

UWA ACADEMY
FOR YOUNG MATHEMATICIANS

Number Theory II: Problems with Solutions

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1. Show $7 \mid 2222^{5555} + 5555^{2222}$.

Solution. By Fermat's Little Theorem, for any natural number n

$$n^7 \equiv n \pmod{7}.$$

So for natural numbers q and r ,

$$\begin{aligned} n^{7q+r} &\equiv (n^7)^q \cdot n^r \pmod{7} \\ &\equiv n^q \cdot n^r \pmod{7} \\ &\equiv n^{q+r} \pmod{7}. \end{aligned}$$

Below we will use this result repeatedly. First though, observe that

$$2222 \equiv 3 \pmod{7} \quad \text{and} \quad 5555 \equiv 4 \equiv -3 \pmod{7}.$$

Thus

$$\begin{aligned} 2222^{5555} + 5555^{2222} &\equiv 3^{5555} + (-3)^{2222} \pmod{7} \\ &\equiv 3^{793+4} + (-3)^{317+3} \pmod{7} \\ &\equiv 3^{113+6} + (-3)^{45+5} \pmod{7} \\ &\equiv 3^{17+0} + (-3)^{7+1} \pmod{7} \\ &\equiv 3^{2+3} + (-3)^{1+1} \pmod{7} \\ &\equiv 3^2(3^3 + 1) \pmod{7} \\ &\equiv 3^2 \cdot 28 \pmod{7} \\ &\equiv 0 \pmod{7}. \end{aligned}$$

Hence $7 \mid 2222^{5555} + 5555^{2222}$.

Solution. (Alternative) The above solution is not a particularly clever way of using Fermat's Little Theorem. Since $n^7 - n$ factorises as $n(n^6 - 1)$, it follows from Fermat's Little Theorem that:

If n is a natural number and $n \not\equiv 0 \pmod{7}$ then $n^6 \equiv 1 \pmod{7}$.

So for natural numbers n , q and r , if $n \not\equiv 0 \pmod{7}$ then

$$\begin{aligned} n^{6q+r} &\equiv (n^6)^q \cdot n^r \pmod{7} \\ &\equiv 1^q \cdot n^r \pmod{7} \\ &\equiv n^r \pmod{7}. \end{aligned}$$

In other words, if $n \not\equiv 0 \pmod{7}$ then we can reduce the power of n modulo 6. (Check the three corollaries to Fermat's Little Theorem in the notes.) Thus

$$\begin{aligned} 2222^{5555} + 5555^{2222} &\equiv 3^{5555} + (-3)^{2222} \pmod{7} \\ &\equiv 3^5 + (-3)^2 \pmod{7} \\ &\equiv 3^2(3^3 + 1) \pmod{7} \\ &\equiv 3^2 \cdot 28 \pmod{7} \\ &\equiv 0 \pmod{7}. \end{aligned}$$

Hence $7 \mid 2222^{5555} + 5555^{2222}$.

2. Show $n^{61} - n$ is divisible by 56 786 730, for all natural numbers n .
Hint: $56\,786\,730 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 61$.

Solution. Here we use Corollary 3 of Fermat's Little Theorem.

Let k, n be natural numbers and let p be a prime. If

$$k \equiv 1 \pmod{p-1}$$

then

$$p \mid n^k - n.$$

We use the theorem several times, with $k = 61$ and p equal to each of the primes 2, 3, 5, 7, 11, 13, 31, 61 in turn.

$$\begin{aligned} 61 &\equiv 1 \pmod{1}. \text{ So } 2 \mid n^{61} - n. \\ 61 &\equiv 1 \pmod{2}. \text{ So } 3 \mid n^{61} - n. \\ 61 &\equiv 1 \pmod{4}. \text{ So } 5 \mid n^{61} - n. \\ 61 &\equiv 1 \pmod{6}. \text{ So } 7 \mid n^{61} - n. \\ 61 &\equiv 1 \pmod{10}. \text{ So } 11 \mid n^{61} - n. \\ 61 &\equiv 1 \pmod{12}. \text{ So } 13 \mid n^{61} - n. \\ 61 &\equiv 1 \pmod{30}. \text{ So } 31 \mid n^{61} - n. \\ 61 &\equiv 1 \pmod{60}. \text{ So } 61 \mid n^{61} - n. \end{aligned}$$

Hence $\text{lcm}(2, 3, 5, 7, 11, 13, 31, 61) = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 61$ divides $n^{61} - n$ for all natural numbers n .

