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Number Theory I: Problems with Solutions

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1. Determine *simple* rules for divisibility by each of the following natural numbers:

   (i) 2
   (ii) 3
   (iii) 4
   (iv) 5
   (v) 6
   (vi) 8
   (vii) 9
   (viii) 10
   (ix) 11
   (x) 12
   (xi) 15

   Note: there is a rule for 7, but it's complicated and it is not much better than straight division.

   Solution. We will use congruences for some of the solutions here. Remember, $m \mid n$ *if and only if* $n \equiv 0 \pmod{m}$.

   (i) Every natural number n can be written as $10q + r$, where r is the remainder after n is divided by 10, i.e. r is the last digit of n. Now $2 \,|\, 10$. So

   $$2 \,|\, n \quad \text{if and only if} \quad 2 \,|\, r,$$

   where r is the last digit of n. In other words, *2 divides n if and only if n ends in 0 or 2 or 4 or 6 or 8.*

   (ii) Suppose the decimal representation of n is $a_k\, a_{k-1}\, \dots\, a_0$. Then

   $$n = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10 a_1 + a_0.$$

   Now $10 \equiv 1 \pmod{3}$; so

   $$10^\ell \equiv 1 \pmod{3},$$

   for *any* natural number ℓ. So

   $$\begin{aligned} n &= 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10 a_1 + a_0 \\ &\equiv a_k + a_{k-1} + \cdots + a_1 + a_0 \pmod{3}. \end{aligned}$$

   Hence,

   $$3 \,|\, n \quad \text{if and only if} \quad 3 \,|\, a_k + a_{k-1} + \cdots + a_1 + a_0.$$

   In other words, *3 divides n if and only if 3 divides the sum of the digits of n.*

   (iii) Every natural number n can be written as $100q + r$, where r is the remainder after n is divided by 100, Now $4 \,|\, 100$. *(Note that 4 does not divide 10.)* So

   $$4 \,|\, n \quad \text{if and only if} \quad 4 \,|\, r,$$

   where r consists of the last two digits of n.

(iv) Every natural number n can be written as $10q + r$, where r is the remainder after n is divided by 10, i.e. r is the last digit of n. Now $5 \,\big|\, 10$. So

$$5 \,\big|\, n \quad \text{if and only if} \quad 5 \,\big|\, r,$$

where r is the last digit of n. In other words, *5 divides* n *if and only if* n *ends in a 0 or a 5.*

(v) $6 = \operatorname{lcm}(2, 3)$, so to check divisibility by 6, we check for divisibility by 2 *and* 3.

(vi) Every natural number n can be written as $1000q + r$, where r is the remainder after n is divided by 1000, Now $8 \,\big|\, 1000$. *(Note that 8 does not divide 100.)* So

$$8 \,\big|\, n \quad \text{if and only if} \quad 8 \,\big|\, r,$$

where r consists of the last three digits of n.

(vii) Just as we did for 3, suppose the decimal representation of n is $a_k\, a_{k-1} \, \ldots \, a_0$. Then

$$n = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10a_1 + a_0.$$

Now $10 \equiv 1 \pmod 9$; so

$$10^\ell \equiv 1 \pmod 9,$$

for *any* natural number ℓ. So

$$\begin{aligned} n &= 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10a_1 + a_0 \\ &\equiv a_k + a_{k-1} + \cdots + a_1 + a_0 \pmod 9. \end{aligned}$$

Hence,

$$9 \,\big|\, n \quad \text{if and only if} \quad 9 \,\big|\, a_k + a_{k-1} + \cdots + a_1 + a_0.$$

In other words, *9 divides* n *if and only if 9 divides the sum of the digits of* n.

(viii) Well of course everyone knows that: *10 divides* n *if and only if the last digit of* n *is 0;* but let's see this another way. Like the case for 6, $10 = \operatorname{lcm}(2, 5)$, so to check divisibility by 10, we check for divisibility by 2 *and* 5. In other words,

$10 \,\big|\, n$ if and only if n has 0 or 2 or 4 or 6 or 8 as last digit *and* n has 0 or 5 as last digit.

