INDUCTION

The *Principle of Mathematical Induction* states:

If we can show that both

- $P(1)$ is true; and
- for a general positive integer k,

$$\text{if } P(k) \text{ is true then } P(k+1) \text{ is also true;}$$

then we can conclude that $P(n)$ is true for all positive integers n.

This is *exactly* like proving that we can climb an infinite ladder (in the sense that given enough time we can get to any rung of the ladder), which we might do by proving the following analogous statements.

- We can get on the *first* (bottom) rung.
- We can get from any one rung (i.e. the kth rung) to the next rung (i.e. the $k+1$st rung).

Problem. Show that $1+3+5+\cdots+2n-1=n^2$ for all $n \in \mathbb{Z}^+$.

Solution. Let $P(n): \quad 1+3+5+\cdots+2n-1=n^2$.

- First we prove $P(1)$.

$$\begin{aligned}\text{LHS of } P(1) &= 1\\ &= 1^2 = \text{RHS of } P(1).\end{aligned}$$

So $P(1)$ is true.

- Now we prove that for any positive integer k *"if $P(k)$ is true then $P(k+1)$ is true."* So assume $P(k)$ is true, i.e.

$$1+3+5+\cdots+2k-1=k^2.$$

Now try to deduce $P(k+1)$:

$$\begin{aligned}\text{LHS of } P(k+1) &= 1+3+5+\cdots+2k-1+2(k+1)-1\\ &= \big(\text{LHS of } P(k)\big)+2(k+1)-1\\ &= \big(\text{RHS of } P(k)\big)+2k+1, \text{(by inductive assumption)}\\ &= k^2+2k+1\\ &= (k+1)^2\\ &= \text{RHS of } P(k+1).\end{aligned}$$

So $P(k+1)$ is true, *if $P(k)$ is true*.

- Hence, we have shown that $P(1)$ is true and that $P(k) \implies P(k+1)$, and so by induction $P(n)$ is true for all $n \in \mathbb{Z}^+$.