.1.
Claim 4.2. $(v, x_2) \in E$.
Since $\chi_1(G) = 3$ and $(A - \{x_2\}) \cup \{z_1, z_2\}$ is 1-independent, it follows that $\{u, z, v, x_2\}$ is not 1-independent. This in turn implies that $(v, x_2) \in E$.

Combining Claims 4.1 and 4.2 with the assumption that G is triangle-free, we have $(z_2, x_2) \notin E$. Now, note that the sets

$$X_1 = A \cup \{z_1, z_2\} \text{ and } V - X_1 = \{u, z, v\} \text{ are both 1-independent}$$

implying that $\chi_1(G) \leq 2$, a contradiction. Thus $(z_1, x_1) \in E$. Using similar arguments we conclude that $(z_1, x_2) \in E$ and $(z_2, x_i) \in E$ for $i = 1, 2$. Now, clearly, $G - v \cong G_4$. This completes Case i.

Case ii. $|A_1| \leq 2$.
Since $\Delta(H) = 2$ and $|B| = 4$, clearly H is either $P_3 \cup K_1$ or P_4 or C_4.

Let us first consider the case that $H \cong P_3 \cup K_1$ or P_4.
If $|A_1| \leq 1$ then the sets $X = A \cup \{z\}$ and $V - X$ partition the vertex set V of G into two 1-independent sets implying that $\chi_1(G) \leq 2$, a contradiction.
Hence $|A_1| = 2$. Let $A_1 = \{u_1, u_2\}$. If $(v, u_1) \notin E$ then the sets $X_1 = \{u, u_1\} \cup (B - \{z\})$ and

$V - X_1$ partition V into 1-independent sets. This implies that $\chi_1(G) \leq 2$, a contradiction. Thus $(v, u_1) \in E$. Similarly $(v, u_2) \in E$.

Now let us assume that $H \cong P_4$ and $(v, z_2) \in E(H)$. The arguments used to conclude that v and z are both adjacent to u_1 and u_2 can now be repeated with reference to the vertices z_1 and z_2 since $d_H(z_2) = 2$. Thus we conclude, without loss of generality, that z_1 and z_2 are both adjacent to say u_3 and u_4 from $A - \{u_1, u_2\}$. Let $\{u_5\} = A - \{u_1, u_2, u_3, u_4\}$. Note that $G - u_5 \cong G_2$.

Now let $H \cong P_3 \cup K_1$. If z_1 has at most one neighbour in $A - \{u_1, u_2\}$ then $\chi_1(G) \leq 2$ since

$$X = A \cup \{z_1\} \text{ and } V - X \text{ are both 1-independent.}$$

Thus z_1 and similarly z_2 have at least two neighbours in $A - \{u_1, u_2\}$. Now let $\{u_3, u_4, u_5\} = A - A_1$. Suppose that z_1 and z_2 have two common neighbours in $\{u_3, u_4, u_5\}$, say u_3 and u_4. Then clearly $G - u_5 \cong G_1$.

Now assume that z_1 and z_2 have exactly one common neighbour. Specifically, assume that z_1 is adjacent to u_3 and u_4; z_2 is adjacent to u_3 and u_5. Now

$$X_1 = (A - \{u_3\}) \cup \{z_1, z_2\} \text{ is 1-independent so that } V - X_1 \text{ is not}$$

as $\chi_1(G) = 3$. This implies that $(v, u_3) \in E$. Similarly, by considering the sets

$$X_2 = \{u_1, u_2, u_3, u_4, z_2\} \text{ and } X_3 = \{u_1, u_2, u_3, u_5, z_1\}$$

we conclude that (v, u_5) and (v, u_4) are in E. The graph $G \cong G_5$ given in Figure 2.

From now onwards we will assume that $H \cong C_4$. Thus every vertex of H has degree $\Delta(H) = 2$ in H. Moreover we assume that z has the largest number of neighbours in A. Recall that $(v, z) \notin E(H)$. Since $|A_1| \leq 2$, we have $|N_G(z) \cap N_G(v) \cap A| \leq 2$.

Firstly if $|N_G(z) \cap N_G(v) \cap A| = 1$ then the sets

$$X_1 = (A - (N_G(z) \cap N_G(v))) \cup \{z, v\} \text{ and } V - X_1$$

provide a (2,1)-colouring of G, a contradiction to the assumption that $\chi_1(G) = 3$.

