The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

MATHEMATICAL OLYMPIADS LECTURE NOTES

AMOC Intermediate Mathematics Extension Programme

Unit One

1. A positive integer is said to be *partitioned* if expressed in the form $n = x_1 + x_2 + \cdots + x_k$, where $k \geq 1$ and each of $x_1, x_2, \ldots, x_k$ is a positive integer. Assuming that order is significant (so that, for example, $1+2$ and $2+1$ are different partitions of 3), determine in terms of n the number of ways of partitioning n.

   Hints. Let p_n be the number of ways of partitioning n.

   The only partition of 1 is 1. So $\ldots p_1 = 1$.
   The partitions of 2 are $2, 1+1$. So $\ldots p_2 = 2$.
   The partitions of 3 are $3, 2+1, 1+2, 1+1+1$. So $\ldots p_3 = 4$.
   Continue in this way until you are able to *guess* a general expression for p_n.
   Now find a connection between the partitions of $n-1$ and the partitions of n and thence prove your *guess* by induction.

   Alternatively, we could write each x_i of a partition as x_i strokes, e.g. write the partition $2+1$ of 3 as $||+|$. In this way each partition of n is written as a sequence of n strokes where between adjacent strokes we choose to put a $+$ or not to put a $+$. So ... for each sequence of n strokes there are $n-1$ gaps between adjacent strokes and for each gap there are 2 possibilities. Now count!

2. A pair of integers (a, b) is *co-prime* when the greatest common divisor of a and b is 1. Find the number of co-prime pairs of positive integers which add to 7250.

   Hints. First show that: *If $a + b = n$, then (a, b) is a co-prime pair if and only if (a, n) is a co-prime pair.*

   This reduces the problem to finding the number of integers a that are co-prime to 7250 such that $1 \leq a \leq 7250$.

   Consider the *complementary* problem of finding the number of integers a that are *not* co-prime to 7250 such that $1 \leq a \leq 7250$. Each such a must be divisible by a prime divisor of 7250.
   The prime divisors of 7250 are 2, 5 and 29.
   Let P, Q, R be the sets of integers between 1 and 7250 inclusive that are divisible by 2, 5 or 29 respectively.
   Find the size of $P \cup Q \cup R$. This gives the answer to the *complementary problem.*

   Now deduce the answer to the original problem.

3. A bank invites term deposits under the following conditions:

   (1) The deposit (or initial principal) may be any amount in dollars and cents, with a minimum of \$1.

   (2) The yearly interest, calculated at the end of each year and added to the principal, is one cent less than 10% of the current principal, fractions of a cent being discarded.

   (3) The deposit with accumulated interest is returned at end of the sixth year.

   Find the smallest initial deposit which would result in no fractions of cents being discarded in any of the six years.

   Hints. Let x_n be the value of the principal at the end of the nth year.
   Derive a relationship between x_n and x_{n-1}.
   Guess a relationship between x_n and x_0 (x_0 is the initial principal). This relationship should remind you of the *compound interest* formula.
   Prove your *guess* by mathematical induction.
   What's the sum of a *geometric series?*
   Now think in terms of congruences.

4. A finite set of straight lines in the plane is said to be in general position if no two of the lines are parallel and no three are concurrent. Determine the number of regions into which the plane is divided by a set of n straight lines in general position.

   Hints. Let R_n be the number of regions with n lines in general position.
   Try a few small values of n, i.e. $0, 1, 2, 3, \ldots$, and get a feel for the problem.
   Guess a general expression for R_n and prove your *guess* by induction, i.e. by finding how much R_{n+1} is bigger than R_n.

5. S_1 is the sequence of natural numbers $1, 2, 3, 4, \ldots$. For each $n \geq 1$, if S_n is the sequence $s_1, s_2, s_3, s_4, \ldots$, then S_{n+1} is the sequence $s'_1, s'_2, s'_3, s'_4, \ldots$, where $s'_i = s_i + 1$ whenever n is a divisor of s_i, and $s'_i = s_i$ otherwise.
   Determine those n for which the first $n - 1$ terms of S_n are all equal to the number n, but the nth term is not.

   Hints. Write down the first few sequences ...

   $$S_1 \text{ is } 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, \ldots$$
   $$S_2 \text{ is } 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, \ldots$$
   $$S_3 \text{ is } 3, 3, 5, 5, 7, 7, 9, 9, 11, 11, 13, 13, 15, 15, \ldots$$

   Keep going until you can see a general pattern.
   Now check whether the given condition is *true* or *false* for each of your sequences and *guess* when the condition is *true* for general n.
   Prove your *guess* by induction.

6. A collection of n^2 integers, each 1 or -1, is arranged into n rows and n columns to form an $n \times n$ square array. Let r_i be the product of the numbers in the ith row and let c_i be

the product of the numbers in the ith column.
Determine for which values of n it is possible that

$$r_1 + r_2 + \cdots + r_n + c_1 + c_2 + \cdots + c_n = 0.$$

Hints. Consider the arrays possible when n is 0, 1 and 2.
Guess for what values of n that arrays are *possible/impossible.*
Now prove it, by providing a *construction* for the *possible* case; and a *contradiction* proof for the *impossible* case.

Unit Two

1. Let n be a positive integer.
   Prove that the average of the numbers $\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}$ exceeds $\frac{2}{3}\sqrt{n}$.

2. A square is divided into 81 small squares by lines parallel to its sides. The numbers $1, 2, \ldots, 81$ are entered in an arbitrary fashion, one in each square.
   Show that, however the numbers are entered, it is possible to find two small squares with an edge in common whose entries differ by more than 5.

3. A set of points is said to have diameter d if, for any $x < d$, there is a pair of points of the set distant at least x apart, but no points of the set is more than distance d apart. Show that a square piece of paper 1 unit by 1 unit (regarded as the set of points inside and on the boundary of a plane unit square) can be divided into three pieces each of diameter not exceeding $\frac{\sqrt{65}}{8}$ but not into three pieces each of diameter less than $\frac{\sqrt{65}}{8}$.

4. We call an L-shaped figure, formed by joining three unit squares $ABCD$, $DCEF$, $GDFH$ edge to edge, an L-tromino. A square courtyard 2^n by 2^n square units is divided into unit squares by lines parallel to the edges. One of the unit squares is reserved for a drainpipe opening. Prove that the rest of the courtyard can be tiled with unbroken, non-overlapping L-tromino shaped tiles.

5. Let n be a positive integer. A $2n-1$ by $2n-1$ array of real numbers, all between -1 and 1, is formed in such a way that the sum of any four that form a 2×2 block is zero. Prove that the sum of all the numbers is at most $2n-1$.

6. Functions f and g are defined by $f(x) = 3^x$ and $g(x) = 100^x$. Two sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are defined as follows:

   (i) $a_1 = 3$ and $a_{n+1} = f(a_n)$ for $n \geq 1$;
   (ii) $b_1 = 100$ and $b_{n+1} = g(b_n)$ for $n \geq 1$.

   Determine the smallest positive integer m for which $b_m > a_{100}$.

7. A finite set of points, not all on the same line, are chosen from the plane and each is labelled with one of the numbers $0, 1, -1$. Suppose that, for each line containing two or more chosen points, the labels on all chosen points on the line add to 0.
   Prove that every chosen point has been labelled 0.