Constant Coefficient Linear Differential Equations

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The meaning of ***linear***.

First we should define the term *linear*.

We say a function f is ***linear*** if and only if f satisfies the following two conditions:

(i) $f(x+y) = f(x) + f(y)$ for any x, y; and
(ii) $f(\alpha x) = \alpha f(x)$ for any scalar α and for any x.

We should really have been more careful with the above definition – we should have spelt out what the domain of f is … usually it is $\mathbb{R}$.

In a similar way we can say an *operator* $\mathfrak{L}$ is *linear*. Essentially it has the same definition as given above, except that the domain of an operator is a set of *functions* rather than a set of *numbers:*

We say an operator $\mathfrak{L}$ is ***linear*** if and only if $\mathfrak{L}$ satisfies the following two conditions:

(i) $\mathfrak{L}(f+g) = \mathfrak{L}(f) + \mathfrak{L}(g)$ for any functions f, g; and
(ii) $\mathfrak{L}(\alpha f) = \alpha\, \mathfrak{L}(f)$ for any scalar α and for any function f.

Of course we need to know how to add two functions and how to multiply a function by a *scalar.* These are defined *pointwise* as follows. Given functions f, g defined on some domain D and a scalar α the functions $f+g$ and αf are the functions such that at each $x \in D$,

$$(f+g)(x) = f(x) + g(x),$$
$$f(\alpha x) = \alpha f(x).$$

Example 1. *The operator* $D \equiv \dfrac{d}{dx}$ *is linear. For example, if* $f(x) = x^2$, $g(x) = \sin(x)$, $\alpha = 3$ *then*

$$\begin{aligned}
D\big(f(x) + g(x)\big) &= \frac{d}{dx}(x^2 + \sin x)\\
&= \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin x)\\
&= D\big(f(x)\big) + D\big(g(x)\big)\\
D\big(\alpha f(x)\big) &= \frac{d}{dx}(3x^2)\\
&= 3\frac{d}{dx}(x^2)\\
&= \alpha D\big(f(x)\big).
\end{aligned}$$

Of course there was nothing special about our choices of f, g and α here … any other differentiable functions and scalar would have done.

There are some conventions we employ with operators. Sometimes we leave brackets out. The convention is the operator operates on the following *term,* where a *term* is something that is delimited by a + or −, e.g.

$$\mathfrak{L}\, fg + h \text{ is the same as } \mathfrak{L}(fg) + h.$$

The *product rule* could be written as follows,

$$Duv = uDv + vDu,$$

where u, v are functions of x. We can form new operators by taking *linear combinations* of old ones, in the same way as with functions. The *constant linear operator* acts by multiplication. Thus,

$$(D + \alpha)p(x)q(x) = D\big(p(x)q(x)\big) + \alpha p(x)q(x),$$

where the brackets around $p(x)q(x)$ on the righthand side could have been omitted.

Also, *products* of *linear operators* turn out to be *linear,* i.e. if $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are *linear* then $\mathfrak{L}_1\,\mathfrak{L}_2$ is also *linear.* (For good practice, you should prove this.)

Okay, that's enough new notation for now. Now we'll list a few problems that come under the heading of *linear d.e.s with constant coefficients* and, I hope, you will see how the new notation helps us to think in the *right* way. As we attempt to solve them we will need to draw on some new ideas which we will list as lemmas or corollaries. We will add to these as we need them.

Problems.

1. Find the *general solution* of: $y' - 3y = 0.$
2. Find the *general solution* of: $y' - 3y = e^{2x}.$
3. Find the *general solution* of: $y' - 3y = x.$
4. Find the *general solution* of: $y' - 3y = 2e^{3x}.$
5. Find the *general solution* of: $y' - 3y = e^{2x} + x + 2e^{3x}.$
6. Find the *general solution* of: $y' - 3y = 1.$
7. Find the *general solution* of: $y'' + 5y' + 6y = e^{x}.$
8. Find the *general solution* of: $y'' + 5y' + 6y = e^{-2x}.$
9. Find the *general solution* of: $y'' + 5y' + 6y = e^{x} + e^{-2x}.$
10. Find the *general solution* of: $2y'' - 2y' + y = e^{-2x} + x.$
11. Solve the *Initial Value Problem:* $y'' + y = 1, \quad y(0) = 0,\ y'(0) = 0.$

Here are the first few lemmas we need.