Next let $|N_G(z) \cap N_G(v) \cap A| = 0$. If $|A_1| \leq 1$ then by the choice z, $|N_G(v) \cap A| \leq 1$. But then the sets $Y_1 = A \cup \{v, z\}$ and $V - Y_1 = \{u, z_1, z_2\}$ provide a (2,1)-colouring of G, a contradiction. Hence $|A_1| = 2$ and let $A_1 = \{u_1, u_2\}$. If v has atmost one neighbour in A then the sets

$$X_2 = \{v, z, u_2, u_3, u_4, u_5\} \text{ and } V - X_2 = \{u, u_1, z_1, z_2\}$$

form a (2,1)-colouring of G, a contradiction. If v has two neighbours in A, say u_3 and u_4, then the sets

$$X_3 = \{z_1, z_2, u_1, u_2, u_3, u_4\} \text{ and } V - X_3 = \{u, u_5, z, v\}$$

provide a (2,1)-colouring of G, a contradiction.

Hence $|N_G(z) \cap N_G(v) \cap A| = 2$. Without any loss of generality we assume that $N_G(z) \cap N_G(v) \cap A = \{u_1, u_2\}$. Similarly we can easily show that $|N_G(z_1) \cap N_G(z_2) \cap A| = 2$. Without any loss of generality, let $N_G(z_1) \cap N_G(z_2) \cap A = \{u_3, u_4\}$. Now let $\{u_5\} = A - \{u_1, u_2, u_3, u_4\}$. It is easy to see that $G - u_5 \cong G_3$.

This completes the proof of the lemma. □

Lemma 5 *Let G be a triangle-free graph of order 10 with $\Delta(G) = 4$ and $3 \le \Delta(H) \le 4$. If $\chi_1(G) = 3$ then there exists a vertex u^* in G such that $G - u^* \cong G_1$ or G_2.*

Proof. We will assume $\chi_1(G) = 3$. Let $A = \{u_1, u_2, u_3, u_4\}$. If $\Delta(H) = 4$ then G is a subgraph of $K_{5,5}$ and $\chi_1(G) \le \chi_0(G) = 2$, a contradiction. Hence we assume $\Delta(H) = 3$.

Let $N_H(z) = \{z_1, z_2, z_3\}$ and $v \in B$ such that $(z, v) \notin E(H)$. We provide a proof of this lemma by making and proving, a sequence of claims.

Claim 5.1. $N_H(v) \ge 2$

Suppose that $N_H(v) \le 1$; then we can partition V into two 1-independent sets, $X = A \cup \{z\}$ and $V - X$. Hence $\chi_1(G) \le 2$, a contradiction. This establishes Claim 5.1.

Without any loss of generality, assume that (v, z_1) and (v, z_2) are in $E(H)$. Note that $|N_G(z) \cap A| \le 1$ and $|N_G(v) \cap A| \le 2$.

Claim 5.2. If $|N_G(z) \cap A| = 1$ then $G - u_1 \cong G_2$.

Suppose that $|N_G(z) \cap A| = 1$ and let $(z, u_1) \in E$. If, in addition, $(v, u_1) \in E$ then the sets

$$X = \{u_2, u_3, u_4, z, v\} \quad \text{and} \quad V - X$$

partition V into 1-independent sets implying $\chi_1(G) \le 2$, a contradiction. Hence $(v, u_1) \notin E$. If $|N_G(v) \cap A| \le 1$ then again $\chi_1(G) \le 2$ since

$$X_1 = A \cup \{v, z\} \text{ and } V - X_1 \text{ are both 1-indpendent,}$$

Hence $|N_G(v) \cap A| = 2$. Let us assume that $N_G(v) \cap A = \{u_2, u_3\}$. The set

$$X_2 = \{u_1, u_3, u_4, z, v\} \text{ is 1 independent, so } \quad V - X_2 \text{ is not 1-independent}$$

as $\chi_1(G) = 3$. This implies that $(u_2, z_3) \in E$. Similarly we conclude that $(u_3, z_3) \in E$.

Since the sets

$$Y_1 = \{u, z, v, z_3\} \text{ and } Y_2 = \{u_1, u_2, u_3, z_1, z_2\} \text{ are 1-indpendent,}$$

$$V - Y_1 = A \cup \{z_1, z_2\} \text{ and } V - Y_2 = \{z, z_3, u, u_4, v\} \text{ are not 1-independent}$$

as $\chi_1(G) = 3$. Hence (u_4, z_1), (u_4, z_2) and (u_4, z_3) are all in E. Now $G - u_1$ is isomorphic to G_2 given in Figure 3.