(ix) Suppose the decimal representation of n is $a_k\, a_{k-1} \, \ldots \, a_0$. Then

$$n = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10a_1 + a_0.$$

Now $10 \equiv -1 \pmod{11}$; so

$$10^\ell \equiv (-1)^\ell \pmod{11},$$

for *any* natural number ℓ. So

$$\begin{aligned} n &= 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10a_1 + a_0 \\ &\equiv (-1)^k a_k + (-1)^{k-1} a_{k-1} + \cdots - a_1 + a_0 \pmod{11}. \end{aligned}$$

Hence,

$$11 \,\big|\, n \quad \text{if and only if} \quad 11 \,\big|\, a_0 - a_1 + a_2 - \cdots + (-1)^k a_k.$$

In other words, *11 divides* n *if and only if 11 divides the* difference *of the sum of* odd-place *digits of* n *and the sum of the* even-place *digits of* n.

(x) $12 = \operatorname{lcm}(3,4)$, so to check divisibility by 12, we check for divisibility by 3 *and* 4.

(xi) $15 = \operatorname{lcm}(3,5)$, so to check divisibility by 15, we check for divisibility by 3 *and* 5.

2. The number $739ABC$ is divisible by 7, 8 and 9. What values can A, B and C take?

Solution. If two natural numbers a, b have *greatest common divisor* equal to 1, then a, b are said to be *relatively prime.* The numbers 7, 8 and 9 are *pairwise relatively prime,* i.e. any pair are relatively prime. So their lowest common multiple is simply the product of all three. Written mathematically:

$$\operatorname{lcm}(7,8,9) = 7.8.9 = 504.$$

We must choose a number of the form $739ABC$ such that it is a multiple of 7, 8 and 9; i.e. we must choose a number of the form $739ABC$ that is divisible by $\operatorname{lcm}(7,8,9) = 504$. Now 739 000 gives remainder 136 on division by 504. Hence the numbers $739ABC$ we are looking for, are of form

$$739\,000 - 136 + k.504$$

where k is an integer. We can see that k can only be 1 or 2. If $k = 1$, we get the number 739 368 so that one solution for A, B, C is

$$A = 3, B = 6, C = 8;$$

and if $k = 2$ we get the number 739 872 so that another solution for A, B, C is

$$A = 8, B = 7, C = 2.$$

3. Show that $x^2 - y^2 = 2$ has no integer solutions.

Solution. We may as well assume that x, y are not negative. Now 2, being prime can only be written as the product of two natural numbers in one way: $2 = 1.2$; and

$$x^2 - y^2 = (x - y)(x + y).$$

By our assumption $x + y \geq x - y$. Hence

$$x - y = 1$$
$$x + y = 2.$$

Solving these equations simultaneously, we get $x = \frac{3}{2}$, $y = \frac{1}{2}$ (which are not integers). So there can be no integer solutions of $x^2 - y^2 = 2$.

4. Prove that for every integer n:

(i) $3 \mid n^3 - n$;
(ii) $6 \mid n(n-1)(2n-1)$;
(iii) $30 \mid n^5 - n$;
(iv) $120 \mid n^5 - 5n^3 + 4n$;
(v) $4 \nmid n^2 + 2$;
(vi) $121 \nmid n^2 + 3n + 5$.

Solution.

(i) Since the integers $n-1, n, n+1$ are consecutive 3 divides exactly one of them. Thus

$$3 \mid (n-1).n.(n+1) = n^3 - n.$$

(ii)
- Either $2 \mid n$ or $2 \mid n-1$; so $2 \mid n(n-1)(2n-1)$.
- Similarly, at least one of the three consecutive integers $n-1, n, n+1$ is divisible by 3. Suppose $3 \mid n+1$; then $3 \mid 2(n+1)-3 = 2n-1$. So, *if* 3 divides $n+1$ *then* 3 divides $2n-1$. Hence, since at least one of $n-1, n, n+1$ is divisible by 3, we have at least one of $n-1, n, 2n-1$ is divisible by 3. So $3 \mid n(n-1)(2n-1)$.

Thus, since $2 \mid n(n-1)(2n-1)$ and $3 \mid n(n-1)(2n-1)$, we have: $6 = \operatorname{lcm}(2,3)$ divides $n(n-1)(2n-1)$.

(iii) 5 divides exactly one of the five consecutive integers $n-2, n-1, n, n+1, n+2$. In terms of congruences, exactly one of $n-2, n-1, n, n+1, n+2$ is *congruent* to 0 *modulo* 5. Thus:

$$\begin{aligned} n^5 - n = n(n^4-1) = n(n^2-1)(n^2+1) &= n(n-1)(n+1)(n^2+1) \\ &\equiv n(n-1)(n+1)(n^2-4) \pmod 5 \\ &\equiv n(n-1)(n+1)(n-2)(n+2) \pmod 5 \\ &\equiv 0 \pmod 5 \end{aligned}$$

So $5 \mid n^5 - n$.

