University of Western Australia
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UWA ACADEMY
FOR YOUNG MATHEMATICIANS

Induction: Problems with Solutions

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1. Prove that for any natural number $n \geq 2$,

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1.$$

Hint: First prove

$$\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{(n-1)n} = \frac{n-1}{n}.$$

Solution. Observe that for $k > 0$

$$\frac{1}{k} - \frac{1}{k+1} = \frac{k+1-k}{k(k+1)} = \frac{1}{k(k+1)}.$$

Hence

$$\begin{aligned}\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{(n-1)n} &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} \\ &= 1 - \frac{1}{n} \\ &= \frac{n-1}{n}.\end{aligned}$$

Now, for all $k > 2$

$$\frac{1}{k^2} < \frac{1}{(k-1)k}.$$

So

$$\begin{aligned}\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} &< \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{(n-1)n} \\ &= \frac{n-1}{n} \\ &< 1.\end{aligned}$$

2. Prove for any natural number n that

(i) $1 + 3 + 5 + \cdots + 2n - 1 = n^2$;
(ii) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$;
(iii) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$;
(iv) $1^2 + 4^2 + 7^2 + \cdots + (3n-2)^2 = \frac{1}{2}n(6n^2 - 3n - 1)$;
(v) $2^2 + 5^2 + 8^2 + \cdots + (3n-1)^2 = \frac{1}{2}n(6n^2 + 3n - 1)$.

Solution.

(i) Let $P(n): \quad 1+3+5+\cdots+2n-1=n^2$.

- First we prove $P(1)$.

$$\begin{aligned}\text{LHS of } P(1) &= 1\\ &= 1^2 = \text{RHS of } P(1).\end{aligned}$$

So $P(1)$ is true.

- Now we prove that for any natural number k *"if $P(k)$ is true then $P(k+1)$ is true."* So assume $P(k)$ is true, i.e.

$$1+3+5+\cdots+2k-1=k^2.$$

Now try to deduce $P(k+1)$:

$$\begin{aligned}\text{LHS of } P(k+1) &= 1+3+5+\cdots+2k-1+2(k+1)-1\\ &= \big(\text{LHS of } P(k)\big)+2(k+1)-1\\ &= \big(\text{RHS of } P(k)\big)+2k+1, \text{(by inductive assumption)}\\ &= k^2+2k+1\\ &= (k+1)^2\\ &= \text{RHS of } P(k+1).\end{aligned}$$

So $P(k+1)$ is true, *if $P(k)$ is true.*

- Hence, by induction $P(n)$ is true for all natural numbers n.

(ii) Let $P(n): \quad 1^2+2^2+3^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$.

- Firstly,

$$\begin{aligned}\text{LHS of } P(1) &= 1^2 = 1\\ &= \tfrac{1}{6}(1+1)(2.1+1) = \text{RHS of } P(1).\end{aligned}$$

So $P(1)$ is true.

- Now assume $P(k)$ is true, for some natural number k, i.e.

$$1^2+2^2+3^2+\cdots+k^2=\tfrac{1}{6}k(k+1)(2k+1),$$

and deduce $P(k+1)$:

$$\begin{aligned}\text{LHS of } P(k+1) &= 1^2+2^2+3^2+\cdots+k^2+(k+1)^2\\ &= \big(\text{LHS of } P(k)\big)+(k+1)^2\\ &= \big(\text{RHS of } P(k)\big)+(k+1)^2, \text{(by inductive assumption)}\\ &= \tfrac{1}{6}k(k+1)(2k+1)+(k+1)^2\\ &= \tfrac{1}{6}(k+1)\big(k(2k+1)+6(k+1)\big)\\ &= \tfrac{1}{6}(k+1)(2k^2+7k+6)\\ &= \tfrac{1}{6}(k+1)(k+2)(2k+3)\\ &= \tfrac{1}{6}(k+1)(k+1+1)\big(2(k+1)+1\big)\\ &= \text{RHS of } P(k+1).\end{aligned}$$

So $P(k+1)$ is true, *if $P(k)$ is true.*

- Hence, by induction $P(n)$ is true for all natural numbers n.

