Row rank = column rank

The purpose of this note is to demonstrate by way of an example why the row rank and column rank of a matrix are equal.

Exercise 1. Generalise the idea in the example and dangerous-bend discussion to produce a proper proof.

Example 2. *Let*

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 5 & 1 \\ 0 & 4 & 4 & -4 \end{bmatrix}$$

Then

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & 4 & 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ *row-rank*$(A) =$ *#non-zero rows of an RE form of* $A = 2$.

To determine whether the columns of A are $\ell.i.$we could solve the *'$\ell.i.$problem'* for the columns $\underset{\sim}{c}_i$ of A:

$$\text{Solve } k_1\underset{\sim}{c}_1 + k_2\underset{\sim}{c}_2 + \cdots + k_n\underset{\sim}{c}_n = \underset{\sim}{\mathbf{0}} \text{ for the } k_i$$

or equivalently, solve $A\underset{\sim}{\boldsymbol{k}} = \underset{\sim}{\mathbf{0}}$ for $\underset{\sim}{\boldsymbol{k}}$, i.e. reduce

$$\left[A \mid \underset{\sim}{\mathbf{0}}\right] \sim \begin{bmatrix} 1 & 0 & 1 & 1 & | & 0 \\ 0 & 1 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\therefore \underset{\sim}{\boldsymbol{k}} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

If $s = 1, t = 0$ we have: $-\underset{\sim}{c}_1 - \underset{\sim}{c}_2 + \underset{\sim}{c}_3 = \underset{\sim}{\mathbf{0}} \quad \therefore \underset{\sim}{c}_3 = \underset{\sim}{c}_1 + \underset{\sim}{c}_2$.
If $s = 0, t = 1$ we have: $-\underset{\sim}{c}_1 + \underset{\sim}{c}_2 + \underset{\sim}{c}_4 = \underset{\sim}{\mathbf{0}} \quad \therefore \underset{\sim}{c}_4 = \underset{\sim}{c}_1 - \underset{\sim}{c}_2$.

Observe ...

(1) Using the solutions for $\underset{\sim}{\boldsymbol{k}}$ in this way shows us how to write the "non-step" vectors as a linear combination of the "step" vectors. (The ith vector is a "step" vector $\iff$ an RE form of the matrix has a leading entry in the ith column.) Thus for a spanning set of colspace(A) we may delete the "non-step" vectors.

(2) To determine whether "step" vectors are $\ell.i.$we need to solve the *'$\ell.i.$problem'* for the subset consisting of the "step" vectors, but the solution of this problem is obtained from the previous solution by setting $s = t = 0$ (this is equivalent to deleting $\underset{\sim}{c}_3$ and $\underset{\sim}{c}_4$). But this leads to $k_1 = k_2 = 0$, showing that the "step" vectors $\underset{\sim}{c}_1$ and $\underset{\sim}{c}_2$ are $\ell.i.$

(3) The previous step shows column-rank$(A) = 2$ and shows that a basis for colspace(A) is $\{\underset{\sim}{c}_1, \underset{\sim}{c}_2\}$.

Corollary 3. *If $A \sim B$ where B is in RE form, then the "step" vectors of A form a basis for colspace(A).*

Corollary 4. *row-rank(A) = column-rank(A)*

Proof. Suppose $A \sim B$ where B is in RE form. Then

$$\begin{aligned} \text{row-rank}(A) &= \#\text{non-zero rows of } B \\ &= \#\text{steps of } B \\ &= \text{column-rank}(A) \end{aligned}$$

□