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FOR YOUNG MATHEMATICIANS

Number Theory I: Problems with Solutions

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1. The number $739ABC$ is divisible by 7, 8 and 9. What values can A, B and C take?

 Solution. If two natural numbers a, b have *greatest common divisor* equal to 1, then a, b are said to be *relatively prime*. The numbers 7, 8 and 9 are *pairwise relatively prime,* i.e. any pair are relatively prime. So their lowest common multiple is simply the product of all three. Written mathematically:

$$\operatorname{lcm}(7, 8, 9) = 7.8.9 = 504.$$

 We must choose a number of the form $739ABC$ such that it is a multiple of 7, 8 and 9; i.e. we must choose a number of the form $739ABC$ that is divisible by $\operatorname{lcm}(7, 8, 9) = 504$. Now 739 000 gives remainder 136 on division by 504. Hence the numbers $739ABC$ we are looking for, are of form

$$739\,000 - 136 + k.504$$

 where k is an integer. We can see that k can only be 1 or 2. If $k = 1$, we get the number 739 368 so that one solution for A, B, C is

$$A = 3, B = 6, C = 8;$$

 and if $k = 2$ we get the number 739 872 so that another solution for A, B, C is

$$A = 8, B = 7, C = 2.$$

2. Determine *simple* rules for divisibility by each of the following natural numbers:

(i) 2	(iv) 5	(vii) 9	(x) 12
(ii) 3	(v) 6	(viii) 10	(xi) 15
(iii) 4	(vi) 8	(ix) 11	

 Note: there is no rule as simple as the above for 7, but rules (other than straight division) do *exist.*

 Solution. We will use congruences for some of the solutions here. Remember, $m \mid n$ *if and only if* $n \equiv 0 \pmod{m}$.

 (i) Every natural number n can be written as $10q + r$, where r is the remainder after n is divided by 10, i.e. r is the last digit of n. Now $2 \mid 10$. So

$$2 \mid n \quad \text{if and only if} \quad 2 \mid r,$$

 where r is the last digit of n. In other words, *2 divides n if and only if n ends in 0 or 2 or 4 or 6 or 8.*

(ii) Suppose the decimal representation of n is $a_k\, a_{k-1}\, \dots\, a_0$. Then

$$n = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10 a_1 + a_0.$$

Now $10 \equiv 1 \pmod 3$; so

$$10^\ell \equiv 1 \pmod 3,$$

for *any* natural number ℓ. So

$$\begin{aligned} n &= 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10 a_1 + a_0 \\ &\equiv a_k + a_{k-1} + \cdots + a_1 + a_0 \pmod 3. \end{aligned}$$

Hence,

$$3 \,\big|\, n \quad \text{if and only if} \quad 3 \,\big|\, a_k + a_{k-1} + \cdots + a_1 + a_0.$$

In other words, *3 divides n if and only if 3 divides the sum of the digits of n.*

(iii) Every natural number n can be written as $100q + r$, where r is the remainder after n is divided by 100, Now $4 \,\big|\, 100$. *(Note that 4 does not divide 10.)* So

$$4 \,\big|\, n \quad \text{if and only if} \quad 4 \,\big|\, r,$$

where r consists of the last two digits of n.

(iv) Every natural number n can be written as $10q + r$, where r is the remainder after n is divided by 10, i.e. r is the last digit of n. Now $5 \,\big|\, 10$. So

$$5 \,\big|\, n \quad \text{if and only if} \quad 5 \,\big|\, r,$$

where r is the last digit of n. In other words, *5 divides n if and only if n ends in a 0 or a 5.*

(v) $6 = \operatorname{lcm}(2, 3)$, so to check divisibility by 6, we check for divisibility by 2 *and* 3.

(vi) Every natural number n can be written as $1000q + r$, where r is the remainder after n is divided by 1000, Now $8 \,\big|\, 1000$. *(Note that 8 does not divide 100.)* So

$$8 \,\big|\, n \quad \text{if and only if} \quad 8 \,\big|\, r,$$

where r consists of the last three digits of n.