(iii) Let $P(n): \quad 1^3+2^3+3^3+\cdots+n^3=\frac{1}{4}n^2(n+1)^2$.

- Firstly,

$$\begin{aligned}\text{LHS of } P(1) &= 1^3 = 1\\ &= \tfrac{1}{4}.1^2(1+1)^2 = \text{RHS of } P(1).\end{aligned}$$

  So $P(1)$ is true.

- Now assume $P(k)$ is true, for some natural number k, i.e.

$$1^3+2^3+3^3+\cdots+k^3=\tfrac{1}{4}k^2(k+1)^2.$$

  and deduce $P(k+1)$:

$$\begin{aligned}\text{LHS of } P(k+1) &= 1^3+2^3+3^3+\cdots+k^3+(k+1)^3\\ &= \big(\text{LHS of } P(k)\big)+(k+1)^3\\ &= \big(\text{RHS of } P(k)\big)+(k+1)^3, \text{(by inductive assumption)}\\ &= \tfrac{1}{4}k^2(k+1)^2+(k+1)^3\\ &= \tfrac{1}{4}(k+1)^2\big(k^2+4(k+1)\big)\\ &= \tfrac{1}{4}(k+1)^2(k^2+4k+4)\\ &= \tfrac{1}{4}(k+1)^2(k+2)^2\\ &= \tfrac{1}{4}(k+1)^2(k+1+1)^2\\ &= \text{RHS of } P(k+1).\end{aligned}$$

  So $P(k+1)$ is true, *if* $P(k)$ *is true.*

- Hence, by induction $P(n)$ is true for all natural numbers n.

(iv) Let $P(n): \quad 1^2+4^2+7^2+\cdots+(3n-2)^2=\frac{1}{2}n(6n^2-3n-1)$.

- Firstly,

$$\begin{aligned}\text{LHS of } P(1) &= 1^2 = 1\\ &= \tfrac{1}{2}.1(6.1^2-3.1-1) = \text{RHS of } P(1).\end{aligned}$$

  So $P(1)$ is true.

- Now assume $P(k)$ is true, for some natural number k, i.e.

$$1^2+4^2+7^2+\cdots+(3k-2)^2=\tfrac{1}{2}k(6k^2-3k-1)$$

  and deduce $P(k+1)$:

$$\begin{aligned}\text{LHS of } P(k+1) &= 1^2+4^2+7^2+\cdots+(3k-2)^2+\big(3(k+1)-2\big)^2\\ &= \big(\text{LHS of } P(k)\big)+\big(3(k+1)-2\big)^2\\ &= \big(\text{RHS of } P(k)\big)+\big(3(k+1)-2\big)^2, \text{(by inductive assumption)}\\ &= \tfrac{1}{2}k(6k^2-3k-1)+9k^2+6k+1\\ &= \tfrac{1}{2}(6k^3-3k^2-k+18k^2+12k+2)\\ &= \tfrac{1}{2}(6k^3+15k^2+11k+2)\\ &= \tfrac{1}{2}(k+1)(6k^2+9k+2)\\ &= \tfrac{1}{2}(k+1)\big(6(k+1)^2-12k-6+9k+2\big)\\ &= \tfrac{1}{2}(k+1)\big(6(k+1)^2-3k-4\big)\\ &= \tfrac{1}{2}(k+1)\big(6(k+1)^2-3(k+1)-1\big)\\ &= \text{RHS of } P(k+1).\end{aligned}$$

  So $P(k+1)$ is true, *if* $P(k)$ *is true.*

- Hence, by induction $P(n)$ is true for all natural numbers n.

(v) Let $P(n): \quad 2^2+5^2+8^2+\cdots+(3n-1)^2 = \frac{1}{2}n(6n^2+3n-1)$.