(iv) Let $N = n^5 - 5n^3 + 4n$. Then

$$\begin{aligned} N = n^5 - 5n^3 + 4n &= n(n^4 - 5n^2 + 4) \\ &= n(n^2-1)(n^2-4) \\ &= n(n-1)(n+1)(n-2)(n+2). \end{aligned}$$

So N is the product of the five consecutive integers: $(n-2), (n-1), n, (n+1), (n+2)$. Exactly one of these integers is divisible by 5, at least one is divisible by 4 and at least one is divisible by 3. Further, if $k \in \{-2,-1,0,1,2\}$ and $n+k$ is a factor of N that is divisible by 4, then either $n+k-2$ or $n+k+2$ is a factor of N both of which are even. That is, either $(n+k)(n+k-2) \mid N$ or $(n+k)(n+k+2) \mid N$; in either case, we see that $8 \mid N$. Hence, $120 = \operatorname{lcm}(3,5,8)$ divides $N = n^5 - 5n^3 + 4n$.

(v) Either $n = 2k$ or $n = 2k+1$ for some integer k. If $n = 2k$ then $n^2 + 2 = 4k^2 + 2 \equiv 2 \pmod 4$. On the other hand, if $n = 2k+1$ then $n^2 + 2 = 4k^2 + 4k + 3 \equiv 3 \pmod 4$. In either case, $4 \nmid n^2 + 2$.

(vi) Observe that

$$n^2 + 3n + 5 = (n+7)(n-4) + 33,$$

so that $11 \mid n^2+3n+5$ if and only if $11 \mid (n+7)(n-4)$. Thus, if $11 \nmid (n+7)(n-4)$ then 11 (and hence 121) does *not* divide n^2+3n+5. So, assume 11 divides $(n+7)(n-4)$. Then $11 \mid n+7$ or $11 \mid n-4$; but then 11 must divide *both* of $n+7$ and $n-4$, since

$$n + 7 \equiv n - 4 \pmod{11}.$$

Thus, $121 \mid (n+7)(n-4)$. However, $121 \nmid 33$. So $121 \nmid n^2+3n+5 = (n+7)(n-4)+33$. Hence, in all cases, $121 \nmid n^2 + 3n + 5$.

5. Prove that for all integers a and b: 3 divides $(a+b)^3 - a^3 - b^3$.

Solution.

$$\begin{aligned} (a+b)^3 - a^3 - b^3 &= a^3 + 3a^2b + 3ab^2 + b^3 - a^3 - b^3 \\ &= 3(a^2b + ab^2) \end{aligned}$$

So, since $3 \mid 3(a^2b+ab^2)$, we have $3 \mid (a+b)^3 - a^3 - b^3$.

6. Is 167 prime?

   Solution. Suppose 167 is composite. Then it has a divisor $m > 1$. Then m and $167/m$ both divide 167. The lesser of m and $167/m$ is at most $\sqrt{167}$ and must have a prime decomposition consisting of primes less than or equal to $\sqrt{167}$. Now $\sqrt{167} < 13$ and it is easy to check that none of the primes 2, 3, 5, 7 or 11 divide 167. So we have a contradiction. That is, 167 cannot be composite; and since it is not 1 it must be prime.

7. Show that if $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime decomposition of the positive integer n, then the number of divisors of n (including 1 and n) is $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$.

   Solution. Observe that every divisor of n is of the form

   $$p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k},$$

   where $f_1, \ldots, f_k$ are all integers, and

   $$\begin{gathered} 0 \le f_1 \le e_1 \\ 0 \le f_2 \le e_2 \\ \vdots \\ 0 \le f_k \le e_k. \end{gathered}$$

   The correct term for a number e that occurs as an index as in p^e is *exponent*. We say that, e is the *exponent* of p in the expression p^e.

   In particular, notice that $1 = p_1^0 p_2^0 \cdots p_k^0$ (i.e. in this case, $f_1 = f_2 = \cdots = f_k = 0$); and n is the divisor of n with $f_1 = e_1$, $f_2 = e_2$, ..., $f_k = e_k$. So the number of divisors of n is *the number of choices of* f_1 times *the number of choices of* f_2 times ... times *the number of choices of* f_k. Now the set of choices for f_1 is $\{0, 1, 2, \ldots, e_1\}$ – there are $e_1 + 1$ such choices. So, in general, the the number of divisors of n is $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$.