3. When Jane is one year younger than Betty will be when Jane is half as old as Betty will be when Jane is twice as old as Betty is now, Betty will be three times as old as Jane was when Betty was as old as Jane is now.
 One is in her teens and ages are in completed years. How old are they?

Solution. Let Jane's age *now* be x and Betty's age *now* be y . Also, we will represent the ages of Jane and Betty at time i by x_i and y_i respectively. Let's rewrite the given information, including these choices of variables.

When Jane [is x_1 , she] is one year younger than Betty will be [when she is y_2 and] when Jane [is x_2 and she] is half as old as Betty will be [when she is y_3 and] when Jane [is x_3 and she] is twice as old as Betty is now [when she is y], Betty will be [y_1 and she will be] three times as old as Jane was when [she was x_4 and when] Betty was [y_4 and she was] as old as Jane is now [when she is x].

So we get the following equations:

$$\begin{aligned}x_1 &= y_2 - 1 \\x_2 &= \frac{1}{2}y_3 \\x_3 &= 2y \\y_1 &= 3x_4 \\y_4 &= x.\end{aligned}$$

Now define a, b, c, d such that $x_1 = x + a$, $x_2 = x + b$, $x_3 = x + c$, $x_4 = x + d$; so that $y_1 = y + a$, $y_2 = y + b$, $y_3 = y + c$, $y_4 = y + d$. Hence the above equations become:

$$\begin{aligned}x + a &= y + b - 1 \\x + b &= \frac{1}{2}(y + c) \\x + c &= 2y \\y + a &= 3(x + d) \\y + d &= x.\end{aligned}$$

Now rearrange these equations (and multiply by a factor where appropriate):

$$\begin{array}{rcll}x + a - b & & & = y - 1 \\x & + & b - \frac{1}{2}c & = \frac{1}{2}y \\ \frac{1}{2}x & & + \frac{1}{2}c & = y \\3x - a & & + 3d & = y \\3x & & - 3d & = 3y.\end{array}$$

Adding these equations we find that a, b, c, d cancel and we get:

$$8\frac{1}{2}x = 6\frac{1}{2}y - 1$$

i.e.

$$13y - 17x = 2. \tag{1}$$

Since x, y are only allowed to be integers, this equation is an example of a *linear Diophantine Equation*. A method for solving such an equation is to first apply the *Euclidean Algorithm* to find the gcd d of 13 and 17 (which is clearly 1). Tracing the algorithm backwards one can express d in terms of 13 and 17. Applying the *Euclidean Algorithm* then (see Notes) we get:

$$3 \left| \begin{array}{cc|c} 13 & 17 & \\ \hline 12 & 13 & 1 \\ \hline 1 & 4 & \end{array} \right|$$

Thus

$$\begin{aligned}1 &= 13 - 12 \\ &= 13 - 3.4 \\ &= 13 - 3.(17 - 13) \\ &= 4.13 - 3.17.\end{aligned}$$

So

$$\begin{aligned}2 &= 8.13 - 6.17 \\ &= 8.13 + 13.17t - 13.17t - 6.17 \\ &= 13(8 + 17t) - 17(6 + 13t).\end{aligned}$$

Comparing the above with (1) we see that we have a solution for (1) if

$$\begin{aligned}x &= 6 + 13t \\ y &= 8 + 17t\end{aligned}$$

for some integer t . (In fact all solutions of (1) are of this form ... but we will not prove this.) We were also given that one girl was in her teens, so that t must be 1 and

$$x = 19, \quad y = 25.$$

Hence Jane is 19 and Betty is 25.

Answers and Selected Solutions to *Fun with Alphemetics and others*

One day I'll put insert the alphemetics ... they were handed out on a handwritten sheet.

The *rules* of alphemetics

1. Each letter represents a distinct digit in $\{0, 1, \dots, 9\}$.
2. The left-most digit of a number is never 0.
3. An alphemetic represents a sum (of decimal numbers).

The *real* problem

It's not too hard to find a solution. It is *much harder* to prove that a solution found is the *only* solution ... you need to consider *all* the possibilities and eliminate them until only one possibility remains ... and check that that possibility is a solution.

Below the answers are given for each sum but only one (the easiest) is actually solved to give an idea of one possible strategy of solution, without spoiling the *fun* too much.

Problems.