- Firstly,

$$\begin{aligned} \text{LHS of } P(1) &= 2^2 = 4 \\ &= \tfrac{1}{2}.1(6.1^2+3.1-1) = \text{RHS of } P(1). \end{aligned}$$

  So $P(1)$ is true.

- Now assume $P(k)$ is true, for some natural number k, i.e.

$$2^2+5^2+8^2+\cdots+(3k-1)^2 = \frac{1}{2}k(6k^2+3k-1)$$

  and deduce $P(k+1)$. We could follow an approach similar to the previous exercise; instead, we will demonstrate another technique: that of expanding an expression in k in powers of $k+1$ by replacing k by $k+1-1$.

$$\begin{aligned}
\text{LHS of } P(k+1) &= 2^2+5^2+8^2+\cdots+(3k-1)^2+\big(3(k+1)-1\big)^2 \\
&= \big(\text{LHS of } P(k)\big)+\big(3(k+1)-1\big)^2 \\
&= \big(\text{RHS of } P(k)\big)+\big(3(k+1)-1\big)^2, \text{(by inductive assumption)} \\
&= \tfrac{1}{2}k(6k^2+3k-1)+9(k+1)^2-6(k+1)+1 \\
&= \tfrac{1}{2}k\big(3k(2k+1)-1\big)+9(k+1)^2-6(k+1)+1 \\
&= \tfrac{1}{2}k\Big(3\big((k+1)-1\big)\big(2(k+1)-1\big)-1\Big)+9(k+1)^2-6(k+1)+1 \\
&= \tfrac{1}{2}k\Big(3\big(2(k+1)^2-3(k+1)+1\big)-1\Big)+9(k+1)^2-6(k+1)+1 \\
&= \tfrac{1}{2}\big((k+1)-1\big)\big(6(k+1)^2-9(k+1)+2\big)+\tfrac{1}{2}\big(18(k+1)^2-12(k+1)+2\big) \\
&= \tfrac{1}{2}\Big((k+1)\big(6(k+1)^2-9(k+1)+2\big) \\
&\qquad\qquad -\big(6(k+1)^2-9(k+1)+2\big) \\
&\qquad + (k+1)\big(18(k+1)-12\big)+2\Big) \\
&= \tfrac{1}{2}\Big((k+1)\big(6(k+1)^2-9(k+1)+2\big) \\
&\qquad - (k+1)\big(6(k+1)-9\big)-2 \\
&\qquad + (k+1)\big(18(k+1)-12\big)+2\Big) \\
&= \tfrac{1}{2}(k+1)\Big(6(k+1)^2-\ 9(k+1)+2 \\
&\qquad\qquad\qquad -\ 6(k+1)+9 \\
&\qquad\qquad\qquad -18(k+1)-12\Big) \\
&= \tfrac{1}{2}(k+1)\big(6(k+1)^2+3(k+1)-1\big) \\
&= \text{RHS of } P(k+1).
\end{aligned}$$

  So $P(k+1)$ is true, *if* $P(k)$ *is true.*

- Hence, by induction $P(n)$ is true for all natural numbers n.

3. Prove that for any natural number n,

$$2(\sqrt{n+1}-1) < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

Solution. Let $P(n): \quad 2(\sqrt{n+1}-1) < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$. Now $P(n)$ should be thought of as two *simultaneous inequalities,* namely:

$$\mathrm{LHS}(n) < \mathrm{M}(n) \text{ and } \mathrm{M}(n) < \mathrm{RHS}(n),$$

where

$$\begin{aligned}
\mathrm{LHS}(n) &:= 2(\sqrt{n+1}-1),\\
\mathrm{M}(n) &:= 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \text{ and}\\
\mathrm{RHS}(n) &:= 2\sqrt{n}.
\end{aligned}$$

(M is mnemonic for *"middle"*.)