8. Which positive integers have exactly three positive divisors?

   Solution. From the previous problem the number of positive divisors is $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$. The only way 3 can be expressed this way is

   $$2 + 1.$$

   That is, we must have $e_1 = 2$ (and $k = 1$). Hence, integers n with three positive divisors are of form p_1^2 (that is, they are *squared primes*).

9. Which positive integers have exactly four positive divisors?

   Solution. 4 can be expressed in the form $(e_1+1)(e_2+1) \cdots (e_k+1)$ in two ways, namely:

   $$3 + 1 \quad \text{and} \quad (1 + 1)(1 + 1).$$

   Hence, integers n with four positive divisors are of form p_1^3 (*cubed primes*) or $p_1^2 p_2^2$ (*squares of products of two primes*).

10. Show that a natural number n is an exact square if and only if it has an odd number of divisors.

Solution. Now n has a prime decomposition of the form

$$p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}.$$

Also, n is an exact square *if and only if* all the exponents, $e_1, \dots, e_k$ are *even*; in which case, the product $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$ is a product of *odd* numbers and so is itself *odd*. However, by the result of the previous question, the number of divisors of n is $(e_1+1)(e_2+1) \cdots (e_k+1)$. Thus n is an exact square *if and only if* the number of divisors of n is *odd*.

Solution. ***(Alternative)*** Observe that the divisors of n occur in pairs $d, n/d$. Thus:

> n has an odd number of divisors *if and only if* $d = n/d$ for some divisor d of n.

However, $d = n/d$ for some divisor d of n *if and only if* $n = d^2$ for some divisor d of n; and $n = d^2$ for some divisor d of n is *exactly* what we mean when we say n is an *exact* (or *perfect*) square. Thus

> n has an odd number of divisors *if and only if* n is an exact square.

*11. There are 50 prisoners in a row of locked cells. With the return of the King from the Crusades, a partial amnesty is declared and it works like this. When the prisoners are still asleep, the jailer walks past the cells 50 times, each time walking from left to right. On the first pass, he turns the lock in every cell (so that every cell is now open). On the second pass he turns the lock on every second cell (meaning that these cells are now locked again). On the third pass, he turns the lock on every third cell, and so on. In general, on the kth pass, he turns the lock on every kth cell. The question is: which cells are unlocked at the end of the process so that the prisoner is free to go?

Solution. The sixth cell lock will be turned 4 times on passes 1, 2, 3 and 6, these being the divisors of 6, and so it will end up locked. The ninth cell lock will be turned on passes 1, 3 and 9 and so will end up unlocked. So we need to know which numbers have an even number of divisors and which have an odd number of divisors.
From the previous question, we know that

> A natural number n has an *odd* number of divisors *if and only if* it is a perfect square.

So the squares have an *odd* number of divisors and the non-squares have an *even* number of divisors. The (natural number) squares less than or equal to 50 are: 1, 4, 9, 16, 25, 36 and 49. Consequently, the prisoners in cells: 1, 4, 9, 16, 25, 36 and 49, will be released.

12. Is the following statement true or false? *The number* $n^2 + n + 41$ *is prime for all positive integers* n*.*

Solution. No, n^2+n+41 is *not* prime for every natural number n. Clearly, whenever n is a multiple of 41, we have $41 \mid n^2 + n + 41$. A similar argument shows that no *polynomial* in n with integer coefficients exists that gives a *prime* for each natural number $n \dots$ multiples of the constant term of the polynomial will always yield counter-example values for n. The values of $n^2 + n + 41$ for $n \in \{1, 2, 3, 4, \dots, 39\}$ are:

$$\begin{array}{rrrrrrrrrr} 43, & 47, & 53, & 61, & 71, & 83, & 97, & 113, & 131, & 151, \\ 173, & 197, & 223, & 251, & 281, & 313, & 347, & 383, & 421, & 461, \\ 503, & 547, & 593, & 641, & 691, & 743, & 797, & 853, & 911, & 971, \\ 1033, & 1097, & 1163, & 1231, & 1301, & 1373, & 1447, & 1523, & 1601. & \end{array}$$

It is an interesting coincidence that these numbers are all prime.

13. Is the list of prime numbers *finite?* i.e. is there a *largest* prime number?

    Solution. We give an argument by contradiction (originally due to Euclid).