Lemma 1. $(D-\beta)e^{\alpha x} = (\alpha-\beta)e^{\alpha x}$.

Proof.

$$\begin{aligned}(D-\beta)e^{\alpha x} &= De^{\alpha x} - \beta e^{\alpha x}\\ &= \alpha e^{\alpha x} - \beta e^{\alpha x}\\ &= (\alpha-\beta)e^{\alpha x}.\end{aligned}$$

□

From Lemma 1 we see that when a *constant coefficient linear operator* operates on $e^{\alpha x}$ the result is obtained by replacing all the Ds of the operator by αs.

Corollary 2. *$e^{\alpha x}$ satisfies $(D-\alpha)y = 0$.*

Proof. The lefthand side of the given d.e. is the operator $D-\alpha$ applied to y. Substituting $e^{\alpha x}$ for y and applying Lemma 1 we obtain

$$\begin{aligned}(D-\alpha)e^{\alpha x} &= (\alpha-\alpha)e^{\alpha x}\\ &= 0.\end{aligned}$$

□

We now have enough to attempt the first couple of problems.

Solution of Problem 1: *Find the* general solution *of:* $y' - 3y = 0$.
In operator notation we wish to solve: $(D-3)y = 0$. By Corollary 2, e^{3x} satisfies this d.e. Using the fact that $D-3$ is a *linear* operator, we see that, for any *real* A

$$\begin{aligned}(D-3)Ae^{3x} &= A.0\\ &= 0.\end{aligned}$$

So we find that a more general solution of the d.e. is $y = Ae^{3x}$. In fact it is the *most general* solution.

Solution of Problem 2: *Find the* general solution *of:* $y' - 3y = e^{2x}$.
In operator notation we wish to solve: $(D-3)y = e^{2x}$. From the previous problem's solution we have seen that $(D-3)Ae^{3x} = 0$. Now, by Lemma 1,

$$\begin{aligned}(D-3)e^{2x} &= (2-3)e^{2x}\\ &= -e^{2x}.\end{aligned}$$

So, using the fact that $D-3$ is a *linear* operator, we see that,

$$\begin{aligned}(D-3)(-e^{2x}) &= -(-e^{2x})\\ &= e^{2x}.\end{aligned}$$

Hence, $y = -e^{2x}$ is a *solution* of the given d.e. However, using the *linear* property of $D-3$ again, we see that

$$\begin{aligned}(D-3)(Ae^{3x} - e^{2x}) &= A.0 + e^{2x}\\ &= e^{2x}.\end{aligned}$$

So we find that a more general solution of the given d.e. is $y = Ae^{3x} - e^{2x}$. In fact it is the *most general* solution.

The solution of Problem 2 exhibits an important general strategy when solving *constant coefficient linear d.e.s.* For Problem 1 the righthand side was 0; such d.e.s are called *homogeneous.* The solution, e^{2x} of Problem 2 is just *one* of its solutions; to emphasise this, we call it a *particular* solution. Thus ...

> To find the *most general* solution of a *constant coefficient linear d.e.* that is *not homogeneous,*
>
> > we find a *particular* solution of the d.e. and *add* to it, the *most general* solution of the corresponding *homogeneous* d.e.

We need a new observation to solve Problem 3:

Lemma 3. *If $p(x)$ is a* polynomial *of* degree n *then $(D-\alpha)\big(p(x)\big)$ is a polynomial of degree at most n.*

Proof. When we differentiate a polynomial of degree at least 1, the result is again a polynomial but with its degree reduced by 1. If $\alpha \neq 0$ then $\alpha p(x)$ is a polynomial of the same degree as $p(x)$. Thus, since the sum of two polynomials is again a polynomial, $(D-\alpha)\big(p(x)\big)$ is a polynomial of degree no more than the degree of $p(x)$. (If $\alpha = 0$ the degree is reduced ... this is the reason for including the phrase *"at most"*.) □

Now we are ready to solve Problem 3.