(vii) Just as we did for 3, suppose the decimal representation of n is $a_k\, a_{k-1}\, \dots\, a_0$. Then

$$n = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10 a_1 + a_0.$$

Now $10 \equiv 1 \pmod 9$; so

$$10^\ell \equiv 1 \pmod 9,$$

for *any* natural number ℓ. So

$$\begin{aligned} n &= 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10 a_1 + a_0 \\ &\equiv a_k + a_{k-1} + \cdots + a_1 + a_0 \pmod 9. \end{aligned}$$

Hence,

$$9 \,\big|\, n \quad \text{if and only if} \quad 9 \,\big|\, a_k + a_{k-1} + \cdots + a_1 + a_0.$$

In other words, *9 divides n if and only if 9 divides the sum of the digits of n.*

(viii) Well of course everyone knows that: *10 divides n if and only if the last digit of n is 0;* but let's see this another way. Like the case for 6, $10 = \mathrm{lcm}(2,5)$, so to check divisibility by 10, we check for divisibility by 2 *and* 5. In other words,

$10 \mid n$ if and only if n has 0 or 2 or 4 or 6 or 8 as last digit *and* n has 0 or 5 as last digit.

(ix) Suppose the decimal representation of n is $a_k\, a_{k-1}\, \dots\, a_0$. Then

$$n = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10 a_1 + a_0.$$

Now $10 \equiv -1 \pmod{11}$; so

$$10^\ell \equiv (-1)^\ell \pmod{11},$$

for *any* natural number ℓ. So

$$\begin{aligned} n &= 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10 a_1 + a_0 \\ &\equiv (-1)^k a_k + (-1)^{k-1} a_{k-1} + \cdots - a_1 + a_0 \pmod{11}. \end{aligned}$$

Hence,

$$11 \mid n \quad \text{if and only if} \quad 11 \mid a_0 - a_1 + a_2 - \cdots + (-1)^k a_k.$$

In other words, *11 divides n if and only if 11 divides the* difference *of the sum of* odd-place *digits of n and the sum of the* even-place *digits of n.*

(x) $12 = \mathrm{lcm}(3,4)$, so to check divisibility by 12, we check for divisibility by 3 *and* 4.

(xi) $15 = \mathrm{lcm}(3,5)$, so to check divisibility by 15, we check for divisibility by 3 *and* 5.

3. Is 167 prime?

Solution. Suppose 167 is composite. Then it has a divisor $m > 1$. Then m and $167/m$ both divide 167. The lesser of m and $167/m$ is at most $\sqrt{167}$ and must have a prime decomposition consisting of primes less than or equal to $\sqrt{167}$. Now $\sqrt{167} < 13$ and it is easy to check that none of the primes 2, 3, 5, 7 or 11 divide 167. So we have a contradiction. That is, 167 cannot be composite; and since it is not 1 it must be prime.

4. Show that $x^2 - y^2 = 2$ has no integer solutions.

Solution. We may as well assume that x, y are not negative. Now 2, being prime can only be written as the product of two natural numbers in one way: $2 = 1.2$; and

$$x^2 - y^2 = (x-y)(x+y).$$

By our assumption $x + y \geq x - y$. Hence

$$\begin{aligned} x - y &= 1 \\ x + y &= 2. \end{aligned}$$

Solving these equations simultaneously, we get $x = \frac{3}{2}$, $y = \frac{1}{2}$ (which are not integers). So there can be no integer solutions of $x^2 - y^2 = 2$.

5. Prove that 6 divides $n(n-1)(2n-1)$.

Solution.

- Either $2 \mid n$ or $2 \mid n-1$; so $2 \mid n(n-1)(2n-1)$.
- Similarly, at least one of the three consecutive integers $n-1, n, n+1$ is divisible by 3. Suppose $3 \mid n+1$; then $n+1 \equiv 0 \pmod 3$ (i.e. $n \equiv -1 \pmod 3$) and hence

$$\begin{aligned} 2n-1 &\equiv 2.-1-1 \pmod 3 \\ &\equiv -3 \pmod 3 \\ &\equiv 0 \pmod 3 \end{aligned}$$

  So, *if* 3 divides $n+1$ *then* 3 divides $2n-1$. Hence, since at least one of $n-1, n, n+1$ is divisible by 3, we have at least one of $n-1, n, 2n-1$ is divisible by 3. So $3 \mid n(n-1)(2n-1)$.

Thus, since $2 \mid n(n-1)(2n-1)$ and $3 \mid n(n-1)(2n-1)$, we have: $6 = \mathrm{lcm}(2,3)$ divides $n(n-1)(2n-1)$.

