

## SETTING OUT A PROOF

The writing of proofs is an art that takes some time to perfect. Nevertheless, there are some guiding principles:

- Start your proof with:  
*We are required to prove “ . . . ”.*
- Check for *trivial* cases, i.e. cases where the statement is obviously true.
- Split off cases (*trivial* or otherwise). This reduces your original problem to something smaller. (Also, when a statement is false one often finds the statement fails for *trivial* cases . . . checking such cases *first* gives a nice short proof of *falsity*.)
- When using theorems actually *write* them down. If possible write them in the form:

***If*** conditions  
***then*** conclusion.

(This is good practice anyway . . . for the final exam you will be expected to remember *important theorems*.)

In your proof, use *conditions* as a checklist. Check them *all* off. Then write

***Therefore*** conclusion.

- Surround your mathematics with enough *English* to make your proof read sensibly. Remember you are communicating with others. (Headings like *Case 1*. etc. *do* help.)
- End your proof with:  
*Therefore “ . . . ”*  
where “ . . . ” is what you set out to prove.
- Go back and check your proof is arranged in a logical sequence. Eliminate *backward logic* etc. Most proofs take *several* drafts to get right . . . and ironing out those wrinkles is what helps you perfect your artistry in proof-writing. (Of course, the proof you see below is only the *final* draft.)

**Problem.** Using the Mean Value Theorem show that  $|\sin a - \sin b| \leq |a - b|$  for all  $a, b \in \mathbb{R}$ .

**Solution.** We are required to prove:

$$|\sin a - \sin b| \leq |a - b|$$

for all  $a, b \in \mathbb{R}$ . There are three cases to consider:  $a = b$ ,  $a < b$ ,  $a > b$ .

**Case 1:**  $a = b$ . Here both sides of the statement are 0, and  $0 \leq 0$  is a true statement. Hence the statement is true if  $a = b$ .

**Case 2:**  $a < b$ . Here we apply the Mean Value Theorem which states:

***If***  $a < b$ ,  
 $f$  is continuous on  $[a, b]$ ,  
 $f$  is differentiable on  $(a, b)$   
***then*** there exists  $c \in (a, b)$  such that

$$\frac{f(a) - f(b)}{a - b} = f'(c).$$

We have:

$a < b$  (assumed),  
 $\sin(x)$  is continuous on  $\mathbb{R}$  and hence continuous on the closed interval  $[a, b]$ ,  
 $\sin(x)$  is differentiable on  $\mathbb{R}$  and hence differentiable on the open interval  $(a, b)$ .

Also,  $d/dx(\sin x) = \cos x$ . Therefore, by the Mean Value Theorem

there exists  $c \in (a, b)$  such that  $\frac{\sin a - \sin b}{a - b} = \cos c$ .

Now, whatever  $c$  is,  $|\cos c|$  is bounded by 1, i.e.

$$\left| \frac{\sin a - \sin b}{a - b} \right| = |\cos c| \leq 1.$$

Thus,

$$|\sin a - \sin b| \leq |a - b|$$

if  $a < b$ .

**Case 3:**  $a > b$ . Here notice that the statement we are required to prove is symmetric in  $a, b$ . Interchanging the roles of  $a, b$  in Case 2, shows the statement is true for this case also.

Therefore (having checked all cases that can arise),

$$|\sin a - \sin b| \leq |a - b| \text{ for all } a, b \in \mathbb{R}.$$

#### FINAL COMMENTS.

Note that it really is important to split off *Case 1*. in the above proof, since if  $a = b$  the open interval  $(a, b)$  is *empty* ... the condition: " $a < b$ " in the Mean Value Theorem *is* necessary – it ensures that there are points  $c$  in the interval  $(a, b)$ . Also, note that if  $a = b$  then  $a - b = 0$  (and so in such a case we *cannot* divide by  $a - b$ ). To cut a long story short, we cannot apply the Mean Value Theorem to prove *Case 1*.