The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

MATHEMATICAL OLYMPIADS LECTURE NOTES

Maths Olympiad question

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Problem

(Q5) The k^{th} entry of list B is the decimal number 10^k written in base 2.
The k^{th} entry of list C is the decimal number 10^k written in base 5.
Prove that for every integer $n > 1$, there is *exactly one* number in *exactly one* of the lists B, C that has *exactly* n digits.

Below we will need to view the lists B, C as *sequences*. It has been traditional in mathematics to write *sequences* as a list of elements separated by commas, e.g.

$$a_1, a_2, a_3, \dots$$

Sequences have a number of properties in common with *sets*, but also they have some important differences. We can create new sets from old sets using the operations $\cap$ (intersection) and $\cup$ (union). A natural question is: can we define similar operations for sequences? The answer is yes, but however we define these operations they won't have all the properties of $\cap$ and $\cup$. In defining such operations another problem is highlighted – sets are delimited by curly braces $\{\dots\}$; traditionally sequences have not had delimiters. So let's decide to delimit sequences by square brackets $[\dots]$. Now let's highlight the similarities and differences of sets and sequences. Two distinguishing features of sets are:

- the way the elements are arranged between the braces $\{\dots\}$ is not important, e.g. $\{1, 2, 3\} = \{3, 2, 1\}$; and
- duplicated elements are counted once, e.g. $\{1, 1, 2, 3, 3, 3\} = \{1, 2, 3\}$.

This motivates the notion of *cardinality* (or size) of a set. The *cardinality* of a *set* is the number of (distinct) elements it contains. We will use the symbol # to mean *the cardinality of*, e.g.

$$\#\{1, 1, 2, 3, 3, 3\} = \#\{1, 2, 3\} = 3.$$

Now *sequences* may be viewed as *sets* with the two distinguishing features of sets mentioned above, waived. That is, for sequences

- the way the elements are arranged between the brackets $[\dots]$ is important, e.g. $[1, 2, 3] \neq [3, 2, 1]$; and
- duplicated elements are counted according to *multiplicity*, e.g. $[1, 1, 2, 3, 3, 3] \neq [1, 2, 3]$.

This suggests we should define *cardinality* for a *sequence* to be the number of elements it contains (counted according to *multiplicity*) and we will reuse the symbol #, e.g.

$$\#[1, 1, 2, 3, 3, 3] = 6, \quad \#[1, 2, 3] = 3.$$

Now we *can* use the set operations $\cap$ and $\cup$ on sequences if we agree to interpret them as *sets* – consequently what we get out has to be a *set*! e.g.

$$[1,1,2,3,3,3] \cap [2,1] = \{1,2\}$$
$$[1,1,2,3,3,3] \cup [2,1] = \{1,2,3\}\,.$$

Similarly we can use the *membership* relation $\in$ on sequences if we agree to interpret them as sets, e.g.

$$1 \in [1,1,2,3,3,3]\,, \quad 3 \notin [2,1]\,.$$

However, we may wish to retain some of the ordering and duplication structure of the original sequences – what we get out could then be meaningfully interpreted as a sequence. Let's call the operations emulating $\cap$ and $\cup$, *meet* (symbol: $\wedge$) and *join* (symbol: $\vee$) respectively. Define for sequences A, B,

$$\begin{aligned} A \wedge B &:= [a : a \in A \mid a \in B] \\ &= \text{the sequence } A \text{ but with elements } \textit{not} \text{ appearing in } B \textit{ deleted}, \end{aligned}$$

(read the ':' as *such that* and the '|' as *and satisfying the additional condition*) e.g.

$$\begin{aligned} [1,1,2,3,3,3] \wedge [2,1] &= [1,1,2] \\ [1,1,2,3,3,3] \wedge [3,1] &= [1,1,3,3,3] \\ [1,2,3] \wedge [3,2,1] &= [1,2,3] \\ [3,2,1] \wedge [1,2,3] &= [3,2,1]\,. \end{aligned}$$

Notice the second operand of $\wedge$ in such a definition is essentially being interpreted as a set. Also, notice that $\wedge$ is *not commutative*, i.e.

$$A \wedge B \neq B \wedge A, \quad \text{for some } A, B.$$

Let's define $\vee$ by *concatenation*, i.e. $A \vee B$ is simply the sequence formed by taking the elements of A followed by the elements of B, e.g.