6. Prove that for any natural number n that

$$17 \text{ divides } 2^n.3^{2n} - 1.$$

Solution. We need to show that $2^n.3^{2n}-1$ can be written as an integer multiple of 17. Now

$$\begin{aligned} 2^n.3^{2n} - 1 &= (2.3^2)^n - 1 \\ &= 18^n - 1 \\ &= (18-1)(18^{n-1} + 18^{n-2} + \cdots + 1) \\ &= 17.N, \end{aligned}$$

for some integer $N = 18^{n-1} + 18^{n-2} + \cdots + 1$. So $17 \mid 2^n.3^{2n} - 1$.

7. Prove that for any natural number n

$$17^n - 12^n - 24^n + 19^n$$

is divisible by 35.

Solution. Let $N = 17^n - 12^n - 24^n + 19^n$. Now $35 = \mathrm{lcm}(5,7)$. So to check that $35 \mid N$, it is enough to show that $5 \mid N$ *and* $7 \mid N$. We could use the method of the previous question, but we shall use congruences. (Remember: $N \equiv 0 \pmod m$ means *exactly* the same thing as $m \mid N$.)

- Firstly,

$$\begin{aligned} N &= 17^n - 12^n - 24^n + 19^n \\ &\equiv 2^n - 2^n - 4^n + 4^n \pmod 5 \\ &\equiv 0 \pmod 5, \end{aligned}$$

  and hence $5 \mid N$.
- Similarly,

$$\begin{aligned} N &= 17^n - 12^n - 24^n + 19^n \\ &\equiv 3^n - 5^n - 3^n + 5^n \pmod 7 \\ &\equiv 0 \pmod 7, \end{aligned}$$

  and hence $7 \mid N$.

Thus, since $5 \mid N$ and $7 \mid N$, we have $35 = \mathrm{lcm}(5,7)$ divides $N = 17^n - 12^n - 24^n + 19^n$.

8. Prove that $5^{99} + 11^{99} + 17^{99}$ is divisible by 33.

Solution. Let $N = 5^{99} + 11^{99} + 17^{99}$. Now $33 = \operatorname{lcm}(3, 11)$. So to check that $33 \mid N$, it is enough to show that $3 \mid N$ *and* $11 \mid N$. Again, we shall use congruences.

- Firstly,

$$\begin{aligned} N &= 5^{99} + 11^{99} + 17^{99} \\ &\equiv 2^{99} + 2^{99} + 2^{99} \pmod 3 \\ &\equiv 3.2^{99} \pmod 3 \\ &\equiv 0 \pmod 3, \end{aligned}$$

  and hence $3 \mid N$.

- Similarly,

$$\begin{aligned} N &= 5^{99} + 11^{99} + 17^{99} \\ &\equiv 5^{99} + 0^{99} + (-5)^{99} \pmod{11} \\ &\equiv 5^{99} + 0 - 5^{99} \pmod{11} \\ &\equiv 0 \pmod{11}, \end{aligned}$$

  and hence $11 \mid N$.

Thus, since $3 \mid N$ and $11 \mid N$, we have $33 = \operatorname{lcm}(3, 11)$ divides $N = 5^{99} + 11^{99} + 17^{99}$.

9. Prove that for every integer n:

(i) $3 \mid n^3 - n$; (ii) $5 \mid n^5 - n$; (iii) $7 \mid n^7 - n$; (iv) $11 \mid n^{11} - n$.

Show that $n^9 - n$ is not necessarily divisible by 9. *Hint:* Try $n = 2$.
What general result is suggested by the above?

Solution.

(i) 3 divides exactly one of the three consecutive integers $n-1, n, n+1$ and

$$n^3 - n = n(n^2 - 1) = n(n-1)(n+1).$$

So $3 \mid n^3 - n$.

(ii) 5 divides exactly one of the five consecutive integers $n-2, n-1, n, n+1, n+2$. In terms of congruences, exactly one of $n-2, n-1, n, n+1, n+2$ is *congruent* to 0 *modulo* 5. Thus:

$$\begin{aligned} n^5 - n = n(n^4 - 1) = n(n^2 - 1)(n^2 + 1) &= n(n-1)(n+1)(n^2 + 1) \\ &\equiv n(n-1)(n+1)(n^2 - 4) \pmod 5 \\ &\equiv n(n-1)(n+1)(n-2)(n+2) \pmod 5 \\ &\equiv 0 \pmod 5 \end{aligned}$$

So $5 \mid n^5 - n$.