1. MOUSE = 13927
2. SEESAW = 366398 (even)
3. SAILS = 10741 (ODD = 633, odd)
4. BANANA = 129292 (even)

5. SHOPS = 12751
6. MONEY = 59731
7. BEANS = 12493 (STAN = 3749, odd)
8. ROMAN = 17568
9. DRESS = 10288
10. HEALER = 103507
11. MONEY = 10587 (odd)
12. MILLION = 5644683
13. GREW = 1039
14. EVEREST = 4042486
15. BOAST = 17253 (odd)
16. HAHA = 1717

Solution.

$$\begin{array}{r}
 A \ H \ A \\
 A \ H \ A \\
 \ A \\
 \hline
 W \ A \ G \\
 \hline
 H \ A \ H \ A
 \end{array}$$

- First let the *carry* from the right column be k then it is easy to see that

$$2A + G = 10k \tag{2}$$

$$H + A + k = 10 \tag{3}$$

$$A + W + 1 = 10 \tag{4}$$

$$H = 1 \tag{5}$$

- Now k is either 1 or 2 (considering the least and largest values possible for A and G ... remembering that $2A + G$ is *exactly* a multiple of 10.)
- Since $H = 1$ and $k \in \{1, 2\}$, it follows from (3) that A is 8 or 7.
- If $A = 8$ then by (4) $W = 1$, in which case H and W are equal (which is not allowed). So $A \neq 8$. Hence $\boxed{A = 7}$. Therefore $k = 2$ and $\boxed{G = 6}$, and by (4) $\boxed{W = 2}$.
- So finally we get (and check)

$$\begin{array}{r}
 7 \ 1 \ 7 \\
 7 \ 1 \ 7 \\
 \ 7 \\
 \hline
 2 \ 7 \ 6 \\
 \hline
 1 \ 7 \ 1 \ 7
 \end{array}$$

- So HAHA = 1717.

17. GROG = 1061
18. TREAT = 14931
19. SLING = 10594
20. BOAT = 1748
21. HURTS = 10265 ($D = 3 \dots$ so that BAD is odd, $\{B, C\} = \{4, 8\}$)
22. BABY = 4249 (BAG = 423, odd)
23. SALLY = 10885
24. FLAMES = 107853 ($T = 6, K = 2$ or $T = 4, K = 6$)
25. PASHA = 10970 (SHAM = 9703, odd)
26. SIKH = 9867 (odd)
27. RAT = 291
28. PLEASE = 103593 (STAMP = 92581, prime \dots the other possibilities are easily eliminated by checking for divisibility by 3)
29. EXTRA = 68475 (SUPER = 13967, prime $\dots 7 \mid 21679$)
30. ENOUGH = 139208 ($\{R, T\} = \{5, 7\}$)
31. COOK = 1004
32. HOT = 931 ($\{G, P\} = \{2, 4\}$)
33. PEPPER = 131135 (odd)
34. APPLE = 81169
35. ELSIE = 13751
36. HEAD = 2048
37. SAMMY = 58442 (even)
38. GONG = 1061
39. DRUM = 1527
40. MONEY = 10728 (MEN = 127, odd)
41. YEAST = 87130 ($\{U, I, F\} = \{2, 5, 9\}$)
42. LAST = 1498 (TAN = 847, odd)
43. SPLIT = 10469

44. FARM = 1256
45. YEMEN = 15753 (odd)
46. Jane is 19, Betty is 25. (Diophantine equation: $13y - 17x = 2$)
47. Helen is 10, Betty is 18. (Diophantine equation: $51x - 28y = 6$)
48. Susie is 26, Pam is 19. (Diophantine equation: $14x - 19y = 3$)
49. Lynda is 18, Dick is 17. (Diophantine equation: $18x = 17y$)
50. Time was 3:36. (Diophantine equation: $12x + 5y = 216$, where $x : y$ represents the time)
51. At 9:48 the hour hand is on the 49th minute mark ($5.9 + \frac{1}{12}48 = 49$). So time is 9:49.
52. **The Problem:** There was little traffic that day, too little to interfere with the steady progress of the 3 km column. Heading the column, Tom turned his jeep and drove back to check the rear. All was well and he was able to maintain a steady speed there and back without any delays. On returning to his lead position, Tom noted that the column had advanced just 4 km while he was away. How far had he driven in that time?

