The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

MATHEMATICAL OLYMPIADS LECTURE NOTES

Maths Olympiad question – further general comments

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- If you read the dangerous bend with respect to how $\wedge$ (*meet*) and $\vee$ (*join*) operations could be defined for sequences, you will also have read that *distribution* rules for these operations (as they have been defined there) do *not* hold *in general*. Yet, in the proof of the previous item, you will notice that effectively a *distribution* rule was o.k. here. This might appear to be a contradiction. *It isn't!* ...

  When we say

  $\boldsymbol{P}$ does *not* hold *in general*

  you should imagine there are brackets as so:

  $\boldsymbol{P}$ does *not* (hold *in general*)

  as opposed to

  $\boldsymbol{P}$ (does *not* hold) *in general*.

  That is, (with the first bracketing) $\boldsymbol{P}$ may hold *sometimes* but it doesn't hold *all* the time – equivalently, we could say *there is at* **least one** *counterexample to* $\boldsymbol{P}$ *holding*; (with the second bracketing we would mean: $\boldsymbol{P}$ *never holds*).

  Exercises.

  1. Give a counter-example to show for sequences A, B, C that, for the given definitions of $\wedge$ and $\vee$, the *distribution* rules

     $$(A \wedge B) \vee C = (A \vee C) \wedge (B \vee C)$$
     $$(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C)$$

     do *not* hold *in general*.
     Hint: focus on the differences between sets and sequences.

  2. Interpret *your* sequences A, B, C (of *Exercise 1.*) as *sets* and replace $\wedge$ by $\cap$ and $\vee$ by $\cup$. Demonstrate that with these changes the distribution rules now hold (for this *one* example).

  3. Explain why the distribution rule is *valid* in the proof of the previous item. What special properties do the sequences have to make it work?
     Hint: focus on the differences between sets and sequences.

- In proving *two* sets are equal, sometimes *direct enumeration* is impractical, e.g. the two sets may be either quite large or infinite. In such cases, we may use the fact that

  $$A = B \iff A \subseteq B \text{ and } B \subseteq A$$

i.e. we may show $A = B$ by:

Choose an *arbitrary* element $a \in A$

Show $a \in B$

From the above can deduce $a \in B$ for all $a \in A$, i.e. $A \subseteq B$.

Then …choose an *arbitrary* element $b \in B$

Show $b \in A$

Deduce $b \in A$ for all $b \in B$, i.e. $B \subseteq A$.

Observe that we did essentially this when proving that, for x, y with the right properties,

$$S(x) \cup S(y) = \mathbb{N}.$$

First we showed $S(\alpha) \subseteq \mathbb{N}$, where α has the right properties, so that $S(x) \subseteq \mathbb{N}$ and $S(y) \subseteq \mathbb{N}$ and hence $S(x) \cup S(y) \subseteq \mathbb{N}$.

Next, we took an arbitrary $n \in \mathbb{N}$ and showed that either $n \in S(x)$ or $n \in S(y)$, i.e. $n \in S(x) \cup S(y)$, and hence we showed that $\mathbb{N} \subseteq S(x) \cup S(y)$.

It may seem that we are cheating here …the argument suggests we take just one element of A and show that it is in B and then deduce that *every* element of A is in B. The crucial word here is *arbitrary* – the argument is really a *proforma*: *we could replace our choice of a with* **any** *other element of A and the* **same** *argument shows $a \in B$.*

- When *two* sets are large and *finite*, we may use the following fact as an alternative to the above approach, in proving the two sets are equal.

$$A = B \iff A \subseteq B \text{ and } \#A = \#B$$

i.e. we may show $A = B$ by:

Choose an *arbitrary* element $a \in A$

Show $a \in B$

Deduce $a \in B$ for all $a \in A$, i.e. $A \subseteq B$.

Then …show $\#A = \#B$.

Observe that we could have used this approach to prove that, for x, y with the right properties,

$$S_n(x) \cup S_n(y) = \{1, 2, \ldots, n-1\}.$$

We showed $S_n(\alpha) \subseteq \{1, 2, \ldots, n-1\}$, when α had the right properties, so that

$$S_n(x) \cup S_n(y) \subseteq \{1, 2, \ldots, n-1\}.$$

We also showed that $S_n(x) \cap S_n(y) = \emptyset$ and $\#S_n(x) + \#S_n(y) = n - 1$, so that

$$\#\big(S_n(x) \cup S_n(y)\big) = n - 1.$$

Since $\#\{1, 2, \ldots, n-1\} = n - 1$, we may deduce

$$S_n(x) \cup S_n(y) = \{1, 2, \ldots, n-1\}.$$

Then, by *induction* we can deduce

$$S(x) \cup S(y) = \mathbb{N}.$$

Notice, we have deduced two *infinite* sets are equal using a method valid only for *finite* sets, by carefully avoiding infinite cardinals!

- In developing the proof to Q5 of the set of Mathematics Olympiad problems a lot of seemingly unnecessary notation was used and the proof ran to several pages. In defence of this, here is a quote from *The Problems of Mathematics* by Ian Stewart:

  > ... Someone once stated a theorem about prime numbers, claiming that it could never be proved because there was no good notation for primes. Carl Friedrich Gauss proved it from a standing start in five minutes, saying (somewhat sourly) 'what he needs is *notions*, not *notations*'. Calculations are merely a means to an end. If a theorem is proved by an enormous calculation, that result is not properly understood until the reasons the calculation works can be isolated and seen to appear as natural and inevitable. Not all ideas are mathematics; but all good mathematics must contain an idea.

  The *notation* $\wedge$ and $\vee$ is not necessary for understanding the proof of Q5 – it was merely a vehicle used to emphasise the differences between the *notions* of *set* and *sequence*. Notation, in general, is introduced in order to express ideas in a succinct way. Just recapping, the key steps in the proof of Beatty's theorem were:

  - For all $n \in \mathbb{N}$,
  $$\#S_n(x) + \#S_n(y) = \left\lfloor \frac{n}{x} \right\rfloor + \left\lfloor \frac{n}{y} \right\rfloor = n - 1.$$
  - If α is x or y then
  $$\#S_{n+1}(\alpha) = \begin{cases} \#S_n(\alpha), & \text{if } n \notin S(\alpha) \\ \#S_n(\alpha) + 1, & \text{if } n \in S(\alpha). \end{cases}$$
  - For all $n \in \mathbb{N}$,
  $$\#S_{n+1}(x) + \#S_{n+1}(y) = \#S_n(x) + \#S_n(y) + 1.$$

  Hopefully, having written it this way you will also see the conclusion (Beatty's Theorem) as both natural and inevitable.

- **Final pointer:** When proving a statement of the form: *If* conditions *then* conclusion, your proof will usually start with

  Let conditions

  and end with

  Therefore conclusion.

  You should identify in your proof where each *condition* is used. If there are some *conditions* that haven't been used you should check that they really are unnecessary (i.e. that they were *red herrings*); a good way to do this is to construct *examples* that satisfy the *conditions* you have used but do not satisfy the supposed *red herring conditions* and check the *conclusion* still holds; if it doesn't then there is something wrong with your proof – it's possible you only need to identify a *"red herring condition"* at one of the steps.