$$\begin{aligned} [1,1,2,3,3,3] \vee [2,1] &= [1,1,2,3,3,3,2,1] \\ [1,1,2,3,3,3] \vee [3,1] &= [1,1,2,3,3,3,3,1] \\ [1,2,3] \vee [3,2,1] &= [1,2,3,3,2,1] \\ [3,2,1] \vee [1,2,3] &= [3,2,1,1,2,3]\,. \end{aligned}$$

This definition is only of any use if the first operand is *finite*, and again $\vee$ is *not commutative.* However, both $\wedge$ and $\vee$ are *associative*, i.e.

$$(A \wedge B) \wedge C = A \wedge (B \wedge C)$$
$$(A \vee B) \vee C = A \vee (B \vee C).$$

Also for $\wedge$ and $\vee$ on sequences, in general, there are *no* counterparts to the *distribution rules* for $\cap$ and $\cup$ on sets:

$$(A \cap B) \cup C = (A \cap C) \cup (B \cap C)$$
$$(A \cup B) \cap C = (A \cup C) \cap (B \cup C).$$

Solution

- For $N \in \mathbb{N}$,

$$\begin{aligned}
\#\left[\text{digits in } 10^k \text{ when } 10^k \text{ is written in base } 2\right] &= \left\lfloor \log_2(10^k) \right\rfloor + 1 \\
&= \lfloor k \log_2(10) \rfloor + 1 \\
\#\left[\text{digits in } 10^k \text{ when } 10^k \text{ is written in base } 5\right] &= \left\lfloor \log_5(10^k) \right\rfloor + 1 \\
&= \lfloor k \log_5(10) \rfloor + 1
\end{aligned}$$

- Define

$$\begin{aligned}
N_B &= [\lfloor \log_2(10) \rfloor + 1, \lfloor 2\log_2(10) \rfloor + 1, \lfloor 3\log_2(10) \rfloor + 1, \ldots], \\
N_C &= [\lfloor \log_2(10) \rfloor + 1, \lfloor 2\log_2(10) \rfloor + 1, \lfloor 3\log_2(10) \rfloor + 1, \ldots],
\end{aligned}$$

  the sequences of the numbers of digits in elements of lists B, C, respectively.

- Define sequence $S(\alpha)$ by

$$S(\alpha) := [\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \ldots]$$

- If $\alpha > 1$ then $S(\alpha) \subseteq \mathbb{N}$.

  since ... $\lfloor \ell\alpha \rfloor \in \mathbb{Z}$ for all $\ell\alpha \in S(\alpha)$ and $\min\big(S(\alpha)\big) = \lfloor \alpha \rfloor \geq 1$.

- If $\alpha > 1$ then $S(\alpha)$ has no duplicates.

  since ...

  Suppose the statement is false. Then for some $\ell, m \in \mathbb{N}$ such that $\ell < m$, we have $\lfloor \ell\alpha \rfloor = \lfloor m\alpha \rfloor$. Hence

$$\begin{aligned}
&0 \leq \ell\alpha - \lfloor \ell\alpha \rfloor < 1 \quad (\text{i.e. } -1 < -\ell\alpha + \lfloor \ell\alpha \rfloor \leq 0) \\
&0 \leq m\alpha - \lfloor m\alpha \rfloor < 1 \\
\text{So } &-1 < m\alpha - \ell\alpha - \lfloor \ell\alpha \rfloor + \lfloor m\alpha \rfloor < 1 \\
\text{i.e. } &\quad m\alpha - \ell\alpha < 1, \quad (\text{since } \lfloor \ell\alpha \rfloor = \lfloor m\alpha \rfloor).
\end{aligned}$$

  But

$$m\alpha - \ell\alpha = (m - \ell)\alpha > 1.1 = 1,$$

  since $m > \ell$ and $\alpha > 1$. (Contradiction)
  So the original statement is true.

- Define for $n \in \mathbb{N}$,

$$\begin{aligned}
S_n(\alpha) &:= \{s \in S(\alpha) \mid s < n\} \\
&= S(\alpha) \wedge \{1, 2, 3, \ldots, n-1\}
\end{aligned}$$

- If $\alpha > 1$ and α is irrational, and $n \in \mathbb{N}$ then $\#S_n(\alpha) = \left\lfloor \dfrac{n}{\alpha} \right\rfloor$.

  since ...