Solution. Let v be the jeep's speed and u be the column's speed (in km/hr), let T be the total time of Tom's round trip (in hrs), and let x be the total distance Tom travelled (in km). Then

$$T = \frac{3}{v+u} + \frac{3}{v-u} \quad (6)$$

$$uT = 4 \quad (7)$$

$$vT = x \quad (8)$$

Rearranging (7) and (8) we get

$$u = \frac{4}{T} \quad (9)$$

$$v = \frac{x}{T} \quad (10)$$

(noting that this is allowed since T cannot be zero). Now substitute (9) and (10) in (6):

$$\begin{aligned} T &= \frac{3}{\frac{x}{T} + \frac{4}{T}} + \frac{3}{\frac{x}{T} - \frac{4}{T}} \\ &= \frac{3T}{x+4} + \frac{3T}{x-4} \end{aligned}$$

Dividing both sides by T (which is non zero), followed by rearranging we get:

$$\begin{aligned} 1 &= \frac{3}{x+4} + \frac{3}{x-4} \\ (x+4)(x-4) &= 3((x-4) + (x+4)) \end{aligned}$$

$$x^2 - 16 = 6x$$

$$x^2 - 6x - 16 = 0$$

$$(x-8)(x+2) = 0.$$

So x is 8 or $-2 \dots$ but x cannot be negative. Hence Tom drove 8 km.

53. **The Problem:**

A testy old Turk in Toulouse,
Spent fifty-six francs on some booze,
At four francs eleven,
And three ninety seven,
How many of each did he choose?

Solution. Let x and y be the numbers of bottles at Fr 4.11 and Fr 3.97, respectively. Then

$$411x + 397y = 5600 \quad (11)$$

Observe that

$$411 - 397 = 14.$$

So

$$411.400 - 397.400 = 5600.$$

So x, y of the *Diophantine Equation* (11) must satisfy

$$\begin{aligned} x &= 400 - 397t \\ y &= -400 + 411t \end{aligned}$$

for some integer t . For x, y to be both positive we need $t = 1$. So $x = 3$ and $y = 11$. That is, the Turk bought 3 bottles at Fr 4.11 and 11 bottles at Fr 3.97.

54. **The Problem.**

There was a young lady called Chris
Who when asked her age answered this:
“Two thirds of its square
Is a cube, I declare.”
So what was the age of this Miss?

Solution. Let her age be x and the cube be y^3 . (Both x and y are integers.) Then

$$\frac{2}{3}x^2 = y^3.$$

Hence

$$2x^2 = 3y^3 \quad (12)$$

Now 2 divides the LHS ... so 2 divides the RHS and hence $2 \mid y$. Thus 8 divides the RHS. So $2 \mid x$.

Now try a similar idea with 3: 3 divides the RHS ... so 3 divides the LHS and hence $3 \mid x$. Thus 9 divides the LHS (and hence the RHS). So $3 \mid y$. So 81 divides the RHS and hence $9 \mid x$.

Thus $\text{lcm}(2, 9) = 18 \mid x$ and $\text{lcm}(2, 3) = 6 \mid y$. Now let $x = 18\alpha$ and $y = 6\beta$. Then substituting in (12) we get:

$$2 \cdot 18^2 \alpha^2 = 3 \cdot 16^3 \beta^3$$

which reduces to

$$\alpha^2 = \beta^3$$

Suppose for a prime p , p^i is the largest power of p that divides the LHS. Then i is a multiple of 2. Also since p^i is the largest power of p that divides the RHS, i is a multiple of 3. Thus i is a multiple of 6. Either i is 0 for every prime p or i is at least 6 for some prime p . If i is at least 6 for some p then at least $p^3 \mid \alpha$ in which case $\alpha \geq 2^3 = 8$ and $x \geq 8.18 = 144$ and by today's standards Chris would not be a *young* lady and a candidate for the Guinness Book of Records. Thus $\alpha = 1$ and $x = 18$. So Chris is 18.

55. The Problem.

Change my last to a nine,
And my first to a five,
Then the square of a third of
A ninth of me you'll derive.
A few seconds of thought
And I'm sure you'll see
My three digits are clear
So just what must they be?

Solution. Let the number be n and let its second digit be y . Then

$$\left(\frac{1}{3} \cdot \frac{1}{9}n\right)^2 = 509 + 10y.$$

The LHS must in fact be the square of an integer, and so the RHS must be 529, i.e. $y = 2$. Thus

$$\frac{1}{27}n = 23.$$

So $n = 621$ (and the second digit y , is indeed 2).

56. **The Problem.** Bob checked the boy's figures. "Nothing wrong with that," he declared. "444 888 is indeed one less than the square of 667. Do you think that's the only 6-digit number that's one less than a square, with its second half just double its first half?" "No, there is one other," Joe replied. "But see if you can find it yourself."