Solution of Problem 3: *Find the* general solution *of:* $y' - 3y = x$.
In operator notation we wish to solve: $(D-3)y = x$. We have already seen that Ae^{3x} is a solution of the corresponding *homogeneous* d.e. Now Lemma 3 suggests that for a *particular* solution of the given d.e., we should try a *polynomial* of degree 1, since x is a *polynomial* of degree 1. So let's try,

$$y_p(x) = ax + b,$$

for some a, b:

$$\begin{aligned}(D-3)(ax+b) &= a - 3(ax+b)\\ &= -3ax + (a-3b).\end{aligned}$$

Since, two polynomials are equal (for all x), if and only if their coefficients are equal, we see that $ax + b$ is a *particular* solution if and only if

$$\begin{aligned}-3a &= 1\\ a - 3b &= 0,\end{aligned}$$

i.e. $a = -\frac{1}{3}, b = -\frac{1}{9}$. So, $y_p(x) = -\frac{1}{3}x - \frac{1}{9}$ is a *particular* solution.
Using the *linear* property of $D-3$, we see that

$$\begin{aligned}(D-3)(Ae^{3x} - \tfrac{1}{3}x - \tfrac{1}{9}) &= A.0 + x\\ &= x.\end{aligned}$$

Thus, the general solution of the given d.e. is $y = Ae^{3x} - \frac{1}{3}x - \frac{1}{9}$.

For Problem 2 the exponential function of the righthand side was not a solution of the corresponding homogeneous d.e. What if it is? If we try to solve Problem 4 using the same ideas that we used for solving Problem 2 then we end up wanting to divide by *zero!* So we need a fresh idea. The following lemma and corollary will help us here.

Lemma 4. $(D-\alpha)p(x).e^{\alpha x} = p'(x)e^{\alpha x}$.

Proof.

$$\begin{aligned}(D-\alpha)p(x).e^{\alpha x} &= Dp(x).e^{\alpha x} - \alpha p(x).e^{\alpha x}\\ &= p'(x)e^{\alpha x} + p(x).\alpha e^{\alpha x} - \alpha p(x).e^{\alpha x}\\ &= p'(x).e^{\alpha x}.\end{aligned}$$

□

Corollary 5. *$xe^{\alpha x}$ satisfies $(D-\alpha)y = e^{\alpha x}$.*

Proof. The lefthand side of the given d.e. is the operator $D-\alpha$ applied to y. Substituting $e^{\alpha x}$ for y and applying Lemma 4 we obtain

$$\begin{aligned}(D-\alpha)xe^{\alpha x} &= x'e^{\alpha x}\\ &= e^{\alpha x}.\end{aligned}$$

□

Now we are ready to solve a few more problems.

Solution of Problem 4: *Find the* general solution *of:* $y' - 3y = 2e^{3x}$.
In operator notation we wish to solve: $(D-3)y = 2e^{3x}$. We have already seen that $(D-3)Ae^{3x} = 0$. By Corollary 5,

$$(D-3)xe^{3x} = e^{3x}.$$

So, using the fact that $D-3$ is a *linear* operator, we see that,

$$(D-3)(2xe^{2x}) = 2e^{3x}.$$

Hence, $2xe^{3x}$ is a *particular* solution of the given d.e. and, using the *linear* property of $D-3$ again, we see that

$$\begin{aligned}(D-3)(Ae^{3x} + 2xe^{3x}) &= A.0 + 2e^{3x}\\ &= 2e^{3x}.\end{aligned}$$

So we find that a more general solution (and, in fact, the *most general* solution) of the given d.e. is $y = Ae^{3x} + 2xe^{3x}$.

Solution of Problem 5: *Find the* general solution *of:* $y' - 3y = e^{2x} + x + 2e^{3x}$.
In operator notation we wish to solve: $(D-3)y = e^{2x} + x + 2e^{3x}$. From previous problem solutions, we know

(i)	Ae^{3x}	is the *general solution* of	$(D-3)y = 0$;
(ii)	$-e^{2x}$	is a *particular solution* of	$(D-3)y = e^{2x}$;
(iii)	$-\frac{1}{3}x - \frac{1}{9}$	is a *particular solution* of	$(D-3)y = x$; and
(iv)	$2xe^{3x}$	is a *particular solution* of	$(D-3)y = 2e^{3x}$.