    - Suppose the list of primes is *finite* and let the list of all primes from smallest to largest be

    $$2 = p_1, 3 = p_2, p_3, \dots, p_n.$$

    - Let

    $$N = p_1.p_2.\cdots.p_n + 1.$$

    - None of $p_1, p_2, \dots, p_n$ divides N since

    $$N \equiv 1 \pmod{p_k}$$

      for each prime p_k in the list.
    - So N is either *prime* itself or has a *prime* divisor other than the primes of our list. In either case, our list of primes is incomplete. So there must be a prime bigger than p_n, contradicting our original assumption. Thus the list of *primes* cannot be *finite.*

14. Suppose p is prime.

    (i) Show that if $p \,|\, a^3$ then $p \,|\, a$.

    (ii) Show that if $p \,|\, b$ and $p \,|\, a^2 + b^2$ then $p \,|\, a$.

    Solution.

    (i) Suppose $p \,|\, a^3 = a.a.a$. Then by (an obvious corollary to) Euclid's Lemma, $p \,|\, a$ or $p \,|\, a$ or $p \,|\, a$, i.e. $p \,|\, a$.

    (ii) Since $p \,|\, b$ and $p \,|\, a^2 + b^2$, we have $p \,|\, a^2 + b^2 - b.b = a^2 = a.a$, and hence by Euclid's Lemma $p \,|\, a$.

15. For each of the following pairs of integers a, b use the *Euclidean Algorithm* to find $d = (a, b)$ and find a pair of integers x, y such that $ax + by = d$.

    (i) $a = 85$, $b = 41$; (ii) $a = 2613$, $b = 637$.

    Solution.

    (i)

	85	41	
2	82	39	13
	3	2	
1	2		
	1		

    Thus

$$\begin{aligned}1 &= 3 - 1.2\\ &= 3 - 1.(41 - 13.3)\\ &= 14.3 - 1.41\\ &= 14.(85 - 2.41) - 1.41\\ &= 14.85 - 29.41\end{aligned}$$

(ii)

$$\begin{array}{r|r|r|l} & 2613 & 637 & \\ 4 & 2548 & 585 & 9 \\ \cline{2-3} & 65 & 52 & \\ 1 & 52 & & \\ \cline{2-2} & 13 & & \end{array}$$

Thus

$$\begin{aligned} 13 &= 65 - 1.52 \\ &= 65 - 1.(637 - 9.65) \\ &= 10.65 - 1.637 \\ &= 10.(2613 - 4.637) - 1.637 \\ &= 10.2613 - 41.637 \end{aligned}$$

16. Show that if there exist integers x, y such that $ax + by = 1$ then $(a, b) = 1$.

    Solution. Let $d = (a, b)$. Then, by Property 1, $d \mid ax + by$. But $ax + by = 1$. So $d \mid 1$, i.e. $d \leq 1$. The gcd is necessarily ≥ 1. Hence $d = 1$.

17. Show that $(3k + 2, 5k + 3) = 1$ for any integer k.

    Solution.

$$\begin{aligned} (3k+2, 5k+3) &= (3k+2, 2k+1) && \text{since } 2k+1 = 5k+3-(3k+2) \\ &= (k+1, 2k+1) && \text{since } k+1 = 3k+2-(2k+1) \\ &= (k+1, k) && \text{since } k = 2k+1-(k+1) \\ &= (1, k) && \text{since } 1 = k+1-k \\ &= 1 \end{aligned}$$

18. Show that $(a, a + 2) = 2$ if a is even and $(a, a + 2) = 1$ otherwise.

    Solution. Since $2 = a + 2 - a$, we have

$$\begin{aligned} (a, a+2) &= (a, 2) \\ &= \begin{cases} 2 & \text{if } 2 \mid a \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

19. Show that if $(a, b) = 1$ then $(a + b, a - b) = 1$ or 2.

    Solution. Let $d = (a+b, a-b)$. Then $d \mid 2a = (a+b)+(a-b)$ and $d \mid 2b = (a+b)-(a-b)$. Hence

$$d \mid (2a, 2b) = 2(a, b) = 2.$$

    That is, $d = (a + b, a - b)$ is either 1 or 2.

20. Find all solutions to the following *Diophantine Equations.*

   (i) $2x + 5y = 11.$ (ii) $12x + 18y = 50.$ (iii) $202x + 74y = 7638.$

   Does equation (iii) have a solution in *positive* integers x, y?

Solution.

(i) Here $(2, 5) = 1$, and since $2.3 + 5.1 = 11$ we have the general solution

$$\begin{aligned} x &= 3 + 5t \\ y &= 1 - 2t \end{aligned}$$

(This one can be done without using the *Euclidean Algorithm* since the solution is almost obvious.)

(ii) Here $(12, 18) = 6$. So $6 \mid 12x + 18y$. But $6 \nmid 50$. Hence there is no solution.

