

University of Western Australia
DEPARTMENT OF MATHEMATICS

UWA ACADEMY
FOR YOUNG MATHEMATICIANS

The Pigeon-Hole Principle

Greg Gamble

The *Pigeon-Hole Principle (PHP)* is easily illustrated by a simple example:

If 5 pigeons fly into 4 pigeon-holes then at least one pigeon-hole contains two or more pigeons.

i.e. if there is at least one more *pigeon* than there are *pigeon-holes* then at least one of the *pigeon-holes* has more than one *pigeon*.

Example. *Suppose that in my dresser drawer I have socks of three colours ... loose. A bit silly, because I have to get up this morning while it's still dark. How do I ensure that I get a matching pair of socks in the most economical way ... without disturbing my partner?*

Solution. I take 4 socks from the drawer ... since then, by the PHP, I must have at least one pair. The idea is that the *colours* (three of them) are the *pigeon-holes* and the *socks* are the *pigeons*.

Of course, this idea can be generalised a bit:

Theorem ((Extended) Pigeon-Hole Principle). *If there are k pigeon-holes and more than mk pigeons then at least one pigeon-hole has at least $m + 1$ pigeons.*

Elections! Elections!

Voting for a member of the House of Representatives

Members for the House of Representatives are elected using the *preferential voting system*. Suppose that in a given *electorate* there are six *candidates*. Then, a *formal vote* (i.e. a vote that obeys the rules and so one that will be counted) numbers the candidates from 1 to 6 in some order.

So ... how is the winning candidate determined? Well ... first the number 1 votes are counted. In our example, this results in 6 piles of ballot papers: one for each candidate. The candidate with the least number 1 votes is then excluded; and that candidate's pile of ballot papers are re-distributed to the other 5 piles according to the number 2 votes on those ballot papers.

Question 1. How many of the number 1 (*primary*) votes must a candidate get to ensure they are not excluded at the first round?

At the second round the candidate with the smallest pile of ballot papers after the first round is excluded; and that candidate's pile of ballot papers are re-distributed to the remaining (4, in our example) piles according to the *next* number preference on those ballot papers, i.e. the *smallest* number vote that doesn't correspond to an excluded candidate – which will be either number 2 votes or number 3 votes.

Subsequent rounds are analogous to the second round. The natural conclusion of this process is a single pile corresponding to the winning candidate.

Question 2. How many of the number 1 (*primary*) votes must a candidate get to ensure they are not excluded at the k th round, (where k is less than the number of candidates)?

Question 3. At the k^{th} round, how big must a candidate's pile be at the beginning of the round to ensure they are not excluded at that round, (where k is less than the number of candidates)?

In answering these questions you will see a number of short-cuts to the process described above, e.g. if after any round a candidate's pile contains more than half the number of ballot papers then that candidate is certainly the winner.

Question 4. Can you think of another short-cut ... using the PHP?

Voting for a member of the Senate

Usually at a *general election* there is a *half-senate* election, i.e. as well as voting for all the *House of Representatives* we vote for *half* the Senate, the other half keep their jobs until there is another general election.¹ The election on the 2nd of March was of the usual type, and so in Western Australia we voted for 6 of the 12 senators that represent our state. To fill these 6 positions there were 29 candidates. So a *formal* Senate vote numbered the candidates in some order from 1 to 29.²

So ... how are the winning 6 candidates determined? Well ... first as for the *House of Representatives* the number 1 votes are counted. In the recent election, this would have resulted in 29 piles of formal ballot papers. Let N be the total number of *formal votes*. Then a candidate is elected once their pile of ballot papers achieves a *quota*

$$\left\lfloor \frac{N}{6 + 1} \right\rfloor + 1$$

of the formal votes cast.

Question 5. What is the significance of the *quota*? (Ideas explored with regard to House of Representatives voting should help you.)

¹Occasionally, exceptional circumstances bring about a *double dissolution*, where both houses of parliament are dissolved and there is a *full-senate* election.

²You may recall advertisements telling voters to either put a 1 in one box *above the line* or to number all boxes below the line. Each party corresponding to a box above the line logged with the Electoral Commission how *they* would number the boxes *below the line*. So really all *formal* votes cast, number the 29 candidates.

Now ... what happens? Like the *House of Representatives* there are a number of *rounds*, but unlike the *House of Representatives* each round has two parts. Firstly, any candidate who has achieved the quota (call it Q) is declared elected. Then the ballot papers of these elected candidates are re-distributed to the next not-so-far elected preference but at a *reduced* value: they are scaled by the factor³

$$\frac{k - Q}{k},$$

where k is the number of ballot papers in the candidate's pile. Once no more candidates can be lifted to a quota this way, the second part of the round begins. This proceeds *exactly* in the way a *House of Representatives* round does: the candidate with the smallest pile of ballot papers is excluded and those ballot papers are re-distributed to the remaining piles according to the *next*⁴ preference on those ballot papers, at *full* value.

Now, to be convinced that this is a viable means of voting we really need only consider the following two questions:

Question 6. Why can no more than 6 be elected in this way?

Question 7. Why are (at least) 6 candidates elected by this method ... unless there is a tie?

³The idea is that the *surplus* votes for elected candidates are passed on to the remaining candidates; but it would be *unfair* to simply take any $k - Q$ votes as the surplus. So *all* k votes are re-distributed but at a reduced weighting.

⁴*Next* preference this time means the least number vote that corresponds neither to an *already-elected* candidate nor to an *excluded* candidate.