(iii) Exactly one of $n-3, n-2, n-1, n, n+1, n+2, n+3$ is *congruent* to 0 *modulo* 7. Thus:

$$\begin{aligned} n^7 - n = n(n^6 - 1) &= n(n^3 - 1)(n^3 + 1) \\ &= n(n-1)(n^2 + n + 1)(n+1)(n^2 - n + 1) \\ &\equiv n(n-1)(n^2 + n - 6)(n+1)(n^2 - n - 6) \pmod 7 \\ &\equiv n(n-1)(n+3)(n-2)(n+1)(n-3)(n+2) \pmod 7 \\ &\equiv 0 \pmod 7 \end{aligned}$$

So $7 \mid n^7 - n$.

(iv) n is *congruent* to exactly one of $-5,-4,-3,-2,-1,0,1,2,3,4,5$ *modulo* 11. It is simply a matter of checking, for each congruence possibility of n, that $n^{11}-n$ (or a factor of $n^{11}-n$) is *congruent* to 0 *modulo* 11. Note, first that:

$$n^{11}-n=n(n^{10}-1).$$

- If $n \equiv 0 \pmod{11}$ there is nothing to check since n is a factor of $n^{11}-n$.
- $(-1)^{10}-1 \equiv 1^{10}-1 \equiv 0 \pmod{11}$.
  So $n^{10}-1 \equiv 0 \pmod{11}$ if $n \equiv \pm 1 \pmod{11}$.
- $2^5 = 32 \equiv -1 \pmod{11}$. So $2^{10} = (2^5)^2 \equiv 1 \pmod{11}$.
  Hence $(-2)^{10}-1 \equiv 2^{10}-1 \equiv 0 \pmod{11}$.
  So $n^{10}-1 \equiv 0 \pmod{11}$ if $n \equiv \pm 2 \pmod{11}$.
- $3^5 = 243 \equiv 1 \pmod{11}$. So $3^{10} = (3^5)^2 \equiv 1 \pmod{11}$.
  Hence $(-3)^{10}-1 \equiv 3^{10}-1 \equiv 0 \pmod{11}$.
  So $n^{10}-1 \equiv 0 \pmod{11}$ if $n \equiv \pm 3 \pmod{11}$.
- $2^5 = 32 \equiv -1 \pmod{11}$. So $4^{10} = (2^5)^4 \equiv 1 \pmod{11}$.
  Hence $(-4)^{10}-1 \equiv 4^{10}-1 \equiv 0 \pmod{11}$.
  So $n^{10}-1 \equiv 0 \pmod{11}$ if $n \equiv \pm 4 \pmod{11}$.
- $5^2 = 25 \equiv 4 \pmod{11}$ and $4^5 = (2^5)^2 \equiv 1 \pmod{11}$.
  So $5^{10} = (5^2)^5 \equiv 4^5 \equiv 1 \pmod{11}$. Hence $(-5)^{10}-1 \equiv 5^{10}-1 \equiv 0 \pmod{11}$.
  So $n^{10}-1 \equiv 0 \pmod{11}$ if $n \equiv \pm 5 \pmod{11}$.

So, for each congruence possibility of n, we find a factor of $n^{11}-n$ is *congruent* to 0 *modulo* 11. So for any integer n, $n^{11}-n \equiv 0 \pmod{11}$. Hence for any integer $n \mid n^{11}-n$.

Now $2^9-2=510$ and $9 \nmid 510$; so 9 need not divide n^9-n. The general result suggested by the above is:

Theorem (Fermat's Little Theorem). *If n is an integer and p is a prime then $p \mid n^p - n$.*

10. Prove that for all integers a and b:

(i) 3 divides $(a+b)^3-a^3-b^3$; (ii) 5 divides $(a+b)^5-a^5-b^5$.

Can you generalise these results?

Solution. Let us use the results of the previous question. There we saw that for a prime p and an integer n that

$$p \mid n^p - n.$$

In terms of congruences, this result can be written as:

If p is prime and n is an integer then $n^p \equiv n \pmod{p}$.