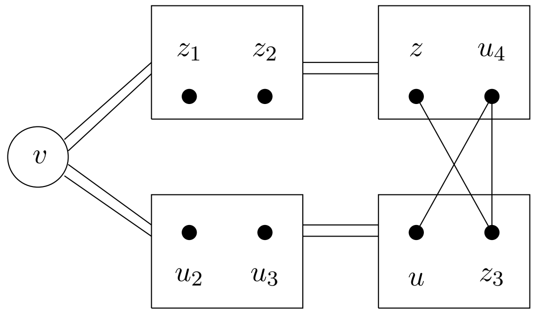


Figure 3: $G - u_1 \cong G_2$

This establishes Claim 5.2. Henceforth we will assume that $|N_G(z) \cap A| = 0$.
Claim 5.3. $|N_G(v) \cap A| = 2$ and $(v, z_3) \notin E(H)$.
Otherwise, that is, if $|N_G(v) \cap A| \leq 1$, then $X = A \cup \{z, v\}$ and $V - X$ provide a partition of V into 1-independent sets, implying $\chi_1(G) \leq 2$. Hence $|N_G(v) \cap A| = 2$. Since $d_G(v) \leq 4$ we now have $(v, z_3) \notin E$. This establishes Claim 5.3.

Without any loss of generality, we now assume that $N_G(v) \cap A = \{u_1, u_2\}$. Clearly there are no edges between $\{z_1, z_2\}$ and $\{u_1, u_2\}$.
Claim 5.4. For $i = 1$ and 2, $(u_i, z_3) \in E$.
Now note that

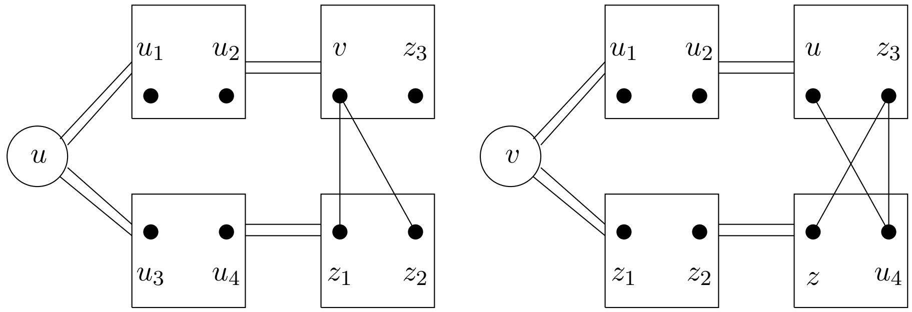
$$X_1 = \{u_2, u_3, u_4, z, v\} \text{ is 1-independent, while } V - X_1 = \{u, u_1, z_1, z_2, z_3\} \text{ is not}$$

as $\chi_1(G) = 3$. This implies $(u_1, z_3) \in E$. Similarly $(u_2, z_3) \in E$. This establishes Claim 5.4.

Since z_3 is adjacent to u_1, u_2 and z and $d_G(z_3) \leq 4$ we can assume, without any loss of generality, that $(z_3, u_3) \notin E$. The set

$$X_1 = \{u, u_3, v, z, z_3\} \text{ is 1-independent, while } V - X_1 = \{u_1, u_2, u_4, z_1, z_2\} \text{ cannot be}$$

as $\chi_1(G) = 3$. This implies that (u_4, z_1) and (u_4, z_2) are both in E. Now if $(z_3, u_4) \notin E$, we can similarly conclude that $(u_3, z_i) \in E$ for $i = 1$ and 2. In this case we can easily verify that $G - z \cong G_1$ (see Figure 4(a)). On the other hand, that is if $(z_3, u_4) \in E$, we can check that $G - u_3 \cong G_2$ (see Figure 4(b)).



(a) $G - z \cong G_1$ (b) $G - u_3 \cong G_2$

Figure 4: *Graph* $G - u^*$

This proves the lemma. □

Suppose that G is a triangle-free graph of order 10 with $\Delta(G) = 4$ and $\chi_1(G) = 3$. As a consequence of Lemmas 2(i) and 5 we can assume that $\Delta(H) = 2$. It is easy to see that H is isomorphic to one of the graphs (i) $P_3 \cup 2K_1$ (ii) $P_3 \cup K_2$ (iii) $P_4 \cup K_1$ (iv) P_5 (v) C_5 and (vi) $C_4 \cup K_1$.

Lemma 6 *Let G be a triangle-free graph of order 10 with $\Delta(G) = 4$ and $\Delta(H) = 2$. Furthermore, let H be isomorphic to $P_3 \cup 2K_1$ or $P_3 \cup K_2$. If $\chi_1(G) = 3$ then there exists a vertex u^* in G such that $G - u^* \cong G_1$ or G_2 or G_3.*

Proof. Assume that $\chi_1(G) = 3$. Let $z \in B$ with $d_H(z) = 2$ and $N_H(z) = \{z_1, z_2\}$. For x in $\{z, z_1, z_2\}$ we have $|N_G(x) \cap A| \geq 2$. Otherwise $X_1 = A \cup \{x\}$ and $V - X_1$ provide a $(2, 1)$-colouring of G, a contradiction. Since $d_H(z) = 2$, we have $|N_G(z) \cap A| = 2$. Since G is K_3-free, this implies $|N_G(z_i) \cap A| = 2$ for $i = 1$ and 2. Without any loss of generality we can write $N_G(z) \cap A = \{u_1, u_2\}$ and $N_G(z_i) \cap A = \{u_3, u_4\}$ for $i = 1$ and 2.