Thus, using the *linear* property of $D-3$, we see that

$$\begin{aligned}(D-3)&(Ae^{3x}-e^{2x}-\tfrac{1}{3}x-\tfrac{1}{9}+2xe^{3x})\\ &=(D-3)(Ae^{3x})+(D-3)(-e^{2x})+(D-3)(-\tfrac{1}{3}x-\tfrac{1}{9})+(D-3)(2xe^{3x})\\ &=A.0+e^{2x}+x+2e^{3x}\\ &=e^{2x}+x+2e^{3x}.\end{aligned}$$

Thus, the general solution of the given d.e. is $y=Ae^{3x}-e^{2x}-\frac{1}{3}x-\frac{1}{9}+2xe^{3x}$.

Solution of Problem 6: *Find the* general solution *of:* $y'-3y=1$.
In operator notation we wish to solve: $(D-3)y=1$. This problem could have been done with much less than we already know. The reason for attempting it here, is that 1 can be interpreted in two ways: *either* as e^{0x} *or* as a constant polynomial. Essentially either interpretation leads to the same method of solution: as a *particular* solution try $y_p=c$. Then

$$(D-3)c=-3c.$$

So $y_p=c$ is a solution of the given d.e. if and only if $c=-\frac{1}{3}$.
Thus, the general solution of the given d.e. is $y=Ae^{3x}-\frac{1}{3}$.

It's now time to do some second order problems. We first need to extend Lemma 1.

Corollary 6. *If $\alpha\neq\beta$ then $e^{\alpha x}$ and $e^{\beta x}$ are independent solutions of $(D-\alpha)(D-\beta)y=0$.*

Proof. Now the operator $D^2-(\alpha+\beta)D+\alpha\beta$ factorises as

$$(D-\alpha)(D-\beta)$$

or

$$(D-\beta)(D-\alpha),$$

i.e. the operators $(D-\alpha)(D-\beta)$ and $(D-\beta)(D-\alpha)$ are apparently identical. They are! This needs more careful examination but essentially this works because α,β are constants. Effectively, this means we need only solve the two first order d.e.s

$$(D-\alpha)y=0(D-\beta)y=0,$$

which have solutions $e^{\alpha x}$ and $e^{\beta x}$ respectively. Now

$$\begin{aligned}(D-\beta)(D-\alpha)e^{\alpha x}&=(D-\beta)(\alpha-\alpha)e^{\alpha x}\\ &=(D-\beta)0\\ &=0.\end{aligned}$$

So $e^{\alpha x}$ satisfies the given d.e. Similarly, by interchanging the roles of α and β, we see that $e^{\beta x}$ satisfies the given d.e. The two solutions are *independent* since $\alpha\neq\beta$. □

Solution of Problem 7: *Find the* general solution *of:* $y''+5y'+6y=e^x$.
In operator notation we wish to solve: $(D^2+5D+6)y=e^x$, which factorises as:

$$(D+3)(D+2)y=e^x.$$

By Corollary 6, the corresponding *homogeneous d.e.:* $(D+3)(D+2)y=0$, has e^{-3x} and e^{-2x} as *independent* solutions. Using the *linear* property of the operator $(D+3)(D+2)$ we see that any *linear combination* of e^{-3x} and e^{-2x} also satisfies the *homogeneous d.e.:*

$$\begin{aligned}(D^2+5D+6)(Ae^{-3x}+Be^{-2x})&=A.0+B.0\\ &=0.\end{aligned}$$

Thus, we see that $Ae^{-3x} + Be^{-2x}$, where $A, B \in \mathbb{R}$, is a *more general* solution of the corresponding *homogeneous d.e.;* in fact, it is the *most general* solution.