(iii) The problem is equivalent to that of finding solutions of $101x+37y = 3819$ (eliminating the common factor: 2). Applying the *Euclidean Algorithm* to 101 and 37 we have:

$$\begin{array}{r|r|r|l}
 & 101 & 37 & \\
2 & 74 & 27 & 1 \\
\cline{2-3}
 & 27 & 10 & \\
2 & 20 & 7 & 1 \\
\cline{2-3}
 & 7 & 3 & \\
2 & 6 & & \\
\cline{2-2}
 & 1 & &
\end{array}$$

Thus

$$\begin{aligned}
1 &= 7 - 2.3 \\
&= 7 - 2.(10 - 1.7) \\
&= 3.7 - 2.10 \\
&= 3.(27 - 2.10) - 2.10 \\
&= 3.27 - 8.10 \\
&= 3.27 - 8.(37 - 1.27) \\
&= 11.27 - 8.37 \\
&= 11.(101 - 2.37) - 8.37 \\
&= 11.101 - 30.37
\end{aligned}$$

So, by the theorem we have

$$\begin{aligned} x &= 11.3819 + 37t \\ y &= -30.3819 - 101t \end{aligned}$$

as the general solution.

(Alternatively) We could instead make the following observations:

$$\begin{array}{rrcrcr}
 & 3700 & = & & & 100.37 \\
 & 101 & = & 1.101 & & \\
 & 18 & = & (18.11).101 & + & (18.{-30}).37 \\
\text{So} \ldots & 3819 & = & 199.101 & + & -440.37
\end{array}$$

giving the following general solution

$$\begin{aligned} x &= 199 + 37s \\ y &= -440 - 101s \end{aligned}$$

where $s \in \mathbb{Z}$. The advantage of the second approach is that it yields a much smaller *starter* solution, making it that much easier to determine whether x, y can both be positive. We see that for y to be positive $s \le -5$, and for x to be positive $s \ge -5$. Thus, if $s = -5$, both x and y are positive, and the solution in this case is:

$$x = 14 \quad \text{and} \quad y = 65,$$

and this is the *unique* solution with this property.

21. A grocer orders apples and oranges at a total cost of \$8.39. If apples cost 25c each and oranges cost 18c each, how many of each type of fruit did the grocer order?

Solution. Let x be the number of apples and y be the number of oranges. Then

$$25x + 18y = 839.$$

Applying a slight modification of the *Euclidean Algorithm* to 25 and 18 we have

$$\begin{array}{r|r|r|l} & 25 & 18 & \\ 1 & 18 & 14 & 2 \\ \cline{2-3} & 7 & 4 & \\ 2 & 8 & & \\ \cline{2-2} & -1 & & \end{array}$$

Observe we allowed a *'negative remainder'* at the last step. Sometimes doing this shortcuts a bit of work ... essentially the *Euclidean Algorithm* still works if we do this. So

$$\begin{aligned} -1 &= 7 - 2.4 \\ &= 7 - 2.(18 - 2.7) \\ &= 5.7 - 2.18 \\ &= 5.(25 - 1.18) - 2.18 \\ &= 5.25 - 7.18 \\ \text{Hence} \ldots \quad 1 &= -5.25 + 7.18 \end{aligned}$$

Observe that

$$\begin{array}{rrcrcr} & 800 & = & 32.25 & & \\ & 36 & = & & & 2.18 \\ & 3 & = & (-5.3).25 & + & (7.3).18 \\ \text{So} \ldots & 839 & = & 17.25 & + & 23.18 \end{array}$$

giving the following general solution

$$\begin{aligned} x &= 17 + 18t \\ y &= 23 - 25t \end{aligned}$$

where $t \in \mathbb{Z}$. Since x, y are both positive, we must have $t = 0$. So there are 17 apples and 23 oranges.

22. An apartment block has units at two rates: most rent at \$87/week, but a few rent at \$123/week. When all are rented the gross income is \$8733/week. How many units of each type are there?

