The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

MATHEMATICAL OLYMPIADS LECTURE NOTES

Notes on what a *theorem* is

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A **theorem** is a mathematical statement that is always *true*. The most common form of a theorem is the following:

If $\boldsymbol{P}$ *then* $\boldsymbol{Q}$.

where $\boldsymbol{P}$ is a list of *conditions* (or more technically *hypotheses*) and $\boldsymbol{Q}$ is the *conclusion*.

We should probably state this common form of a theorem as:

If $\boldsymbol{P}$ **is true** *then* $\boldsymbol{Q}$ **is true**.

This is certainly what we mean. Consider the following example:

If $\boldsymbol{i = j}$ *then* $\boldsymbol{2i = 2j}$.

Observe that adding the clause *"is true"* to both the hypothesis and conclusion of this statement does not change the meaning of the statement. Essentially, the clause *"is true"* has been suppressed for the sake of brevity ... but one should imagine that it is there in spirit.

Consider the following example. We know that a *spider usually has eight legs*. Stating this in the form of a theorem, we might write:

If $\boldsymbol{x}$ **is a normal healthy spider** *then* $\boldsymbol{x}$ **has 8 legs**.

The *converse* of the theorem, *If* $\boldsymbol{P}$ *then* $\boldsymbol{Q}$, is

If $\boldsymbol{Q}$ *then* $\boldsymbol{P}$.

(a similar looking statement but with the $\boldsymbol{P}$ and the $\boldsymbol{Q}$ interchanged). The converse of our "theorem" about spiders is:

If $\boldsymbol{x}$ **has 8 legs** *then* $\boldsymbol{x}$ **is a normal healthy spider**.

This statement is not always true – $\boldsymbol{x}$ *could be a crab, say.* So it is not a theorem, and we would say in such a case that the *converse (of the original "theorem" about spiders) is false.*

The statement, *If* $\boldsymbol{P}$ *then* $\boldsymbol{Q}$, may also be stated in each of the following ways:

$\boldsymbol{P}$ *only if* $\boldsymbol{Q}$.

$\boldsymbol{P} \implies \boldsymbol{Q}$. (the '$\implies$' symbol is read *implies*)

$\boldsymbol{Q}$ *if* $\boldsymbol{P}$.

We will avoid using any of these forms – but it helps to explain the notation/language used in the next most common form of a theorem.

Sometimes a statement

If $\boldsymbol{P}$ *then* $\boldsymbol{Q}$.

and its converse

If $\boldsymbol{Q}$ *then* $\boldsymbol{P}$.

are both always true. In such cases we have a shorter way of expressing this:

$\boldsymbol{P}$ *if and only if* $\boldsymbol{Q}$.

or alternatively:

$\boldsymbol{P} \iff \boldsymbol{Q}$.

(You will have to read the dangerous bend above for the language to make *English* sense.)