Now we must find a *particular* solution of the given d.e. The righthand side function e^x is not a solution of the corresponding *homogeneous d.e.;* so we need only check the effect of the operator $(D+3)(D+2)$ on e^x; this is most easily done by using Lemma 1 twice (see the *dangerous bend* that follows Lemma 1):

$$\begin{aligned}(D+3)(D+2)e^x &= (D+3)(1+2)(e^x)\\ &= (1+3)(1+2)(e^x) = 12e^x.\end{aligned}$$

Thus using the *linear* property of $(D+3)(D+2)$ we deduce that

$$(D+3)(D+2)(\tfrac{1}{12}e^x) = e^x.$$

So putting together the *most general* solution of the corresponding homogeneous d.e. and the *particular* solution of the given d.e. we deduce that the *most general* solution of the given d.e. is $y = Ae^{-3x} + Be^{-2x} + \frac{1}{12}e^x$.

Solution of Problem 8: *Find the* general solution *of:* $y'' + 5y' + 6y = e^{-2x}$.
In operator notation we wish to solve: $(D^2+5D+6)y = e^{-2x}$. From the previous problem we know that $Ae^{-3x}+Be^{-2x}$, where $A, B \in \mathbb{R}$, is the *most general* solution of the corresponding *homogeneous d.e.*

Now we must find a *particular* solution of the given d.e. This time, the righthand side function e^{-2x} *is* a solution of the corresponding *homogeneous d.e.;* so we let Corollary 5 guide us and observe the effect of the operator $(D+3)(D+2)$ on xe^{-2x}:

$$\begin{aligned}(D+3)(D+2)xe^{-2x} &= (D+3)e^{-2x}\\ &= (-2+3)e^{-2x}\\ &= e^{-2x}.\end{aligned}$$

(Notice we used Lemma 1 here too.) Thus, we don't have to adjust xe^{-2x} by any *constant* factor; it is already a *particular* solution of our given d.e.

So putting together the *most general* solution of the corresponding homogeneous d.e. and the *particular* solution of the given d.e. we deduce that the *most general* solution of the given d.e. is $y = Ae^{-3x} + Be^{-2x} + xe^{-2x}$.

Solution of Problem 9: *Find the* general solution *of:* $y'' + 5y' + 6y = e^x + e^{-2x}$.
In operator notation we wish to solve: $(D^2+5D+6)y = e^x+e^{-2x}$. From previous problem solutions, we know

(i)	$Ae^{-3x} + Be^{-2x}$	is the *general solution* of	$(D^2+5D+6)y = 0$;
(ii)	$\frac{1}{12}e^x$	is a *particular solution* of	$(D^2+5D+6)y = e^x$; and
(iii)	xe^{-2x}	is a *particular solution* of	$(D^2+5D+6)y = e^{-2x}$.

Thus, using the *linear* property of D^2+5D+6, we see that

$$\begin{aligned}&(D^2+5D+6)(Ae^{-3x} + Be^{-2x} + \tfrac{1}{12}e^x + xe^{-2x})\\ &\qquad= (D^2+5D+6)(Ae^{-3x}+Be^{-2x}) + (D^2+5D+6)(\tfrac{1}{12}e^x) + (D^2+5D+6)(xe^{-2x})\\ &\qquad= A.0 + B.0 + e^x + x + e^{-2x}\\ &\qquad= e^x + x + e^{-2x}.\end{aligned}$$

Thus, the general solution of the given d.e. is $y = Ae^{-3x} + Be^{-2x} + \frac{1}{12}e^x + xe^{-2x}$.

At this point we can already solve Problem 10; but doing so with the theory we have developed so far, will lead to solutions that involve the complex number i. It's not particularly nice to give *complex*-looking solutions which turn out, on closer examination, to be *real*. If the coefficients of a given *linear constant coefficient d.e.* are all *real* then it must possess *real* solutions. If a *homogeneous linear d.e.* with *constant real coefficients* has a *complex* solution then the *complex conjugate* of that solution is also a solution. This leads us to the next lemma.