First note that no integer multiple of α equals n, since α is irrational. So for some $\ell \in \mathbb{N}$, each of

$$\alpha, 2\alpha, \ldots, \ell\alpha$$

is less than n, and each of

$$(\ell+1)\alpha, (\ell+2)\alpha, \ldots$$

is greater than n. That is,

$$\begin{aligned} \ell\alpha < n &< (\ell+1)\alpha \\ \text{So } \ \ell < \frac{n}{\alpha} &< \ell+1, \quad (\text{since } \alpha > 1 > 0). \end{aligned}$$

Hence $\ell = \left\lfloor \frac{n}{\alpha} \right\rfloor$. Also

$$\lfloor \ell\alpha \rfloor < n \le \lfloor (\ell+1)\alpha \rfloor .$$

So

$$S_n(\alpha) = [\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \ldots, \lfloor \ell\alpha \rfloor] .$$

Hence $\#S_n(\alpha) = \ell = \left\lfloor \frac{n}{\alpha} \right\rfloor$.

- Suppose x, y are positive irrational real numbers such that

$$\frac{1}{x} + \frac{1}{y} = 1.$$

Then $\#S_n(x) + \#S_n(y) = n - 1$ for all $n \in \mathbb{N}$.

since ...

$$\begin{aligned} x > 0 \text{ and } y > 0 &\implies \frac{1}{x} > 0 \text{ and } \frac{1}{y} > 0 \\ &\implies \frac{1}{x} < 1 \text{ and } \frac{1}{y} < 1, \quad (\text{since } \frac{1}{x} + \frac{1}{y} = 1) \\ &\implies x > 1 \text{ and } y > 1. \end{aligned}$$

So $\#S_n(x) = \left\lfloor \frac{n}{x} \right\rfloor$, $\#S_n(y) = \left\lfloor \frac{n}{y} \right\rfloor$.

Now, x, y are irrational. So $\frac{n}{x} \neq \left\lfloor \frac{n}{x} \right\rfloor$, $\frac{n}{y} \neq \left\lfloor \frac{n}{y} \right\rfloor$, and thus

$$\begin{aligned} \frac{n}{x} - 1 < \left\lfloor \frac{n}{x} \right\rfloor < \frac{n}{x} \\ \frac{n}{y} - 1 < \left\lfloor \frac{n}{y} \right\rfloor < \frac{n}{y}. \end{aligned}$$

Hence

$$\frac{n}{x} + \frac{n}{y} - 2 < \left\lfloor \frac{n}{x} \right\rfloor + \left\lfloor \frac{n}{y} \right\rfloor < \frac{n}{x} + \frac{n}{y}.$$

Now

$$\frac{n}{x} + \frac{n}{y} = n\Big(\frac{1}{x} + \frac{1}{y}\Big) = n.$$

Hence

$$n - 2 < \left\lfloor \frac{n}{x} \right\rfloor + \left\lfloor \frac{n}{y} \right\rfloor < n.$$

But $\left\lfloor \frac{n}{x} \right\rfloor + \left\lfloor \frac{n}{y} \right\rfloor \in \mathbb{Z}$. So $\left\lfloor \frac{n}{x} \right\rfloor + \left\lfloor \frac{n}{y} \right\rfloor = n - 1$.

- If $\alpha > 1$ then

$$S_{n+1}(\alpha) = \begin{cases} S_n(\alpha) & \text{if } n \notin S(\alpha) \\ S_n(\alpha) \vee [n] & \text{if } n \in S(\alpha) \end{cases}$$

$$\text{i.e. } \#S_{n+1}(\alpha) = \begin{cases} \#S_n(\alpha) & \text{if } n \notin S(\alpha) \\ \#S_n(\alpha) + 1 & \text{if } n \in S(\alpha) \end{cases}$$

since ...

Now $\alpha > 1$ so $S(\alpha)$ has no duplicated elements. Hence

$$\begin{aligned} S_{n+1}(\alpha) &= S(\alpha) \wedge \{1, \ldots, n-1, n\} \\ &= \big(S(\alpha) \wedge \{1, \ldots, n-1\}\big) \vee \big(S(\alpha) \wedge \{n\}\big) \\ &= S_n(\alpha) \vee \big(S(\alpha) \wedge \{n\}\big) \\ &= \begin{cases} S_n(\alpha) & \text{if } n \notin S(\alpha) \\ S_n(\alpha) \vee [n] & \text{if } n \in S(\alpha). \end{cases} \end{aligned}$$

- **(Beatty's Theorem)** Suppose x, y are positive irrational real numbers such that

$$\frac{1}{x} + \frac{1}{y} = 1.$$

Then

$$\begin{aligned} &S(x) \cup S(y) = \mathbb{N} \text{ and} \\ &S(x) \cap S(y) = \emptyset. \end{aligned}$$

since ...