Let $\{z_3, z_4\} = V(H) - \{z, z_1, z_2\}$. If (z_3, u_1) and (z_3, u_2) are in E then $G - z_4 \cong G_1$ or G_2 or G_3 according as the number of edges between $\{z_3\}$ and $\{u_3, u_4\}$ is 0 or 1 or 2. Hence we will assume, without loss of generality, that $(z_3, u_2) \notin E$. Suppose $(z_4, u_2) \notin E$ then

$$X_1 = \{u, u_2, z_1, z_2, z_3, z_4\} \text{ and } V - X_1$$

form a (2,1)-colouring of G, a contradiction. Hence $(z_4, u_2) \in E$. If $(z_4, u_1) \in E$ then $G - z_3 \cong G_1$ or G_2 or G_3. Hence we assume that $(z_4, u_1) \notin E$. Now since $d_G(u_3) \leq 4$, we can assume that $(u_3, z_3) \notin E$, from which it follows that the sets

$$X_1 = \{u_2, u_3, u_4, z, z_3\} \text{ and } V - X_1$$

form a (2,1)-colouring of G, a contradiction.

This proves the lemma. □

Lemma 7 *Let G be a triangle-free graph of order 10 with $\Delta(G) = 4$ and $\Delta(H) = 2$. Furthermore, let H be isomorphic to $P_4 \cup K_1$. If $\chi_1(G) = 3$ then there exists a vertex u^* in G such that $G - u^* \cong G_i$, for some i, $1 \le i \le 3$.*

Proof. Let us suppose that $\chi_1(G) = 3$. Let z and z_1 be vertices in B with $d_H(z) = d_H(z_1) = 2$. Note that $(z, z_1) \in E(H)$. Let z_2 (z_3) be the other neighbour of z (z_1) . Finally, let $\{z_4\} = V(H) - \{z, z_1, z_2, z_3\}$.
Claim 7.1. For $x = z$ and z_1, $|N_G(x) \cap A| = 2$.
This claim can be proved using arguments similar to the ones used in Lemma 6.

Now,without any loss of generality, let $N_G(z) \cap A = \{u_1, u_2\}$ and $N_G(z_1) \cap A = \{u_3, u_4\}$. Since $\chi_1(G) = 3$ and $V - A - \{z_2, z_3\}$ is 1-independent it follows that $A \cup \{z_2, z_3\}$ is not 1-independent. Note that z_2 and z_3 do not have a common neighbour in A. Thus we conclude that either $(z_2, u_i) \in E$ for $i = 3$ and 4 or $(z_3, u_i) \in E$ for $i = 1$ and 2. Suppose, without loss of generality, $(z_2, u_i) \in E$ for $i = 3$ and 4.

If z_3 is adjacent to both u_1 and u_2, then it is easy to verify that $G - z_4 \cong G_2$.

Hence $(z_3, u_i) \notin E$ for $i = 1$ or 2. Without any loss of generality assume that $(z_3, u_1) \notin E$. Now

$$X_1 = \{u_2, u_3, u_4, z\} \text{ and } \quad X_2 = \{u_1, u_3, u_4, z, z_3\} \text{ are 1-independent.}$$

Since $\chi_1(G) = 3$, the sets

$$V - X_1 = \{u, u_1, z_1, z_2, z_3, z_4\} \text{ and } V - X_2 = \{u, u_2, z_1, z_2, z_4\} \text{ are not 1-independent.}$$

This in turn implies that $(u_i, z_4) \in E$ for $i = 1$ and 2. Now it is easy to verify that $G - z_3 \cong G_1$ or G_2 or G_3.

Hence it follows that there exists a u^* such that $G - u^* \cong G_1$ or G_2 or G_3.

This proves the lemma. □

Lemma 8 *Let G be a triangle-free graph of order 10 with $\Delta(G) = 4$ and $\Delta(H) = 2$. Furthermore, let H be isomorphic to P_5. If $\chi_1(G) = 3$ then there exists a vertex u^* such that $G - u^* \cong G_i$, for some i, $1 \le i \le 3$.*

Proof. We assume that $\chi_1(G) = 3$. Let z be the central vertex of H. Since $\Delta(G) = 4$, $|N_G(z) \cap A| \le 2$. If $|N_G(z) \cap A| \le 1$ then $X = A \cup \{z\}$ and $V - X$ form a partition of V into 1-independent sets implying $\chi_1(G) \le 2$. Thus $|N_G(z) \cap A| = 2$ and let $N_G(z) \cap A = \{u_1, u_2\}$. Also let $N_H(z) = \{z_1, z_2\}$. Furthermore, let z_3 and z_4 be the neighbours of z_1 and z_2 respectively.