Lemma 7. *If* $\{e^{\alpha x}, e^{\beta x}\}$ *is a* basis *of solutions for* $(D^2 + mD + n)y = 0$*, where* $m, n \in \mathbb{R}$ *and* α, β *are* complex *then*

(i) α, β *are* complex conjugates; *and*

(ii) we can take $\{e^{ax}\cos bx, e^{ax}\sin bx\}$*, where* $\alpha = a + bi$*, as an* alternative basis *of solutions.*

Proof. Since $e^{\alpha x}$ and $e^{\beta x}$ are solutions of $(D^2+mD+n)y = 0$, the operator D^2+mD+n factorises as

$$(D-\alpha)(D-\beta),$$

i.e.

$$m = -(\alpha + \beta) \tag{1}$$

$$n = \alpha\beta. \tag{2}$$

If we suppose that $\alpha = a + bi$ and $\beta = c + di$ then (1) tells us that $d = -b$, since m is *real;* and so

$$\begin{aligned} n = \alpha\beta &= (a + bi)(c - bi) \\ &= (ac + b^2) + (a - c)bi \end{aligned}$$

is *real* only if $a = c$ ($b \neq 0$, since we were given that α is *complex*). Thus, $\beta = a - bi$, the *complex conjugate* of α.

Since the given d.e. is *linear homogeneous* any *linear combination* of $e^{\alpha x}, e^{\beta x}$ is also a solution. This is why we chose to say that $\{e^{\alpha x}, e^{\beta x}\}$ was a *basis* of solutions in the *hypothesis* of the lemma. By standard theory of complex numbers

$$\begin{aligned} e^{\alpha x} &= e^{(a+ib)x} = e^{ax}.e^{ibx} = e^{ax}(\cos bx + i\sin bx) \\ e^{\beta x} &= e^{(a-ib)x} = e^{ax}.e^{-ibx} = e^{ax}(\cos bx - i\sin bx). \end{aligned}$$

Thus we see that the functions $e^{\alpha x}, e^{\beta x}$ are *linear combinations* of the functions

$$e^{ax}\cos bx,\ e^{ax}\sin bx.$$

Standard linear algebra tells us that this situation can only have arisen simultaneously with the fact that $e^{ax}\cos bx, e^{ax}\sin bx$ are *linear combinations* of $e^{\alpha x}, e^{\beta x}$:

$$\begin{aligned} e^{ax}\cos bx &= \tfrac{1}{2}(e^{\alpha x} + e^{\beta x}) \\ e^{ax}\sin bx &= \tfrac{1}{2i}(e^{\alpha x} - e^{\beta x}). \end{aligned}$$

Thus, any *linear combination* of $e^{\alpha x}, e^{\beta x}$ can be written as a *linear combination* (but with different coefficients) of $e^{ax}\cos bx, e^{ax}\sin bx$, and *vice versa*. In other words we may replace the *basis* of solutions by

$$\{e^{ax}\cos bx, e^{ax}\sin bx\}.$$

□

Now we are in a good position to solve Problem 10.

Solution of Problem 10: *Find the* general solution *of:* $2y'' - 2y' + y = e^{-2x} + x$.
In operator notation we wish to solve: $(2D^2 - 2D + 1)y = e^{-2x} + x$. Thus we need to find the *general solution* of

$$(2D^2 - 2D + 1)y = 0, \tag{3}$$

and *particular* solutions of each of

$$(2D^2 - 2D + 1)y = e^{-2x}, \tag{4}$$
$$(2D^2 - 2D + 1)y = x. \tag{5}$$

Factorising $2D^2 - 2D + 1$:

$$\begin{aligned} 2D^2 - 2D + 1 = 2(D^2 - D + \tfrac{1}{2}) &= 2\big((D - \tfrac{1}{2})^2 - \tfrac{1}{4} + \tfrac{1}{2}\big) \\ &= 2\big((D - \tfrac{1}{2})^2 + \tfrac{1}{4}\big) \\ &= 2(D - \tfrac{1}{2} + \tfrac{1}{2}i)(D - \tfrac{1}{2} - \tfrac{1}{2}i), \end{aligned}$$

we see that $\{e^{(\frac{1}{2} - \frac{1}{2}i)x}, e^{(\frac{1}{2} + \frac{1}{2}i)x}\}$ is a *basis* of solutions for (3). By Lemma 7, we may choose $\{e^{\frac{1}{2}x}\cos\frac{1}{2}x, e^{\frac{1}{2}x}\sin\frac{1}{2}x\}$ as an *alternative* basis of solutions of (3). Thus the *general solution* of (3) is

$$Ae^{\frac{1}{2}x}\cos\tfrac{1}{2}x + Be^{\frac{1}{2}x}\sin\tfrac{1}{2}x.$$

Now, e^{-2x} is not a solution of the corresponding *homogeneous d.e.* (3); so, in order to solve (4), we investigate the effect of the operator $2D^2 - 2D + 1$ on e^{-2x} (see the *dangerous bend* following Lemma 1):

$$\begin{aligned} (2D^2 - 2D + 1)e^{-2x} &= (2(-2)^2 - 2(-2) + 1)e^{-2x} \\ &= 13e^{-2x}. \end{aligned}$$

