In *Setting out a proof*[1] some guidelines on how to set out a proof were given. Two of the guidelines given were:

- When using theorems actually *write* them down. If possible write them in the form:
  ***If*** *conditions*
  ***then*** *conclusion.*
  (This is good practice anyway ... for the final exam you will be expected to remember *important theorems*.)
  In your proof, use *conditions* as a checklist. Check them *all* off. Then write
  ***Therefore*** *conclusion.*
- Go back and check your proof is arranged in a logical sequence. Eliminate *backward logic* etc. Most proofs take *several* drafts to get right ... and ironing out those wrinkles is what helps you perfect your artistry in proof-writing.

The first of these guidelines applies equally well to *definitions* as it does *theorems*. Below you will get some idea of how the multiple-draft procedure should work for you. Comments are italicised in square brackets.

**Problem.** If a square matrix $A$ satisfies $A^2 - 3A + I = O$ show that $A^{-1} = 3I - A$.

**Solution. ("First draft")** *[First we start with the most obvious way to start the problem.]*
We are required to prove:
If a square matrix $A$ satisfies $A^2 - 3A + I = O$ then $A^{-1} = 3I - A$.
Assume $A$ is a square matrix and $A^2 - 3A + I = O$. Then

$$\begin{aligned} I &= 3A - A^2 \\ A^{-1}.I &= A^{-1}(3A - A^2), && \text{if } A^{-1} \text{ exists} \\ A^{-1} &= 3I - A. \end{aligned}$$

*[At first we might think we are done ... but hang on we have used the condition "if $A^{-1}$ exists" in our "proof" and this was not a given condition. We should now be alerted that we need to look up conditions under which an inverse matrix exists or perhaps how the inverse of a matrix is defined.]*

**Solution. ("Final draft")**
We are required to prove:
If a square matrix $A$ satisfies $A^2 - 3A + I = O$ then $A^{-1} = 3I - A$.
By definition,
***If*** for a matrix $A$ there is a matrix $B$ satisfying

$$AB = BA = I$$

***then*** $B$ is called the *inverse* of $A$ and we may use the *notation* $A^{-1}$ for $B$.
Assume $A$ is a square matrix and $A^2 - 3A + I = O$. Then

$$I = 3A - A^2$$

[1]See `http://maths.uwa.edu.au/~gregg/FirstYear/` on the World Wide Web

Hence,

$$I = A.3I - A.A = 3I.A - A.A$$
$$I = A(3I - A) = (3I - A).A$$

So $3I - A$ satisfies the necessary conditions for the matrix inverse of $A$. Therefore

$$A^{-1} = 3I - A.$$

## Final Comments

You may notice that the adjective *square* was omitted in describing $A$ for the definition of $A^{-1}$ ... in fact, this is forced by the requirement that $AB = BA$. Also you may wonder about the *notation* $A^{-1}$ ... the *inverse* matrix better be *unique* when it exists – in fact, Theorem 1.3 of the notes guarantees this. Lastly, one might speculate as to whether it is enough to show that $AB = I$ (i.e. perhaps $BA = I$ follows once we have $AB = I$). The following example shows this is not necessarily the case if $A$ is *not square*:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which shows that for

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

there is a matrix $B$ such that $AB = I$ (where $I$ is the $2 \times 2$ identity matrix) but for such a $B$, $BA$ is not even defined. However if the matrix is *square* and the coefficients are in $\mathbb{R}$ it *is* true that once we have $AB = I$ then we must also have $BA = AB$ and $BA = I$.