The University of Western Australia
DEPARTMENT OF MATHEMATICS & STATISTICS

UWA ACADEMY
FOR YOUNG MATHEMATICIANS

Number Theory II: Problems

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1. For each of the following pairs of integers a, b use the *Euclidean Algorithm* to find $d = (a, b)$ and find a pair of integers x, y such that $ax + by = d$.

   (i) $a = 85$, $b = 41$; (ii) $a = 2613$, $b = 637$.

2. Show that if there exist integers x, y such that $ax + by = 1$ then $(a, b) = 1$.

3. Show that $(3k + 2, 5k + 3) = 1$ for any integer k.

4. Show that $(a, a + 2) = 2$ if a is even and $(a, a + 2) = 1$ otherwise.

5. Show that if $(a, b) = 1$ then $(a + b, a - b) = 1$ or 2.

6. Find all solutions to the following *Diophantine Equations*.

   (i) $2x + 5y = 11$. (ii) $12x + 18y = 50$. (iii) $202x + 74y = 7638$.

   Does equation (iii) have a solution in *positive* integers x, y?

7. A grocer orders apples and oranges at a total cost of \$8.39. If apples cost 25c each and oranges cost 18c each, how many of each type of fruit did the grocer order?

8. An apartment block has units at two rates: most rent at \$87/week, but a few rent at \$123/week. When all are rented the gross income is \$8733/week. How many units of each type are there?

∗9. Find all integers x, y satisfying: $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{14}$.

∗10. When Jane is one year younger than Betty will be when Jane is half as old as Betty will be when Jane is twice as old as Betty is now, Betty will be three times as old as Jane was when Betty was as old as Jane is now.
One is in her teens and ages are in completed years. How old are they?

11. Solve the adjacent *alphametic* (an addition in which: each letter stands for a different digit; and left-most digits of a number are not allowed to be 0).

$$\begin{array}{cccc} & A & H & A \\ & A & H & A \\ & & & A \\ & W & A & G \\ \hline H & A & H & A \end{array}$$

12. About all we know of Diophantus' life is his epitaph from which his age at death is to be deduced:

> Diophantus spent one-sixth of his life in childhood, one-twelfth in youth, and another one-seventh in bachelorhood. A son was born five years after his marriage and died four years before his father at half his father's age.

13. Augustus de Morgan, a nineteenth-century mathematician, stated:

> I was x years old in the year x^2.

When was he born?

14. Prove that for every integer n:

(i) $3 \mid n^3 - n$;
(ii) $5 \mid n^5 - n$;
(iii) $7 \mid n^7 - n$;
(iv) $11 \mid n^{11} - n$.

Show that $n^9 - n$ is not necessarily divisible by 9. *Hint:* Try $n = 2$.
What general result is suggested by the above?

15. Prove that $3^{6n} - 2^{6n}$ is divisible by 35, for every positive integer n.

∗16. What is the final digit of $7^{7^{7^{7^{7^{7^{7}}}}}}$.

17. Prove that for any natural number n that

$$17 \text{ divides } 2^n.3^{2n} - 1.$$

18. Prove that for any natural number n

$$17^n - 12^n - 24^n + 19^n$$

is divisible by 35.

∗19. Prove that $5^{99} + 11^{99} + 17^{99}$ is divisible by 33.

∗20. What is the final digit of $(((((((((7^7)^7)^7)^7)^7)^7)^7)^7)^7)$? (7 occurs as a power 10 times.)

∗21. *(17th International Olympiad, 1975, Problem 4)* When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B.
Hints: First show that the sum of the digits of B is fairly small (in fact: less than 16). Then use the fact that, for any natural number N,

$$N \equiv (\text{sum of the digits of } N) \pmod 9.$$