Thus using the *linear* property of $2D^2 - 2D + 1$ a *particular* solution of (4) is

$$\tfrac{1}{13}e^{-2x}.$$

Now we find a *particular* solution of (5). Since x is a *polynomial* of degree 1 we should try a *polynomial* of degree 1, i.e. try $y_p = ax + b$:

$$\begin{aligned} (2D^2 - 2D + 1)(ax + b) &= 2D^2(ax + b) - 2D(ax + b) + (ax + b) \\ &= 0 - 2a + (ax + b) \\ &= ax + (b - 2a). \end{aligned}$$

Thus $y_p = ax + b$ is a *particular* solution of (5) if and only if

$$\begin{aligned} a &= 1 \\ b - 2a &= 0, \end{aligned}$$

i.e. $a = 1, b = 2$. So $y_p = x + 2$.

In summary, we have discovered:

(i)	$Ae^{\frac{1}{2}x}\cos\frac{1}{2}x + Be^{\frac{1}{2}x}\sin\frac{1}{2}x$	is the *general solution* of	$(2D^2 - 2D + 1)y = 0$;
(ii)	$\frac{1}{13}e^{-2x}$	is a *particular solution* of	$(2D^2 - 2D + 1)y = e^{-2x}$; and
(iii)	$x + 2$	is a *particular solution* of	$(2D^2 - 2D + 1)y = x$.

Using the *linear* property of $2D^2 - 2D + 1$, we see that

$$\begin{aligned}(2D^2 - 2D + 1)(Ae^{\frac{1}{2}x}\cos\tfrac{1}{2}x + Be^{\frac{1}{2}x}\sin\tfrac{1}{2}x + \tfrac{1}{13}e^{-2x} + x + 2)\\ &= A.0 + B.0 + e^{-2x} + x\\ &= e^{-2x} + x.\end{aligned}$$

Thus, the general solution of the given d.e. is $y = Ae^{\frac{1}{2}x}\cos\frac{1}{2}x + Be^{\frac{1}{2}x}\sin\frac{1}{2}x + \frac{1}{13}e^{-2x} + x + 2$.

You will have observed that the number of undetermined constants of a *general solution* of each d.e. has equalled the *degree* of the d.e., and that the undetermined constants always turned up as part of the *general solution* of the corresponding *homogeneous d.e.* We have one problem left to do and it is an *Initial Value Problem.* What's that? Essentially it is similar to the *general solution* problems except there are sufficient additional conditions to determine the constants; it is called an *"initial value"* problem because the extra conditions are all values of functions of x at *zero,* namely $y(0), y'(0), \dots, y^{(n-1)}(0)$ where n is the *degree* of the d.e.

Solution of Problem 11: *Find the* general solution *of:* $y'' + y = 1,\ y(0) = 0,\ y'(0) = 0.$
We start by finding the *general solution* of the given d.e., which in operator notation is: $(D^2+1)y = 1$. Thus we need to find the *general solution* of

$$(D^2 + 1)y = 0, \tag{6}$$

and a *particular* solution of

$$(D^2 + 1)y = 1. \tag{7}$$

The operator $D^2 + 1$ factorises as: $(D + i)(D - i)$. So we see that $\{e^{-ix}, e^{ix}\}$ is a *basis* of solutions for (6). By Lemma 7, we may choose $\{\cos x, \sin x\}$ as an *alternative* basis of solutions of (6). Thus the *general solution* of (6) is

$$A\cos x + B\sin x.$$

Now, $1 = e^{0x}$ is not a solution of the corresponding *homogeneous d.e.* (6); so, in order to solve (7), we investigate the effect of the operator $D^2 + 1$ on 1:

$$(D^2 + 1)1 = 1.$$

Thus, we see that there is no need to adjust 1 by any *constant* factor, $y_p = 1$ is already a *particular* solution of (7).