Since $\chi_1(G) = 3$ and

$$X = \{u, z, z_3, z_4\} \text{ is 0-independent, the set } \quad V - X = A \cup \{z_1, z_2\} \text{ is not 1-independent.}$$

Since $\{u_1, u_2, z_1, z_2\}$ is totally disconnected, it follows that $\Delta(L) = 2$ where $L = G[\{u_3, u_4, z_1, z_2\}]$. Suppose that $d_L(u_3) = \Delta(L) = 2$. This means that $(u_3, z_i) \in E$ for $i = 1$ and 2. Since G is triangle-free $(u_3, z_i) \notin E$ for $i = 3$ and 4.

Now note that $d_G(z) = \Delta(G) = 4$. Let

$$F = G[V - N_G[z]] = G[\{u, u_3, u_4, z_3, z_4\}].$$

Clearly either
(i) $\Delta(F) = d_F(u_4) = 3$ or
(ii) $\Delta(F) = 2$ and $F \cong P_4 \cup K_1$ or $P_3 \cup 2K_1$.
Hence Lemma 8 is established using Lemmas 5 to 7 in the case $d_L(u_3) = \Delta(L) = 2$. Similarly, the lemma is established when $d_L(u_4) = \Delta(L) = 2$: in other words when $(u_4, z_i) \in E$ for $i = 1$ and 2.

Now let us assume that $d_L(z_1) = \Delta(L) = 2$, that is $(z_1, u_i) \in E$ for $i = 3$ and 4. Therefore $(z_3, u_i) \notin E$ for $i = 3$ and 4. Now note that $d_G(z) = \Delta(G) = 4$.

Note that $F = G[V - N_G[z]] = G[\{u, u_3, u_4, z_3, z_4\}] \cong P_3 \cup 2K_1$ or $P_4 \cup K_1$ or $C_4 \cup K_1$ according as z_4 is adjacent to 0 or 1 or 2 vertices from $\{u_3, u_4\}$.

If $F \cong P_3 \cup 2K_1$ or $P_4 \cup K_1$ then Lemma 8 is established using Lemmas 6 and 7.

Hence we assume that $F \cong C_4 \cup K_1$. This implies that (z_4, u_i) is in E for $i = 3$ and 4. Since $\chi_1(G) = 3$ and the set

$$X_1 = \{u, z, z_1, z_4\} \text{ is 1-independent, the set } \quad V - X_1 \text{ is not 1-independent.}$$

Thus $(z_3, u_i) \in E$ for $i = 1,\ 2$. Now it is easy to verify that $G - z_2 \cong G_1$ or G_2 according as the number of edges between $\{z_4\}$ and $\{u_1, u_2\}$ is 0 or 1. This establishes the lemma when $d_L(z_1) = \Delta(L) = 2$.

Since the vertices z_1 and z_2 are similar, the lemma is established when $d_L(z_2) = \Delta(L) = 2$ in a similar manner.

This completes the proof of Lemma 8. □

Lemma 9 *Let G be a triangle-free graph of order 10 with $\Delta(G) = 4$ and $\Delta(H) = 2$. Furthermore, let H be isomorphic to C_5. If $\chi_1(G) = 3$ then there exists a vertex u^* in G such that $G - u^* \cong G_i$ for some i, $1 \leq i \leq 3$.*

Proof. Let $V(H) = \{z_1, z_2, z_3, z_4, z_5\}$. Assume that $(z_i, z_{i+1}) \in E(H)$ for $i = 1, 2, 3, 4$ and $(z_5, z_1) \in E(H)$. Assume that $\chi_1(G) = 3$. The set

$$X_1 = \{u, z_2, z_4, z_5\} \text{ is 1-independent and so } \ V - X_1 = A \cup \{z_1, z_3\} \text{ is not 1-independent.}$$

This implies that $\Delta(L) = 2$ where $L = G[A \cup \{z_1, z_3\}]$. Now, either $d_L(u_i) = 2$ for some i, $1 \leq i \leq 4$ or $d_L(z_i) = 2$ for $i = 1$ or 3.
Case i. $d_L(u_i) = 2$ for some i, say $i = 1$.

Hence $(u_1, z_i) \in E$ for $i = 1$ and 3. Since G is triangle-free, $(u_1, z_i) \notin E$ for $i = 2$, 4, 5. Since $\chi_1(G) = 3$ and the set $Y_1 = \{u, u_1, z_2, z_4, z_5\}$ is $1-$independent, the set $V - Y_1 = \{u_2, u_3, u_4, z_1, z_3\}$ is not 1-independent. This in turn implies that, for some $i \in \{2, 3, 4\}$, $(u_i, z_j) \in E$ for $j = 1$ and 3. Without any loss of generality we assume that $(u_2, z_j) \in E$ for $j = 1$ and 3. Now note that $(u_2, z_j) \notin E$ for $j = 2$, 4 and 5.