(i) Thus $(a+b)^3 \equiv (a+b) \pmod{3}$, $a^3 \equiv a \pmod{3}$ and $b^3 \equiv b \pmod{3}$. Hence

$$\begin{aligned}(a+b)^3-a^3-b^3 &\equiv a+b-a-b \pmod{3}\\ &\equiv 0 \pmod{3}\end{aligned}$$

So $3 \mid (a+b)^3-a^3-b^3$.

(ii) Similarly, $(a+b)^5 \equiv (a+b) \pmod 5$, $a^5 \equiv a \pmod 5$ and $b^5 \equiv b \pmod 5$. Hence

$$\begin{aligned}(a+b)^5 - a^5 - b^5 &\equiv a + b - a - b \pmod 5\\ &\equiv 0 \pmod 5\end{aligned}$$

So $5 \mid (a+b)^5 - a^5 - b^5$.

Using Fermat's Little Theorem (i.e. using a general prime p in place of 3 or 5 in either of the previous arguments) we get the following general result:

For any integers a, b and for any prime p,

$$p \text{ divides } (a+b)^p - a^p - b^p.$$

11. Prove the following:

(i) $3^{6n} - 2^{6n}$ is divisible by 35, for every positive integer n;

(ii) $n^5 - 5n^3 + 4n$ is divisible by 120, for every integer n.

Solution.

(i) Let $N = 3^{6n} - 2^{6n}$. Now $35 = \text{lcm}(5,7)$. So to check that $35 \mid N$, it is enough to show that $5 \mid N$ *and* $7 \mid N$.

- Firstly,

$$\begin{aligned}N = 3^{6n} - 2^{6n} &= 9^{3n} - 4^{3n}\\ &\equiv 4^{3n} - 4^{3n} \pmod 5\\ &\equiv 0 \pmod 5,\end{aligned}$$

and hence $5 \mid N$.

- Similarly,

$$\begin{aligned}N = 3^{6n} - 2^{6n} &= 27^{2n} - 8^{2n}\\ &\equiv (-1)^{2n} - 1^{2n} \pmod 7\\ &\equiv 1^n - 1^n \pmod 7\\ &\equiv 0 \pmod 7,\end{aligned}$$

and hence $7 \mid N$.

Thus, since $5 \mid N$ and $7 \mid N$, we have $35 = \text{lcm}(5,7)$ divides $N = 3^{6n} - 2^{6n}$.

(ii) Let $N = n^5 - 5n^3 + 4n$. Then

$$\begin{aligned}N = n^5 - 5n^3 + 4n &= n(n^4 - 5n^2 + 4)\\ &= n(n^2-1)(n^2-4)\\ &= n(n-1)(n+1)(n-2)(n+2).\end{aligned}$$

So N is the product of the five consecutive integers: $(n-2), (n-1), n, (n+1), (n+2)$. Exactly one of these integers is divisible by 5, at least one is divisible by 4 and at least one is divisible by 3. Further, if $k \in \{-2,-1,0,1,2\}$ and $n+k$ is a factor of N that is divisible by 4, then either $n+k-2$ or $n+k+2$ is a factor of N both of which are even. That is, either $(n+k)(n+k-2) \mid N$ or $(n+k)(n+k+2) \mid N$; in either case, we see that $8 \mid N$. Hence, $120 = \text{lcm}(3,5,8)$ divides $N = n^5 - 5n^3 + 4n$.

12. Prove that $n^2 + 3n + 5$ is never divisible by 121 for any positive integer n.

Solution. Observe that

$$n^2 + 3n + 5 = (n+7)(n-4) + 33,$$

so that $11 \mid n^2 + 3n + 5$ if and only if $11 \mid (n+7)(n-4)$. Thus, if $11 \nmid (n+7)(n-4)$ then 11 (and hence 121) does *not* divide $n^2 + 3n + 5$. So, assume 11 divides $(n+7)(n-4)$. Then $11 \mid n+7$ or $11 \mid n-4$; but then 11 must divide *both* of $n+7$ and $n-4$, since

$$n + 7 \equiv n - 4 \pmod{11}.$$

Thus, $121 \mid (n+7)(n-4)$. However, $121 \nmid 33$. So $121 \nmid n^2 + 3n + 5 = (n+7)(n-4) + 33$. Hence, in all cases, $121 \nmid n^2 + 3n + 5$.