- Firstly,

  $$\mathrm{LHS}(1) = 2(\sqrt{2}-1) = \frac{2(\sqrt{2}-1)(\sqrt{2}+1)}{\sqrt{2}+1} = \frac{2}{\sqrt{2}+1} < \frac{2}{1+1} = 1 = \mathrm{M}(1),$$

  and

  $$\mathrm{M}(1) = 1 < 2 = 2\sqrt{1} = \mathrm{RHS}(1).$$

  So $P(1)$ is true.
- Now assume $P(k)$ is true, for some natural number k, i.e.

  $$\mathrm{M}(k) > \mathrm{LHS}(k) \text{ and } \mathrm{M}(k) < \mathrm{RHS}(k),$$

  and deduce $P(k+1)$:

  $$\begin{aligned}
  \mathrm{M}(k+1) &= 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}\\
  &= \mathrm{M}(k) + \frac{1}{\sqrt{k+1}}\\
  &> \mathrm{LHS}(k) + \frac{1}{\sqrt{k+1}}, \text{ (by inductive assumption)}\\
  &= 2(\sqrt{k+1}-1) + \frac{1}{\sqrt{k+1}}\\
  &= 2(\sqrt{k+2}-1) - 2(\sqrt{k+2}-\sqrt{k+1}) + \frac{1}{\sqrt{k+1}}\\
  &= 2(\sqrt{k+2}-1) - \frac{2(\sqrt{k+2}-\sqrt{k+1})(\sqrt{k+2}+\sqrt{k+1})}{\sqrt{k+2}+\sqrt{k+1}} + \frac{1}{\sqrt{k+1}}\\
  &= 2(\sqrt{k+2}-1) - \frac{2}{\sqrt{k+2}+\sqrt{k+1}} + \frac{1}{\sqrt{k+1}}\\
  &> 2(\sqrt{k+2}-1) - \frac{2}{\sqrt{k+1}+\sqrt{k+1}} + \frac{1}{\sqrt{k+1}}\\
  &= 2(\sqrt{k+2}-1) - \frac{2}{2\sqrt{k+1}} + \frac{1}{\sqrt{k+1}} = 2(\sqrt{k+2}-1) = \mathrm{LHS}(k+1),
  \end{aligned}$$

and

$$
\begin{aligned}
\mathrm{M}(k+1) &= 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \\
&= \mathrm{M}(k) + \frac{1}{\sqrt{k+1}} \\
&< \mathrm{RHS}(k) + \frac{1}{\sqrt{k+1}}, \text{ (by inductive assumption)} \\
&= 2\sqrt{k} + \frac{1}{\sqrt{k+1}} \\
&= 2\sqrt{k+1} - 2(\sqrt{k+1} - \sqrt{k}) + \frac{1}{\sqrt{k+1}} \\
&= 2\sqrt{k+1} - \frac{2(\sqrt{k+1} - \sqrt{k})(\sqrt{k+1} + \sqrt{k})}{\sqrt{k+1} + \sqrt{k}} + \frac{1}{\sqrt{k+1}} \\
&= 2\sqrt{k+1} - \frac{2}{\sqrt{k+1} + \sqrt{k}} + \frac{1}{\sqrt{k+1}} \\
&< 2\sqrt{k+1} - \frac{2}{\sqrt{k+1} + \sqrt{k+1}} + \frac{1}{\sqrt{k+1}} \\
&= 2\sqrt{k+1} - \frac{2}{2\sqrt{k+1}} + \frac{1}{\sqrt{k+1}} = 2\sqrt{k+1} = \mathrm{RHS}(k+1).
\end{aligned}
$$

i.e. $\mathrm{LHS}(k+1) < \mathrm{M}(k+1) < \mathrm{RHS}(k+1)$.

So $P(k+1)$ is true, *if* $P(k)$ *is true.*

- Hence, by induction $P(n)$ is true for all natural numbers n.

4. Prove $3^n > 2^n$ for all natural numbers n.

Solution. Let $P(n): \quad 3^n > 2^n$.

