Really all one needs to know about finding derivatives are the *product* and *chain* rules, a few standard derivatives and that sometimes that a little rearrangement of your original function can help a lot.

In particular, it should be remembered that the *quotient* rule is derived from the other rules:

$$\begin{aligned}\left(\frac{u}{v}\right)' &= (u.v^{-1})'\\ &= u'.v^{-1} + u.(-1)v^{-2}v'\\ &= \frac{u'v - uv'}{v^2}.\end{aligned}$$

If the denominator function, v is a (non-trivial) power of another function, then the result of applying the *quotient* rule in differentiating u/v is an expression that can *always* be reduced. Rewriting u/v as uv^{-1} and applying the *product* rule is a better option, in such cases.

Example 1. Find the derivative of $f(x) = x/(\log x)^2$.

Solution. ***By the quotient rule***

$$f'(x) = \frac{1.(\log x)^2 - x.2(\log x).1/x}{(\log x)^4}$$

Notice that the *quotient* rule gives a somewhat messy expression that needs a good deal of tidying up. In particular, there is a common factor of $\log x$ in numerator and denominator. Students who use the *quotient* rule in such cases generally fail to observe that a common factor can be cancelled.

Solution. ***By the product rule***

$$\begin{aligned}f(x) &= \frac{x}{(\log x)^2} = x.(\log x)^{-2}\\ f'(x) &= 1.(\log x)^{-2} + x.(-2)(\log x)^{-3}.1/x\\ &= \frac{\log x - 2}{(\log x)^3}.\end{aligned}$$

Another useful idea is to rearrange the expression in x being differentiated so that x appears in as *few* places as possible.

Example 2. Find the derivative of $f(x) = \arctan\left(\frac{1+x^2}{1-x^2}\right)$.

Solution. Observe that

$$\frac{1+x^2}{1-x^2} = \frac{2-1+x^2}{1-x^2} = \frac{2}{1-x^2} - 1.$$

The right-most expression has x in just one place. Let's see how this is useful.

$$\begin{aligned}
f'(x) &= \frac{1}{1+\left(\frac{1+x^2}{1-x^2}\right)^2} \cdot \left(\frac{2}{1-x^2} - 1\right)' \\
&= \frac{1}{1+\left(\frac{1+x^2}{1-x^2}\right)^2} \cdot \frac{-2}{(1-x^2)^2} \cdot -2x \\
&= \frac{1}{(1-x^2)^2 + (1+x^2)^2} \cdot 4x \\
&= \frac{4x}{2+2x^4} \\
&= \frac{2x}{1+x^4}.
\end{aligned}$$

Now let's consider an example with an exponential in the denominator. Again the quotient rule leaves us with a common factor in numerator and denominator.

Example 3. Find the derivative of $f(x) = \dfrac{x+1}{e^x}$.

Solution. ***By the quotient rule***

$$f'(x) = \frac{1.e^x - (x+1).e^x}{(e^x)^2}$$

We should cancel the common factor e^x in numerator and denominator and tidy up. Rearranging $f(x)$ first and then applying the *product* rule is more direct.

Solution. ***By the product rule***

$$\begin{aligned}
f(x) &= \frac{x+1}{e^x} = (x+1).e^{-x} \\
f'(x) &= 1.e^{-x} + (x+1). - e^{-x} \\
&= (1-x-1)e^{-x} \\
&= -xe^{-x}
\end{aligned}$$

The advantages of using the *product* rule in this last example become even more apparent when we have to find the second and higher order derivatives of $f(x)$. Observe that each derivative of $f(x)$ is of the form:

$$\text{polynomial} \times e^{-x}$$

and the *product* rule is most conducive to finding the derivatives in this form.