Observe that $d_G(z_1) = \Delta(G) = 4$. Let $F = G[V - N_G[z_1]] = G[\{u, u_3, u_4, z_3, z_4\}]$. Clearly either
(i) $\Delta(F) = 3$, or
(ii) $F \cong P_3 \cup K_2$ or P_5.
Thus Lemma 9 is established using Lemmas 5, 6 and 8, in Case i.
Case ii. $d_L(z_i) = 2$ for $i = 1$ or 3.
Let us assume that $(z_1, u_i) \in E$ for $i = 1$ and 2. Note that $d_G(z_1) = 4$ and consider the subgraph $G[V - N_G[z_1]] = G[\{u, u_3, u_4, z_3, z_4\}] = F$, say. Since G is triangle-free, the vertex u_3 (also u_4) is adjacent to at most one of z_3 and z_4. If u_3 (or u_4) is adjacent to neither z_3 nor z_4 then $F \cong P_3 \cup K_2$ or P_5. Thus the lemma is established using Lemmas 6 and 8. Suppose that both u_3 and u_4 are adjacent to the same vertex, say z_3, then $\Delta(F) = 3$ and the lemma is established using Lemma 5. Hence without any loss of generality assume that (u_3, z_3) and (u_4, z_4) are in E. Hence (u_3, z_2) and (u_4, z_5) are not in E. Now, it is easy to check that

$$Y_1 = \{u_1, u_2, u_3, z_2, z_4\} \text{ and } V - Y_1 = \{u, u_4, z_1, z_3, z_5\}$$

provide a (2,1)-colouring of G, a contradiction. This proves Lemma 9.

Lemma 10 *Let G be a triangle-free graph of order 10 with $\Delta(G) = 4$ and $\Delta(H) = 2$. Furthermore, let H be isomorphic to $C_4 \cup K_1$. If $\chi_1(G) = 3$ then there exists a vertex u^* such that $G - u^* \cong G_i$ for some i, $1 \le i \le 3$.*

Proof. Let us assume that $\chi_1(G) = 3$. Recall that $u \in V$ with $d_G(u) = \Delta(G) = 4$, $N_G(u) = A = \{u_1, u_2, u_3, u_4\}$, $B = \{z_1, z_2, z_3, z_4, z_5\}$ and $H = G[B] = C_4 \cup K_1$. Assume that $(z_i, z_{i+1}) \in E(H)$ for $i = 1$, 2, 3 and $(z_4, z_1) \in E(H)$. Hence z_5 has degree 0 in H.

The sets

$$Y_1 = \{u, z_2, z_4, z_5\} \text{ and } Y_2 = \{u, z_1, z_3, z_5\} \text{ are 1-independent.}$$

Since $\chi_1(G) = 3$ the sets

$$V - Y_1 = \{z_1, z_3\} \cup A \text{ and } V - Y_2 = \{z_2, z_4\} \cup A \text{ are not 1-independent.}$$

Hence $F_1 = G[V - Y_1]$ and $F_2 = G[V - Y_2]$ both have maximum degree 2.
Case i. The subgraph F_i, $i = 1$, 2, attains its maximum degree at a z_j for some j in $\{1, 2, 3, 4\}$. We assume without loss of generality that

$$d_{F_1}(z_1) = 2,\ N_{F_1}(z_1) = \{u_1, u_2\},\ \ d_{F_2}(z_2) = 2,\ N_{F_2}(z_2) = \{u_3, u_4\}.$$

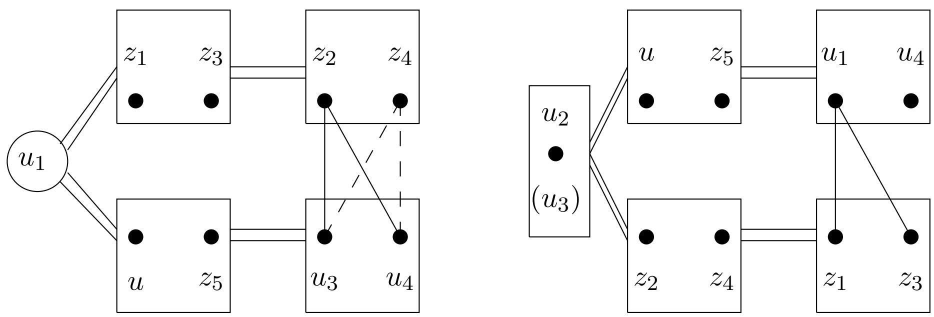
Note that $d_G(z_i) = 4$ for $i = 1$ and 2. Now we can assume that the subgraphs