13. What is the final digit of

$$(((((((((7^7)^7)^7)^7)^7)^7)^7)^7)^7)^7?$$

(7 occurs as a power 10 times.)

Solution. The final digit of a (decimal) number is its remainder *modulo* 10. Now $7^2 = 49 \equiv -1 \pmod{10}$. So $7^7 = (7^2)^3.7 \equiv -7 \pmod{10}$, and

$$(7^7)^7 \equiv (-7)^7 \equiv -(7^7) \equiv -(-7) \equiv 7 \pmod{10}.$$

Proceeding in this way, we see that $((7^7)^7)^7 \equiv 7 \pmod{10}$, and in general

$$(\cdots(((7^7)^7)^7)\cdots^7) \equiv \pm 7 \pmod{10},$$

where the sign is + if all together there is an *even* number of 7s appearing as powers in the formula, and − if there is an *odd* number of 7s appearing as powers in the formula. Now, 10 is even. So the final digit of the given formula is 7.

14. What is the final digit of $7^{7^{7^{7^{7^{7^{7}}}}}}$.

Solution. Firstly, we will call an expression of the form

$$7^{7^{7^{7^{\cdot^{\cdot^{\cdot^{7}}}}}}}$$

a *tower* of 7s. Our problem has a tower of 7 7s. Observe that

$$7^4 = (7^2)^2 \equiv (-1)^2 \equiv 1 \pmod{10}.$$

Hence, *modulo* 10,

$$7^k \equiv \begin{cases} 1 & \text{if } k \equiv 0 \pmod{4} \\ 7 & \text{if } k \equiv 1 \pmod{4} \\ -1 & \text{if } k \equiv 2 \pmod{4} \\ -7 & \text{if } k \equiv 3 \pmod{4}, \end{cases}$$

where k is a natural number. Thus to determine the last digit of a tower of 7 7s, we need to determine what a tower of 6 7s is congruent to *modulo* 4. Now, $7 \equiv -1 \pmod{4}$. Hence, *modulo* 4,

$$7^m \equiv \begin{cases} 1 & \text{if } m \text{ is } \textit{even} \\ -1 & \text{if } m \text{ is } \textit{odd}, \end{cases}$$

where m is a natural number. A tower of 5 7s is certainly odd. So, a tower of 6 7s is congruent to -1 *modulo* 4 (and $-1 \equiv 3 \pmod 4$). So, a tower of 7 7s is congruent to -7 *modulo* 10 (and $-7 \equiv 3 \pmod{10}$). Hence, a tower of 7 7s must end in a 3.

15. When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B.
Hints: First show that the sum of the digits of B is fairly small (in fact: less than 16). Then use the fact that, for any natural number N,

$$N \equiv (\text{sum of the digits of } N) \pmod 9.$$

Solution.

- First we will show that the sum of the digits of B is fairly small. Now $4444 < 10\,000 = 10^4$. Hence

$$4444^{4444} < 10^{4.4444} = 10^{17776},$$

and so 4444^{4444} cannot have more than 17 776 digits. Thus, A the sum of the digits of 4444^{4444}, cannot be more than $17\,776.9 = 159\,984$, (since each digit is at most a 9). Of the natural numbers less than or equal to 159 984, the number with the largest digit sum is 99 999. So B is not more than 45. Of the natural numbers less than or equal to 45, the number with the largest digit sum is 39. So the sum of the digits of B is not more than 12.

- Now we use the result given in the hint:

$$\text{For any natural number } N, \quad N \equiv (\text{sum of the digits of } N) \pmod 9.$$

(Note that this result was proved when the divisibility rule for 9 was proved in question 2(vii).) Using this result we see that 4444^{4444} is *congruent* to its digit sum A, *modulo* 9. Using the result again, we see that A is *congruent* to its digit sum B, *modulo* 9. Using the result one further time, we see that B is *congruent* to its digit sum, *modulo* 9. That is,

$$\begin{aligned} 4444^{4444} &\equiv A && \pmod 9 \\ &\equiv B && \pmod 9 \\ &\equiv (\text{sum of the digits of } B) && \pmod 9 \end{aligned}$$

- Now we determine what 4444^{4444} is congruent to *modulo* 9.