Solution. Let x, y be the numbers of apartments at the rates \$87/week and \$123/week, respectively. Then

$$87x + 123y = 8733.$$

Cancelling the common factor 3 we have

$$29x + 41y = 2911.$$

Applying the *Euclidean Algorithm* to 29 and 41 we have

$$\begin{array}{r|r|r|l}
 & 29 & 41 & \\
2 & 24 & 29 & 1 \\
\cline{2-3}
 & 5 & 12 & \\
2 & 4 & 10 & 2 \\
\cline{2-3}
 & 1 & 2 &
\end{array}$$

So

$$\begin{aligned}
1 &= 5 - 2.2 \\
&= 5 - 2.(12 - 2.5) \\
&= 5.5 - 2.12 \\
&= 5.(29 - 2.12) - 2.12 \\
&= 5.29 - 12.12 \\
&= 5.29 - 12.(41 - 1.29) \\
&= 17.29 - 12.41
\end{aligned}$$

Observe that

$$\begin{array}{rrcrcr}
 & 2900 & = & 100.29 & & \\
 & 12 & = & -1.29 & + & 1.41 \\
 & -1 & = & -17.29 & + & 12.41 \\
\text{So} \ldots & 2911 & = & 82.29 & + & 13.41
\end{array}$$

giving the following general solution

$$\begin{aligned}
x &= 82 + 41t \\
y &= 13 - 29t
\end{aligned}$$

where $t \in \mathbb{Z}$. Since x, y are both positive, we must have $t = 0$ or $t = -1$. If $t = -1$ then $x = 41 < 42 = y$, in which case there are more of the \$123/week apartments (contrary to the given information). So $t = 0$ and there are 82 apartments renting at \$87/week and 13 apartments renting at \$123/week.

$*$23. Find all integers x, y satisfying: $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{14}$.

Solution. First let us multiply both sides of the given equation by xy. This gives:

$$y + x = \frac{xy}{14}$$

Since $x, y \in \mathbb{Z}$ we must have $14 \,|\, xy$ and hence $7 \,|\, xy$ whence $7 \,|\, x$ or $7 \,|\, y$. Without loss of generality assume $7 \,|\, x$, and write $x = 7k$. Then

$$y + 7k = \frac{ky}{2}$$

and we have $2 \,|\, ky$ and hence $2 \,|\, k$ or $2 \,|\, y$.

Case 1: $2 \,|\, k$. Write $k = 2\ell$, so that

$$y + 14\ell = \ell y.$$

Since $\ell \,|\, 14\ell$ and $\ell \,|\, \ell y$ we have $\ell \,|\, y$. Hence $y = \ell m$, say and so

$$m + 14 = \ell m$$

and we have that $m \,|\, 14$ (i.e. $m \in \{\pm 1, \pm 2, \pm 7, \pm 14\}$) and $\ell = 1 + 14/m$, giving:

$$\begin{aligned} x &= 14(1 + 14/m) \\ y &= m(1 + 14/m). \end{aligned}$$

Furthermore, since x, y are non-zero, $m \neq -14$. Enumerating the possibilities for the pairs (x, y) we get

$$(210, 15), (-182, 13), (112, 16), (-84, 12), (42, 21), (-14, 7), (28, 28).$$

Case 2: $2 \nmid k$ and $2 \,|\, y$. Write $y = 2s$, so that

$$2s + 7k = ks.$$

Since $k \,|\, 7k$ and $k \,|\, ks$, but $2 \nmid k$ we must have $k \,|\, s$. Write $s = kt$. Then

$$2t + 7 = kt$$

and we have that $t \,|\, 7$ (i.e. $t \in \{\pm 1, \pm 7\}$) and $k = 2 + 7/t$, giving:

$$\begin{aligned} x &= 7(2 + 7/t) \\ y &= 2(2 + 7/t)t. \end{aligned}$$

Like the previous case, since x, y are non-zero, $t \neq -7$. Enumerating the possibilities for the pairs (x, y) we get

$$(63, 18), (-35, 10), (21, 42).$$

We found the pair $(21, 42)$ (in the reverse order) in Case 1.

Thus the complete list of pairs for (x, y) is

$$(210, 15), (-182, 13), (112, 16), (-84, 12), (42, 21), (-14, 7), (28, 28), (63, 18), (-35, 10),$$

or of course the same pairs in reverse order.

*24. When Jane is one year younger than Betty will be when Jane is half as old as Betty will be when Jane is twice as old as Betty is now, Betty will be three times as old as Jane was when Betty was as old as Jane is now.
One is in her teens and ages are in completed years. How old are they?

Solution. Let Jane's age *now* be x and Betty's age *now* be y. Also, we will represent the ages of Jane and Betty at time i by x_i and y_i respectively. Let's rewrite the given information, including these choices of variables.

> When Jane [is x_1, she] is one year younger than Betty will be [when she is y_2 and] when Jane [is x_2 and she] is half as old as Betty will be [when she is y_3 and] when Jane [is x_3 and she] is twice as old as Betty is now [when she is y], Betty will be [y_1 and she will be] three times as old as Jane was when [she was x_4 and when] Betty was [y_4 and she was] as old as Jane is now [when she is x].

