Rank-nullity Theorem Applications

The following are some further examples of applications of the Rank-nullity Theorem:

Theorem. ***(Rank-nullity)*** *Let A be an $m \times n$ matrix. Then*

$$\operatorname{rank}(A) + \operatorname{null}(A) = n = \#\textit{columns of } A.$$

Example 1. *Let B be a 6×4 matrix of rank 4.*
Then $\operatorname{null}(B) = 4 - 4 = 0$. *So* $B\underset{\sim}{x} = \underset{\sim}{\mathbf{0}} \iff \underset{\sim}{x} = \underset{\sim}{\mathbf{0}}$.

Example 2. *Let B be a 4×6 matrix of rank 4.*
Then $\operatorname{rank}(B) + \operatorname{null}(B) = 6$. *So* $\operatorname{null}(B) = 6 - 4 = 2$,
and hence the solution set of $B\underset{\sim}{x} = \underset{\sim}{\mathbf{0}}$ depends on 2 parameters.

Example 3. *Let A be an $m \times n$ matrix.*
A has m rows $\implies \operatorname{rank} A \leq m$.
A has n columns $\implies$ column-rank$(A) = \operatorname{rank}(A) \leq n$.
$\therefore \operatorname{rank}(A) \leq \min(m, n)$.

So if $m < n$ then $\operatorname{rank}(A) < n$ *and* $\operatorname{null}(A) = n - \operatorname{rank}(A) > 0$.
This proves again that a homogeneous *system with more unknowns (n) than equations (m) always has a* non-trivial *solution.*

Example 4. *Given*

$$A = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 3 & 8 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad A \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

find a basis for rowspace(A).

> ***Solution.*** $\operatorname{rank}(A) + \operatorname{null}(A) = 3$ *and* $\operatorname{null}(A) \geq 1$. *So* $\operatorname{rank}(A) \leq 2$.
> *But first two rows are $\ell.i.$*
> *Hence* $\operatorname{rank}(A) = 2$ *and* $\{(0\ 2\ 4), (2\ 3\ 8)\}$ *is a basis for* rowspace(A).

Remark 5. Let A be a (square) $n \times n$ matrix.
We proved that: A is invertible $\iff \operatorname{null}(A) = 0$.
Now we have from the Rank-nullity Theorem:

$$\begin{aligned} A \text{ invertible} &\iff \operatorname{null}(A) = n \\ &\iff \operatorname{rank}(A) = n \\ &\iff \text{row vectors of } A \text{ are } \ell.i. \\ &\iff \text{column vectors of } A \text{ are } \ell.i. \end{aligned}$$