First Year students often find the concept of *linear independence* hard to grasp. This is not surprising, since the definition was arrived at after much hind-sight had been developed. So, the definition is somewhat unnatural for the novice. To make the definition a little more palatable we introduce the following term.

> For a set of vectors $\{\underset{\sim}{\boldsymbol{u}}_1, \underset{\sim}{\boldsymbol{u}}_2, \dots, \underset{\sim}{\boldsymbol{u}}_n\}$ define the[1] *"linear independence equation"* to be
>
> $$\alpha_1 \underset{\sim}{\boldsymbol{u}}_1 + \alpha_2 \underset{\sim}{\boldsymbol{u}}_2 + \cdots + \alpha_n \underset{\sim}{\boldsymbol{u}}_n = \underset{\sim}{\boldsymbol{0}}.$$

Now we may express the definition for *linear independence* in the following way:

> A set of vectors $\{\underset{\sim}{\boldsymbol{u}}_1, \underset{\sim}{\boldsymbol{u}}_2, \dots, \underset{\sim}{\boldsymbol{u}}_n\}$ is *linearly independent* $\iff$ the *linear independence equation* has a *unique* solution (namely, the *trivial* solution).

This says, that to test whether a set of vectors is *linearly independent* we should first write down "the" corresponding *linear independence equation* and then *solve* it. We then decide the set of vectors is *linearly independent* if we find only one solution; otherwise we decide the set of vectors is *linearly dependent.* Let us apply the definition to a *proof* type problem.

Problem. Given $\{\underset{\sim}{\boldsymbol{u}}_1, \underset{\sim}{\boldsymbol{u}}_2, \underset{\sim}{\boldsymbol{u}}_3\}$ is *linearly independent*, is the set $\{\underset{\sim}{\boldsymbol{u}}_1+\underset{\sim}{\boldsymbol{u}}_2, \underset{\sim}{\boldsymbol{u}}_2+\underset{\sim}{\boldsymbol{u}}_3, \underset{\sim}{\boldsymbol{u}}_3+\underset{\sim}{\boldsymbol{u}}_1\}$ *linearly independent*?

Solution. We are given that $\{\underset{\sim}{\boldsymbol{u}}_1, \underset{\sim}{\boldsymbol{u}}_2, \underset{\sim}{\boldsymbol{u}}_3\}$ is *linearly independent* so it's *linear independence equation,*

$$\alpha_1 \underset{\sim}{\boldsymbol{u}}_1 + \alpha_2 \underset{\sim}{\boldsymbol{u}}_2 + \alpha_3 \underset{\sim}{\boldsymbol{u}}_3 = \underset{\sim}{\boldsymbol{0}}$$

has only the solution $\alpha_1 = 0, \quad \alpha_2 = 0, \quad \alpha_3 = 0.$

Is the set $\{\underset{\sim}{\boldsymbol{u}}_1+\underset{\sim}{\boldsymbol{u}}_2, \underset{\sim}{\boldsymbol{u}}_2+\underset{\sim}{\boldsymbol{u}}_3, \underset{\sim}{\boldsymbol{u}}_3+\underset{\sim}{\boldsymbol{u}}_1\}$ *linearly independent*? We write down it's *linear independence equation,* using variables that don't conflict with any already chosen:

$$\alpha(\underset{\sim}{\boldsymbol{u}}_1+\underset{\sim}{\boldsymbol{u}}_2) + \beta(\underset{\sim}{\boldsymbol{u}}_2+\underset{\sim}{\boldsymbol{u}}_3) + \gamma(\underset{\sim}{\boldsymbol{u}}_3+\underset{\sim}{\boldsymbol{u}}_1) = \underset{\sim}{\boldsymbol{0}}$$

which on rearrangement gives

$$(\alpha+\gamma)\underset{\sim}{\boldsymbol{u}}_1 + (\alpha+\beta)\underset{\sim}{\boldsymbol{u}}_2 + (\beta+\gamma)\underset{\sim}{\boldsymbol{u}}_3 = \underset{\sim}{\boldsymbol{0}}.$$

Comparing this equation with $\{\underset{\sim}{\boldsymbol{u}}_1, \underset{\sim}{\boldsymbol{u}}_2, \underset{\sim}{\boldsymbol{u}}_3\}$'s *linear independence equation,* which has just the trivial solution, we have:

$$\alpha+\gamma = 0, \quad \alpha+\beta = 0, \quad \beta+\gamma = 0$$

which itself is a system of equations (in the variables α, β, γ). Representing this system as an augmented matrix and reducing to RE we have:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

From the final RE matrix we see the system for α, β, γ has *full rank* (i.e. there are as many steps (echelons) as there can possibly be). So the solution is *unique* (and that solution must be the trivial one: $\alpha = 0, \quad \beta = 0, \quad \gamma = 0$).

Hence the set $\{\underset{\sim}{\boldsymbol{u}}_1+\underset{\sim}{\boldsymbol{u}}_2, \underset{\sim}{\boldsymbol{u}}_2+\underset{\sim}{\boldsymbol{u}}_3, \underset{\sim}{\boldsymbol{u}}_3+\underset{\sim}{\boldsymbol{u}}_1\}$ is *linearly independent.*

[1] The word *the* is perhaps improperly used here, since any list of n distinct non-conflicting variables can be used in place of the vector coefficients, $\alpha_1, \alpha_2, \dots, \alpha_n$.