- Firstly,

$$
\begin{aligned}
\text{LHS of } P(1) = 3^1 &= 3 \\
&> 2 = 2^1 = \text{RHS of } P(1).
\end{aligned}
$$

So $P(1)$ is true.

- Now assume $P(k)$ is true, for some natural number k, i.e.

$$3^k > 2^k$$

and deduce $P(k+1)$:

$$
\begin{aligned}
\text{LHS of } P(k+1) &= 3^{k+1} \\
&= 3^k.3 \\
&> 2^k.3, \text{(by inductive assumption)} \\
&> 2^k.2 \\
&= 2^{k+1} \\
&= \text{RHS of } P(k+1).
\end{aligned}
$$

i.e. LHS of $P(k+1) >$ RHS of $P(k+1)$.

So $P(k+1)$ is true, *if* $P(k)$ *is true.*

- Hence, by induction $3^n > 2^n$ for all natural numbers n.

5. Prove *Bernouilli's Inequality* which states:

If $x \geq -1$ then $(1+x)^n \geq 1 + nx$ for all natural numbers n.

Solution. Let $P(n): \quad (1+x)^n \geq 1+nx$, if $x \geq -1$.

- Firstly,

$$\begin{aligned} \text{LHS of } P(1) = (1+x)^1 &= 1+x \\ &= 1 + 1.x = \text{RHS of } P(1). \end{aligned}$$

  So $P(1)$ is true.
- Now assume $P(k)$ is true, for some natural number k, i.e.

$$(1+x)^k \geq 1+kx, \text{ if } x \geq -1$$

  and deduce $P(k+1)$:

$$\begin{aligned} \text{LHS of } P(k+1) &= (1+x)^{k+1} \\ &= (1+x)^k.(1+x) \\ &= \big(\text{LHS of } P(k)\big).(1+x) \\ &\geq \big(\text{RHS of } P(k)\big).(1+x), (\text{by inductive assumption} \ldots\ 1+x \geq 0 \text{ since } x \geq -1) \\ &= (1+kx)(1+x) \\ &= 1 + (k+1)x + kx^2 \\ &\geq 1 + (k+1)x, (\text{since } k > 0, x^2 \geq 0, \text{ so that } kx^2 \geq 0) \\ &= \text{RHS of } P(k+1). \end{aligned}$$

  So $P(k+1)$ is true, *if* $P(k)$ *is true.*
- Hence, by induction $P(n)$ is true for all natural numbers n.

6. Prove that, if $\sin x \neq 0$ and n is a natural number then

$$\cos x \cdot \cos 2x \cdots \cos 2^{n-1}x = \frac{\sin 2^n x}{2^n \sin x}.$$

Solution. Here we need the identity

$$\sin 2\alpha = 2 \sin\alpha \cos\alpha,$$

for any number α.

Let $P(n): \quad \cos x \cdot \cos 2x \cdots \cos 2^{n-1}x = \dfrac{\sin 2^n x}{2^n \sin x}$, if $\sin x \neq 0$.

- Firstly,

$$\begin{aligned} \text{LHS of } P(1) = \cos x &= \frac{2\sin x \cos x}{2 \sin x}, \ (\text{since } \sin x \neq 0) \\ &= \frac{\sin 2x}{2^1 \sin x} = \text{RHS of } P(1). \end{aligned}$$

  So $P(1)$ is true.

- Now assume $P(k)$ is true, for some natural number k, i.e.

$$\cos x \cdot \cos 2x \cdots \cos 2^{k-1}x = \frac{\sin 2^k x}{2^k \sin x}, \text{ if } \sin x \neq 0$$

and deduce $P(k+1)$:

$$\begin{aligned}
\text{LHS of } P(k+1) &= \cos x \cdot \cos 2x \cdots \cos 2^{k-1}x . \cos 2^k x \\
&= \big(\text{LHS of } P(k)\big) . \cos 2^k x \\
&= \big(\text{RHS of } P(k)\big) . \cos 2^k x, \text{(by inductive assumption)} \\
&= \frac{\sin 2^k x}{2^k \sin x} \cdot \cos 2^k x \\
&= \frac{\sin 2^{k+1} x}{2.2^k \sin x} \\
&= \frac{\sin 2^{k+1} x}{2^{k+1} \sin x} \\
&= \text{RHS of } P(k+1).
\end{aligned}$$

So $P(k+1)$ is true, *if $P(k)$ is true.*

- Hence, by induction $P(n)$ is true for all natural numbers n.