$$L_1 = G[V - N_G[z_1]] = G[\{u, u_3, u_4, z_3, z_5\}] \text{ and } L_2 = G[V - N_G[z_2]] = G[\{u, u_1, u_2, z_4, z_5\}]$$

are both isomorphic to $C_4 \cup K_1$. For otherwise by Lemmas 5 to 9 there exists a vertex u^* in G such that $G - u^* \cong G_i$ for some i, $1 \leq i \leq 3$. Thus $(z_5, u_i) \in E$ for $i = 1$, 2, 3 and 4. Now the set

$$X_1 = \{z_1, z_2, z_5, u\} \text{ is 1-independent and so } V - X_1 = A \cup \{z_3, z_4\} \text{ is not.}$$

Hence we can assume, without loss of generality, that $(z_3, u_1) \in E$. It is easy to verify that the graph $G - u_2 \cong G_1$ or G_2 or G_3 according as the number of edges between z_4 and $\{u_3, u_4\}$ is 0 or 1 or 2. The graph $G - u_2$ is illustrated in Figure 5(a). The dotted lines indicate that the edges may or may not be in G. This completes the proof of Lemma 10 in Case i.



(a) $G - u_2 \cong G_i$, $1 \leq i \leq 3$ (b) $G - u_3 \cong G_1$ (resp.$G - u_2 \cong G_1$)

Figure 5: *Graph* $G - u^*$

Case ii. The subgraph F_1 attains its maximum degree at a u_j for some j in $\{1, 2, 3, 4\}$ and F_2 attains its maximum degree at a z_j for some j in $\{2, 4\}$. Furthermore $d_{F_1}(z_i) \leq 1$ for $i = 1$ and 3.

We assume without loss of generality that $(u_1, z_i) \in E(F_1)$ for $i = 1$ and 3; $(u_j, z_2) \in E(F_2)$ for $j = 2$ and 3. Note that $N_{F_1}(z_1) = N_{F_1}(z_3) = \{u_1\}$.

Since $d_G(z_2) = 4$, the subgraph

$$M_1 = G[V - N_G[z_2]] = G[\{u, u_1, u_4, z_4, z_5\}]$$

can be assumed to be isomorphic to $C_4 \cup K_1$. For otherwise, the lemma is established using Lemmas 5 to 9. Hence $(z_5, u_i) \in E$ for $i = 1$ and 4 and $(z_4, u_4) \notin E$. Thus $d_G(u_1) = 4$. Again

$$M_2 = G[V - N_G[u_1]] = G[\{u_2, u_3, u_4, z_2, z_4\}]$$

is assumed to be isomorphic to $C_4 \cup K_1$, by Lemmas 5 to 9. Hence (z_4, u_2) and (z_4, u_3) are in E. The set

$$X_1 = \{u, u_1, z_2, z_4\} \text{ is 1-independent and so } V - X_1 = \{u_2, u_3, u_4, z_1, z_3, z_5\} \text{ is not}$$

as $\chi_1(G) = 3$. This implies that z_5 is adjacent to at least one of $\{u_2, u_3\}$. If z_5 is adjacent to u_2 (resp. u_3) then it is easy to check that $G - u_3 \cong G_1$ (resp. $G - u_2 \cong G_1$). The graph $G - u_3$ (resp. $G - u_2$) is illustrated in Figure 5(b). This completes the proof of Lemma 10 in Case ii.

Case iii. Each subgraph F_i, $i = 1,\ 2$, attains its maximum degree at a u_j for some j in $\{1, 2, 3, 4\}$. Furthermore, every z_j has degree at most 1 in the corresponding F_i. We assume without loss of generality that

$$d_{F_1}(u_1) = 2,\ N_{F_1}(u_1) = \{z_1, z_3\},\quad d_{F_2}(u_2) = 2,\ N_{F_2}(u_2) = \{z_2, z_4\}.$$

Note that there are no other edges between A and $\{z_1, z_2, z_3, z_4\}$.

The set

$$X_1 = \{u_2, u_3, u_4, z_1, z_3\} \text{ is 1-independent and so } V - X_1 = \{u, u_1, z_2, z_4, z_5\} \text{ is not}$$

as $\chi_1(G) = 3$. Hence $(z_5, u_1) \in E$. Now note that $d_G(u_1) = 4$. But

$$N_1 = G[V - N_G[u_1]] = G[\{u_2, u_3, u_4, z_2, z_4\}]$$

is isomorphic to $P_3 \cup 2K_1$. Hence by Lemma 6 there exists a vertex u^* such that $G - u^* \cong G_i$ for some i, $1 \le i \le 3$.