$$\begin{aligned} 4444^{4444} &\equiv (4+4+4+4)^{4444} && \pmod 9 \\ &\equiv 16^{4444} && \pmod 9 \\ &\equiv (-2)^{4444} && \pmod 9 \\ &\equiv (-2)^{3.1481+1} && \pmod 9 \\ &\equiv \left((-2)^3\right)^{1481}.(-2) && \pmod 9 \\ &\equiv (-8)^{1481}.(-2) && \pmod 9 \\ &\equiv 1^{1481}.(-2) && \pmod 9 \\ &\equiv 1.(-2) && \pmod 9 \\ &\equiv 7 && \pmod 9 \end{aligned}$$

Putting these three facts together we get

$$\text{(the sum of the digits of } B) \equiv 7 \pmod 9$$

and *the sum of the digits of* B is a *natural number* less than or equal to 12. Thus

$$\text{(the sum of the digits of } B) = 7.$$

16. Show that if $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime decomposition of the positive integer n, then the number of divisors of n (including 1 and n) is $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$.

Solution. Observe that every divisor of n is of the form

$$p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k},$$

where $f_1, \ldots, f_k$ are all integers, and

$$\begin{aligned} 0 &\le f_1 \le e_1 \\ 0 &\le f_2 \le e_2 \\ &\vdots \\ 0 &\le f_k \le e_k. \end{aligned}$$

The correct term for a number e that occurs as an index as in p^e is *exponent*. We say that, e is the *exponent* of p in the expression p^e.

In particular, notice that $1 = p_1^0 p_2^0 \cdots p_k^0$ (i.e. in this case, $f_1 = f_2 = \cdots = f_k = 0$); and n is the divisor of n with $f_1 = e_1,\ f_2 = e_2,\ \ldots,\ f_k = e_k$. So the number of divisors of n is *the number of choices of* f_1 times *the number of choices of* f_2 times ... times *the number of choices of* f_k. Now the set of choices for f_1 is $\{0, 1, 2, \ldots, e_1\}$ – there are $e_1 + 1$ such choices. So, in general, the the number of divisors of n is $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$.

17. Show that a natural number n is an exact square if and only if it has an odd number of divisors.

Remark: The statement in the last problem easily follows from the previous problem. Another way to prove it is the following.

If $d \mid n$ then n/d is an integer which also divides n. From this it follows that n has an odd number of divisors if and only if $d = n/d$ for some divisor d of n.

Solution. Now n has a prime decomposition of the form

$$p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}.$$

Also, n is an exact square *if and only if* all the exponents, $e_1, \ldots, e_k$ are *even*; in which case, the product $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$ is a product of *odd* numbers and so is itself *odd*. However, by the result of the previous question, the number of divisors of n is $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$. Thus n is an exact square *if and only if* the number of divisors of n is *odd*.

Solution. (Alternative) Using the remark above, we have

n has an odd number of divisors *if and only if* $d = n/d$ for some divisor d of n.

However, $d = n/d$ for some divisor d of n *if and only if* $n = d^2$ for some divisor d of n; and $n = d^2$ for some divisor d of n is *exactly* what we mean when we say n is an *exact* (or *perfect*) square. Thus

n has an odd number of divisors *if and only if* n is an exact square.

18. There are 50 prisoners in a row of locked cells. With the return of the King from the Crusades, a partial amnesty is declared and it works like this. When the prisoners are still asleep, the jailer walks past the cells 50 times, each time walking from left to right. On the first pass, he turns the lock in every cell (so that every cell is now open). On the second pass he turns the lock on every second cell (meaning that these cells are now locked again). On the third pass, he turns the lock on every third cell, and so on. In general, on the kth pass, he turns the lock on every kth cell. The question is: which cells are unlocked at the end of the process so that the prisoner is free to go?

Solution. The sixth cell lock will be turned 4 times on passes 1, 2, 3 and 6, these being the divisors of 6, and so it will end up locked. The ninth cell lock will be turned on passes 1, 3 and 9 and so will end up unlocked. So we need to know which numbers have an even number of divisors and which have an odd number of divisors.

From the previous question, we know that

A natural number n has an *odd* number of divisors *if and only if* it is a perfect square.

So the squares have an *odd* number of divisors and the non-squares have an *even* number of divisors. The (natural number) squares less than or equal to 50 are: 1, 4, 9, 16, 25, 36 and 49. Consequently, the prisoners in cells: 1, 4, 9, 16, 25, 36 and 49, will be released.