7. Prove that for any natural number $n \geq 2$,

$$\big(1 - \frac{1}{\sqrt{2}}\big)\big(1 - \frac{1}{\sqrt{3}}\big) \cdots \big(1 - \frac{1}{\sqrt{n}}\big) < \frac{2}{n^2}.$$

Solution. We will need the following inequality:

$$\text{For } n \geq 2, \quad n \geq \sqrt{n+1}.$$

We prove this as follows. Assume $n \geq 2$.

$$n^2 - 2n + 1 = (n-1)^2 \geq 0$$

$$\begin{aligned}
\text{So } n^2 &\geq 2n - 1 \\
&= n + n - 1 \\
&\geq n + 2 - 1 \\
&= n + 1
\end{aligned}$$

$$\text{Hence } n \geq \sqrt{n+1}.$$

We will use a slight variation on the usual induction. In ladder terminology our *"first"* rung occurs for $n = 2$.

Let $P(n)$: $\quad \big(1 - \frac{1}{\sqrt{2}}\big)\big(1 - \frac{1}{\sqrt{3}}\big) \cdots \big(1 - \frac{1}{\sqrt{n}}\big) < \frac{2}{n^2}.$

- Firstly,

$$\begin{aligned}
\text{LHS of } P(2) = 1 - \frac{1}{\sqrt{2}} &= \frac{\sqrt{2}-1}{\sqrt{2}} \\
&= \frac{(\sqrt{2}-1)(\sqrt{2}+1)}{\sqrt{2}(\sqrt{2}+1)} \\
&= \frac{1}{2+\sqrt{2}} \\
&< \frac{1}{2} = \frac{2}{2^2} = \text{RHS of } P(2).
\end{aligned}$$

  So $P(2)$ is true.

- Now assume $P(k)$ is true, for some natural number $k \geq 2$, i.e.

$$\big(1 - \frac{1}{\sqrt{2}}\big)\big(1 - \frac{1}{\sqrt{3}}\big)\cdots\big(1 - \frac{1}{\sqrt{k}}\big) < \frac{2}{k^2}$$

  and deduce $P(k+1)$:

$$\begin{aligned}
\text{LHS of } P(k+1) &= \big(1 - \frac{1}{\sqrt{2}}\big)\big(1 - \frac{1}{\sqrt{3}}\big)\cdots\big(1 - \frac{1}{\sqrt{k}}\big)\cdot\big(1 - \frac{1}{\sqrt{k+1}}\big) \\
&= \big(\text{LHS of } P(k)\big)\cdot\big(1 - \frac{1}{\sqrt{k+1}}\big) \\
&< \big(\text{RHS of } P(k)\big)\cdot\big(1 - \frac{1}{\sqrt{k+1}}\big), \text{ (by inductive assumption)} \\
&= \frac{2}{k^2}\cdot\big(1 - \frac{1}{\sqrt{k+1}}\big) \\
&= \frac{2(\sqrt{k+1}-1)}{k^2\sqrt{k+1}} \\
&= \frac{2(\sqrt{k+1}-1)(\sqrt{k+1}+1)}{k^2\sqrt{k+1}(\sqrt{k+1}+1)} \\
&= \frac{2(k+1-1)}{k^2(k+1+\sqrt{k+1})} \\
&= \frac{2}{k(k+1+\sqrt{k+1})} \\
&= \frac{2}{k(k+1)+k\sqrt{k+1}} \\
&\leq \frac{2}{k(k+1)+\sqrt{k+1}.\sqrt{k+1}}, \text{ (since } k \geq \sqrt{k+1} \text{ for } k \geq 2) \\
&= \frac{2}{k(k+1)+k+1} \\
&= \frac{2}{(k+1)^2} \\
&= \text{RHS of } P(k+1).
\end{aligned}$$

  i.e. LHS of $P(k+1) <$ RHS of $P(k+1)$.