This completes the proof of the lemma. □

Combining Lemmas 2 to 10 we have the following result.

Theorem 3 *Let G be a triangle-free graph of order 10 with $\chi_1(G) = 3$. Then either $G \cong G_5$ given in Figure 2 or there exists a vertex u^* such that $G - u^* \cong G_i$ for some i, $1 \le i \le 4$.*

We observe that there are exactly four triangle-free graphs of order 9, namely G_i, $1 \le i \le 4$ which are $(3, 1)$-critical. The graphs G_1 and G_4 are also $(3, 1)$-edge-critical. The next theorem determines all the $(3, 1)$-edge-critical triangle-free graphs of order 10.

Theorem 4 *A triangle-free graph G of order 10 is $(3, 1)$-edge-critical if and only if it is isomorphic to G_5 or $G_1 \cup K_1$ or $G_4 \cup K_1$.* □

Proof. Let G be a $(3, 1)$-edge-critical triangle-free graph of order10. By Theorem 3, either $G \cong G_5$ or there is a vertex u^* in G such that $G - u^* \cong G_i$ for $1 \le i \le 4$. Clearly the vertex u^* must be an isolated vertex and i must be equal to 1 or 4. Hence G is isomorphic to G_5 or $G_1 \cup K_1$ or $G_4 \cup K_1$.

It is easy to see that $G_1 \cup K_1$ and $G_4 \cup K_1$ are $(3,1)$-edge-critical. To complete the proof of the theorem we will show that $\chi_1(G_5 - e) = 2$ for every edge e of G_5. Clearly $\chi_1(G_5 - e) \geq 2$ for every edge e of G_5.

Suppose that $e = (u, u_1)$. The sets

$$X_1 = \{u, v, u_1, z_1, z_2\} \text{ and } V(G_5) - X_1 = \{u_2, u_3, u_4, u_5, z\}$$

are 1-independent and hence provide a (2,1)-colouring of $G_5 - e$. The edges (u, u_2), (v, u_1) and (v, u_2) are similar to (u, u_1) and it is easy to show that the removal of any one of these edges reduces $\chi_1(G_5)$.

Next suppose that $e = (v, u_3)$ or (u, u_3). The sets

$$X_1 = \{u, v, u_3, z\} \text{ and } V(G_5) - X_1 = \{u_1, u_2, u_4, u_5, z_1, z_2\}$$

provide a partition of $V(G_5) - e$ into 1-independent sets and hence $\chi_1(G_5 - e) = 2$. Suppose that $e = (v, u_4)$ or (u, u_4). The sets

$$X_2 = \{u, v, u_4, z, z_2\} \text{ and } V(G_5) - X_2 = \{u_1, u_2, u_3, u_5, z_1\}$$

are 1-independent and hence $\chi_1(G_5 - e) = 2$. Similarly if $e = (v, u_5)$ or (u, u_5) the sets

$$X_3 = \{u, v, u_5, z, z_1\} \text{ and } V(G_5) - X_3 = \{u_1, u_2, u_3, u_4, z_2\}$$

are 1-independent and so $\chi_1(G_5 - e) = 2$.

If $e = (u_3, z_1)$ (resp. (u_3, z_2)), then the sets $X_1 = \{u_1, u_2, u_3, u_4, u_5, z_1$ (resp. z_2)$\}$ and $V(G_5) - X_1$ provide a $(2,1)$-colouring of $G_5 - e$. If $e = (u_4, z_1)$ (resp. (u_5, z_2)), then the sets $X_2 = \{u_1, u_2, u_3, u_4, u_5, z_1$ (resp. z_2)$\}$ and $V(G_5) - X_2$ provide a $(2,1)$-colouring of $G_5 - e$.

Now if $e = (z, z_i)$ for $i = 1$ or 2 the sets $X_1 = \{u, v, z, z_1, z_2\}$ and $V(G_5) - X_1$ provide a $(2,1)$-colouring of $G_5 - e$.

Finally if $e = (z, u_i)$ for $i = 1$ or 2 the sets

$$X_1 = \{u, v, z_1, z_2\} \text{ and } V(G_5) - X_1$$

provide a $(2,1)$-colouring of $G_5 - e$.

Thus we have shown that for each e in G_5 we have $\chi_1(G_5 - e) = 2$.

This completes the proof of the theorem. □

It is easy to see that if a graph with no isolated vertices is $(3,1)$-edge-critical then it is also $(3,1)$-critical. From Theorem 3 it follows that if $G \not\cong G_5$ is a triangle-free graph of order 10 with $\chi_1(G) = 3$ then G is not $(3,1)-$critical. Hence we have the following theorem.

Theorem 5 *A triangle-free graph G of order 10 is $(3,1)-$critical if and only if it is isomorphic to G_5 given in Figure 2.* □

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