For any given $n \in \mathbb{N}$, one of four possibilities may arise:

$$\begin{array}{rlll} \text{(i)} & \#S_{n+1}(x) = \#S_n(x) & \text{and} & \#S_{n+1}(y) = \#S_n(y); \\ \text{(ii)} & \#S_{n+1}(x) = \#S_n(x) + 1 & \text{and} & \#S_{n+1}(y) = \#S_n(y); \\ \text{(iii)} & \#S_{n+1}(x) = \#S_n(x) & \text{and} & \#S_{n+1}(y) = \#S_n(y) + 1; \text{ or} \\ \text{(iv)} & \#S_{n+1}(x) = \#S_n(x) + 1 & \text{and} & \#S_{n+1}(y) = \#S_n(y) + 1. \end{array}$$

However, for all $n \in \mathbb{N}$,

$$\begin{aligned} \#S_{n+1}(x) + \#S_{n+1}(y) &= n \\ &= n - 1 + 1 \\ &= \#S_n(x) + \#S_n(y) + 1. \end{aligned}$$

So cases (i) and (iv) cannot occur. Thus, for all $n \in \mathbb{N}$, either

- case (ii) occurs, so that $n \in S(x)$ and $x \notin S(y)$; or
- case (iii) occurs, so that $n \notin S(x)$ and $x \in S(y)$.

So for all $n \in \mathbb{N}$, $n \in S(x) \cup S(y)$ i.e. $S(x) \cup S(y) = \mathbb{N}$; and
for all $n \in \mathbb{N}$, $n \notin S(x) \cap S(y)$ i.e. $\mathbb{N} \cap S(x) \cap S(y) = S(x) \cap S(y) = \emptyset$.

- $\dfrac{1}{\log_2(10)} + \dfrac{1}{\log_5(10)} = 1$

  since ... $\log_{10}(2) + \log_{10}(5) = \log_{10}(10) = 1$,
  $\log_{10}(2) = \dfrac{1}{\log_2(10)}$, $\log_{10}(5) = \dfrac{1}{\log_5(10)}$.

- If $a, x \in \mathbb{N}$ and $a > 1$, $x > 1$ such that there is a prime divisor p dividing *exactly* one of a and x then $\log_a(x)$ is irrational.

  since ...

  Let $a, x \in \mathbb{N}$ and $a > 1$, $x > 1$ and let p be a prime dividing *exactly* one of a and x. Firstly, $\log_a(x)$ is positive, since $a > 1$ and $x > 1$.
  Now suppose (for a contradiction) that $\log_a(x)$ is rational. Then

  $$\log_a(x) = m/n,$$

  for some $m, n \in \mathbb{N}$ (m, n can be chosen to be positive since $\log_a(x)$ is positive). Thus

  $$a^{m/n} = x$$

  and hence $a^m = x^n$. By assumption p divides *exactly* one of a and x, and so p divides *exactly* one of a^m and x^n. Since $a^m = x^n$ we have a contradiction. So $\log_a(x)$ is rational.

- $\log_2(10)$ and $\log_5(10)$ are positive and irrational real numbers.

  since ... 2, 5, 10 are integers greater than 1; 5 divides exactly one of 2 and 10; and 2 divides exactly one of 5 and 10.

- Let $x = \log_2(10)$ and $y = \log_5(10)$. Then

  $$S(x) \cup S(y) = \mathbb{N},$$
  $$S(x) \cap S(y) = \emptyset.$$

- Let $x = \log_2(10)$ and $y = \log_5(10)$. Then

  $$N_B = [s + 1 : s \in S(x)],$$
  $$N_C = [s + 1 : s \in S(y)],$$

  and hence

  $$N_B \cup N_C = \{n \in \mathbb{N} \mid n > 1\},$$
  $$N_B \cap N_C = \emptyset,$$

  which is what was required to be proved.