  So $P(k+1)$ is true, *if* $P(k)$ *is true.*

- Hence, by induction $P(n)$ is true for all natural numbers $n \geq 2$.

8. Prove that for any natural number n,

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\le\frac{1}{\sqrt{3n+1}}.$$

Solution. Let $P(n):\quad \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\le\frac{1}{\sqrt{3n+1}}.$

- Firstly,

$$\begin{aligned}\text{LHS of } P(1) &= \frac{1}{2}=\frac{1}{\sqrt{4}}\\ &= \frac{1}{\sqrt{3.1+1}} = \text{RHS of } P(1).\end{aligned}$$

  So $P(1)$ is true.

- Now assume $P(k)$ is true, for some natural number k, i.e.

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2k-1}{2k}\le\frac{1}{\sqrt{3k+1}}$$

  and deduce $P(k+1)$:

$$\begin{aligned}\text{LHS of } P(k+1) &= \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2k-1}{2k}\cdot\frac{2(k+1)-1}{2(k+1)}\\
&= \big(\text{LHS of } P(k)\big)\cdot\frac{2k+1}{2k+2}\\
&\le \big(\text{RHS of } P(k)\big)\cdot\frac{2k+1}{2k+2},\ \text{(by inductive assumption)}\\
&= \frac{1}{\sqrt{3k+1}}\cdot\frac{1}{1+\frac{1}{2k+1}}\\
&= \frac{1}{\sqrt{3k+1}}\cdot\frac{1}{\sqrt{(1+\frac{1}{2k+1})^2}}\\
&= \frac{1}{\sqrt{3k+1}}\cdot\frac{1}{\sqrt{1+\frac{2}{2k+1}+\frac{1}{(2k+1)^2}}}\\
&= \frac{1}{\sqrt{3k+1+\frac{2(3k+1)}{2k+1}+\frac{3k+1}{(2k+1)^2}}}\\
&< \frac{1}{\sqrt{3k+1+\frac{6k+2}{2k+1}+\frac{2k+1}{(2k+1)^2}}}\\
&= \frac{1}{\sqrt{3k+1+\frac{6k+2+1}{2k+1}}}\\
&= \frac{1}{\sqrt{3k+1+3}}\\
&= \frac{1}{\sqrt{3(k+1)+1}}\\
&= \text{RHS of } P(k+1).\end{aligned}$$

  i.e. LHS of $P(k+1) <$ RHS of $P(k+1)$.

  So $P(k+1)$ is true, *if* $P(k)$ *is true.*

- Hence, by induction $P(n)$ is true for all natural numbers n.

9. Prove that $7^{2n} - 48n - 1$ is divisible by 2304 for every natural number n.

Solution. Let $P(n): \quad 2304 \,|\, f(n)$ where $f(n) = 7^{2n} - 48n - 1$.

- Firstly, $f(1) = 7^{2.1} - 48.1 - 1 = 0$ and $2304 \,|\, 0$. So $P(1)$ is true.
- Now assume $P(k)$ is true, for some natural number k, i.e.

$$2304 \,|\, f(k).$$

We now deduce $P(k+1)$.

$$\begin{aligned} f(k+1) &= 7^{2(k+1)} - 48(k+1) - 1 \\ &= 7^{2k}.7^2 - 48(k+1) - 1 \\ &= (7^{2k} - 48k - 1).49 + (48k+1).49 - 48(k+1) - 1 \\ &= 49.f(k) + (48k+1).49 - 48(k+1) - 1 \\ &= 49.f(k) + (49-1).48k + 49 - 48 - 1 \\ &= 49.f(k) + 2304k \\ &\equiv 0 \pmod{2304}, \text{ since } 2304 \,|\, f(k) \text{ by the inductive assumption.} \end{aligned}$$

So $P(k+1)$ is true, *if* $P(k)$ *is true.*

- Hence, by induction $P(n)$ is true for all natural numbers n.