In summary, we have discovered:

(i)	$A\cos x + B\sin x$	is the *general solution* of	$(D^2 + 1)y = 0$; and
(ii)	1	is a *particular solution* of	$(D^2 + 1)y = 1$.

Using the *linear* property of $D^2 + 1$, we see that

$$\begin{aligned}(D^2 + 1)(A\cos x + B\sin x + 1) &= A.0 + B.0 + 1\\ &= 1.\end{aligned}$$

Thus, the *general solution* of the given d.e. is $y = A\cos x + B\sin x + 1$.

Now we use the *initial conditions:* $y(0) = 0,\ y'(0) = 0$, to evaluate A and B. Firstly, $y(0) = 0$ implies that

$$\begin{aligned} 0 &= A\cos 0 + B\sin 0 + 1 \\ &= A + 1. \end{aligned}$$

So $A = -1$. Thus

$$\begin{aligned} y'(x) &= -A\sin x + B\cos x \\ &= \sin x + B\cos x. \end{aligned}$$

Hence, $y'(0) = 0$ implies that $0 = \sin 0 + B\cos 0 = B$. So we have determined the constants of the *general solution* to be: $A = -1, B = 0$, i.e. the solution to the given *Initial Value Problem* is: $y = -\cos x + 1$.

Well, now we have solved all the problems that we intended to solve, but we haven't covered all the variations possible. What solutions should we expect if the linear operator factorises as: $(D-\alpha)^2$? ... and what form of *particular* solution should we try if the righthand side function involves sin and cos? Really, all the ideas needed have already been presented ... they just need to be extended a bit. For example, the first question can be answered using the following generalisation of Corollary 5, which can be proved by induction.

Corollary 8. *Let $k \in \mathbb{N}$. Then $x^k e^{\alpha x}$ satisfies $(D-\alpha)^{k+1}y = 0$.*

My favourite way of doing problems with righthand side functions involving sin and cos is to introduce two new operators $\mathfrak{Re}$ (the *real part* function) and $\mathfrak{Im}$ (the *imaginary part* function). These turn out to be *linear* over $\mathbb{R}$ (i.e. all our scalars need to be *real*) and what's more they *commute* with the *constant coefficient* differential operators (i.e. you can swap the order in which they appear in an equation). The idea is, given a d.e. of form

$$\mathfrak{L}y = f(x),$$

where say, for example, $f(x) = 5\cos 3x + \sin 3x$, we recognise that

$$\begin{aligned} \cos 3x &= \mathfrak{Re}(e^{i3x}) \\ \sin 3x &= \mathfrak{Im}(e^{i3x}). \end{aligned}$$

A *particular* solution can be pieced together by first finding a *particular* solution over $\mathbb{C}$ to

$$\mathfrak{L}y = e^{i3x}.$$

Call such a solution y_p. Then $5\,\mathfrak{Re}(y_p) + \mathfrak{Im}(y_p)$ is a *particular* solution of

$$\mathfrak{L}y = 5\cos 3x + \sin 3x.$$

Why? ... Well, this is how it all works. Starting with:

$$\mathfrak{L}y_p = e^{i3x}.$$

Then

$$\begin{aligned}
\mathfrak{L}\big(5\,\Re(y_p) + \Im(y_p)\big) &= 5\,\mathfrak{L}(\Re\, y_p) + \mathfrak{L}(\Im\, y_p) && \text{since } \mathfrak{L} \text{ is } \textit{linear} \\
&= 5\,\Re(\mathfrak{L}\, y_p) + \Im(\mathfrak{L}\, y_p) && \text{since } \Re \text{ and } \Im \textit{ commute} \text{ with } \mathfrak{L} \\
&= 5\,\Re(e^{i3x}) + \Im(e^{i3x}) \\
&= 5\cos 3x + \sin 3x.
\end{aligned}$$

The beauty of this method is that it reduces such problems to the case where the righthand side functions are *exponential,* problems that we already know how to do! ... We don't have to work out a whole new set of techniques to deal with them.