So we get the following equations:

$$\begin{aligned} x_1 &= y_2 - 1 \\ x_2 &= \tfrac{1}{2} y_3 \\ x_3 &= 2y \\ y_1 &= 3x_4 \\ y_4 &= x. \end{aligned}$$

Now define a, b, c, d such that $x_1 = x + a$, $x_2 = x + b$, $x_3 = x + c$, $x_4 = x + d$; so that $y_1 = y + a$, $y_2 = y + b$, $y_3 = y + c$, $y_4 = y + d$. Hence the above equations become:

$$\begin{aligned} x + a &= y + b - 1 \\ x + b &= \tfrac{1}{2}(y + c) \\ x + c &= 2y \\ y + a &= 3(x + d) \\ y + d &= x. \end{aligned}$$

Now rearrange these equations (and multiply by a factor where appropriate):

$$\begin{array}{ccccccccccccc} x & + & a & - & b & & & & & = & y & - & 1 \\ x & & & + & b & - & \tfrac{1}{2}c & & & = & \tfrac{1}{2}y & & \\ \tfrac{1}{2}x & & & & & + & \tfrac{1}{2}c & & & = & y & & \\ 3x & - & a & & & & & + & 3d & = & y & & \\ 3x & & & & & & & - & 3d & = & 3y. & & \end{array}$$

Adding these equations we find that a, b, c, d cancel and we get:

$$8\tfrac{1}{2}x = 6\tfrac{1}{2}y - 1$$

i.e.

$$13y - 17x = 2. \tag{1}$$

Since x, y are only allowed to be integers, this equation is an example of a *linear Diophantine Equation.* A method for solving such an equation is to first apply the *Euclidean*

Algorithm to find the gcd d of 13 and 17 (which is clearly 1). Tracing the algorithm backwards one can express d in terms of 13 and 17. Applying the *Euclidean Algorithm* then (see Notes) we get:

$$\begin{array}{r|r|r|l} & 13 & 17 & \\ 3 & 12 & 13 & 1 \\ \cline{2-3} & 1 & 4 & \end{array}$$

Thus

$$\begin{aligned} 1 &= 13 - 12 \\ &= 13 - 3.4 \\ &= 13 - 3.(17 - 13) \\ &= 4.13 - 3.17. \end{aligned}$$

So

$$\begin{aligned} 2 &= 8.13 - 6.17 \\ &= 8.13 + 13.17t - 13.17t - 6.17 \\ &= 13(8 + 17t) - 17(6 + 13t). \end{aligned}$$

Comparing the above with (1) we see that we have a solution for (1) if

$$\begin{aligned} x &= 6 + 13t \\ y &= 8 + 17t \end{aligned}$$

for some integer t, which by the theorem in the notes is the general solution. We were also given that one girl was in her teens, so that t must be 1 and

$$x = 19, \quad y = 25.$$

Hence Jane is 19 and Betty is 25.

25. Solve the adjacent *alphametic* (an addition in which: each letter stands for a different digit; and left-most digits of a number are not allowed to be 0).

$$\begin{array}{rrrr} & A & H & A \\ & A & H & A \\ & & & A \\ & W & A & G \\ \hline H & A & H & A \end{array}$$

Answer. HAHA = 1717 ($W = 2$, $G = 6$). The solution is unique.

Solution.

- First let the *carry* from the right column be k then it is easy to see that

$$\begin{aligned} 2A + G &= 10k \qquad (2) \\ H + A + k &= 10 \qquad (3) \\ A + W + 1 &= 10 \qquad (4) \\ H &= 1 \qquad (5) \end{aligned}$$

- Now k is either 1 or 2 (considering the least and largest values possible for A and G … remembering that $2A + G$ is *exactly* a multiple of 10.)

- Since H = 1 and $k \in \{1, 2\}$, it follows from (3) that A is 8 or 7.
- If A = 8 then by (4) $W = 1$, in which case H and W are equal (which is not allowed). So $A \neq 8$. Hence $\boxed{A = 7}$. Therefore $k = 2$ and $\boxed{G = 6}$, and by (4) $\boxed{W = 2}$.
- So finally we get (and check)

$$\begin{array}{cccc} & 7 & 1 & 7 \\ & 7 & 1 & 7 \\ & & & 7 \\ & 2 & 7 & 6 \\ \hline 1 & 7 & 1 & 7 \end{array}$$

- So HAHA = 1717.