10. For every natural number n, show that

$$u_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n.\sqrt{5}}$$

is a natural number.
In fact, u_n is the nth Fibonacci number.

Solution. Before we apply induction we will find the following helpful. Observe that, for any numbers a, b and natural number n,

$$a^{n+1} - b^{n+1} = (a+b)(a^n - b^n) - ab(a^{n-1} - b^{n-1}).$$

Let $\alpha = 1 + \sqrt{5}$ and $\beta = 1 - \sqrt{5}$. Then

$$\begin{aligned} \alpha + \beta &= 2 \\ \alpha\beta &= 1 - 5 = -4. \end{aligned}$$

Also u_n can be expressed more compactly as

$$u_n = \frac{\alpha^n - \beta^n}{2^n.\sqrt{5}}.$$

Now we are ready to apply induction. We shall employ a variant of the usual induction. In ladder terminology, we show:

- we can get onto the *first* **two** rungs; and that
- ***if*** we can get onto the $(k-1)$st and kth rungs ***then*** we can get onto the $(k+1)$st rung.

(At least, this is the general idea! ... we don't quite prove we can get onto the first rung, but we do the next best thing! ... and consequently we need to modify our inductive assumption slightly as well!)

Let $P(n): \quad u_n \in \mathbb{N}$ where $u_n := \dfrac{\alpha^n - \beta^n}{2^n.\sqrt{5}}$.

- Firstly,

$$
\begin{aligned}
u_0 &= \frac{(1+\sqrt{5})^0 - (1-\sqrt{5})^0}{2^0.\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0 \in \mathbb{N} \cup \{0\} \\
u_1 &= \frac{(1+\sqrt{5})^1 - (1-\sqrt{5})^1}{2^1.\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1 \in \mathbb{N}.
\end{aligned}
$$

- Now assume that for some natural number k,

$$
\begin{aligned}
u_{k-1} &\in \mathbb{N} \cup \{0\}, \text{ and} \\
u_k &\in \mathbb{N}.
\end{aligned}
$$

  We now deduce $u_{k+1} \in \mathbb{N}$.

$$
\begin{aligned}
u_{k+1} = \frac{\alpha^{k+1} - \beta^{k+1}}{2^{k+1}.\sqrt{5}} &= \frac{(\alpha+\beta)(\alpha^k - \beta^k) - \alpha\beta(\alpha^{k-1} - \beta^{k-1})}{2^{k+1}.\sqrt{5}} \\
&= \frac{2(\alpha^k - \beta^k) + 4(\alpha^{k-1} - \beta^{k-1})}{2^{k+1}.\sqrt{5}} \\
&= \frac{\alpha^k - \beta^k}{2^k.\sqrt{5}} + \frac{\alpha^{k-1} - \beta^{k-1}}{2^{k-1}.\sqrt{5}} \\
&= u_k + u_{k-1}.
\end{aligned}
$$

  Now u_k, u_{k-1} are nonnegative integers (by our *inductive assumption*); so their sum is again a nonnegative integer. Also, u_k, u_{k-1} are not both zero (since we have assumed $u_k \in \mathbb{N}$); so their sum is a positive integer. Hence,

$$
u_{k+1} \in \mathbb{N}, \ \textbf{\textit{if}} \ u_{k-1} \in \mathbb{N} \cup \{0\} \ \textbf{\textit{and}} \ u_k \in \mathbb{N}.
$$

- Hence, by induction $u_n \in \mathbb{N}$ for all natural numbers n ... and $u_0 = 0$.