Solution. Represent the number as $1002(100x + 10y + z)$. Then

$$1002(100x + 10y + z) = n^2 - 1$$

for some integer n . So $1002 \mid (n-1)(n+1)$ and

$$10^5 \leq n^2 < 10^6.$$

Hence $316 < n < 1000$. Since $1002 = 2 \cdot 3 \cdot 167$, each of 2, 3, 167 divides at least one of $n-1$ or $n+1$. Since

$$n-1 \equiv n+1 \pmod{2}$$

2 must divide both $n - 1$ and $n + 1$. Thus $2.167 = 334$ divides one of $n - 1$ or $n + 1$. So

$$n = 334k + 1 \text{ or } n = 334k - 1$$

for some integer k , such that $316 < n < 1000$. Hence $3 \in \{333, 667, 335, 669\}$. But $3 \mid n - 1$ or $3 \mid n + 1 \dots$ so $3 \nmid n$ and so $n \neq 333$ and $n \neq 669$. If $n = 667$ then $n^2 - 1 = 444\,888$. If $n = 335$ then

$$\begin{aligned}(n - 1)(n + 1) &= 334.336 \\ &= 1002.112 \\ &= 112\,224.\end{aligned}$$

Thus the other number is 112 224 which is one less than the square of 335.

57. **The Problem.** Bob emptied the box onto the table. “You’ve got 56 coins here, all 10 c, 5 c and 1 c. Do you know how much it is?”

“Yes, it’s just 3 c short of a dollar,” Betty replied.

How much of each?

Solution. Let x, y, z be the numbers of 10 c, 5 c and 1 c coins, respectively. Then

$$\begin{aligned}10x + 5y + z &= 97 \\ x + y + z &= 56.\end{aligned}$$

Subtracting these equations, eliminates z , giving:

$$9x + 4y = 41 \tag{13}$$

Observe that

$$9.1 - 4.2 = 1.$$

So

$$9.41 - 4.82 = 41.$$

Hence, in order to satisfy the *Diophantine Equation* (13), we need

$$\begin{aligned}x &= 41 - 4t \\ y &= -82 + 9t,\end{aligned}$$

for some integer t . Now $x > 0$ and $y > 0 \dots$ so $t = 10$ and hence $x = 1$ and $y = 8$. So $z = 56 - 8 - 1 = 47$. So there were one 10 c, eight 5 c and forty-seven 1 c coins.

58. **The Problem.** “You and Bill taken last week, eh?” said John, looking at the photo on the desk. “Very nice too. How old is he now?”

Mike thought a moment. “A teaser for you,” he replied. “If you divide your age by mine and subtract the result from your age divided by his, you get just one seventh of your age.”

How old was Mike?

Solution. Let m be Mike's age, b be Bill's age and j be John's age. Then

$$\frac{j}{b} - \frac{j}{m} = \frac{1}{7}j.$$

So, since $j \neq 0$,

$$\begin{aligned}\frac{1}{b} - \frac{1}{m} &= \frac{1}{7} \\ m - b &= \frac{mb}{7}.\end{aligned}\tag{14}$$

Hence $7 \mid m$ or $7 \mid b$. Also, since $m > 0$ and $b > 0$, $\frac{1}{7}mb > 0$, and hence, by (14), $m > b$. Suppose $b = 7k$, for some integer k . Then

$$\begin{aligned}b &= m - km \\ &= (1 - k)m.\end{aligned}$$

Now $b > 0 \dots$ so $k = 0$ in which case $b = m$, which is impossible. So $7 \nmid b$. Hence $m = 7k$, for some integer k , and so by (14)

$$\begin{aligned}7k &= b + kb \\ &= (k + 1)b.\end{aligned}$$

Now $m > 0$. So k , and hence $k + 1$, is positive. So

$$b = \frac{7k}{k + 1}.$$

Hence $k + 1 \mid 7k$. But $(k + 1, k) = 1$. Hence $k + 1 \mid 7$. So $k + 1$ is 1 or 7. Now $k + 1 \neq 1$, since then $k = 0$ and $b = m$ which is impossible. Thus $k + 1 = 7$, i.e. $k = b = 6$ and $m = 7k = 42$. So Mike's age is 42 (and Bill is 6). (There isn't quite enough information to determine John's age ... he could be 42 or 84 – a common multiple of 6 and 42.)

59. **The Problem.** Stan looked up from the page he'd been working on. "Bill try this one. A four-digit number – by swapping the first and the second digits, and the third and the fourth you get four times the original number." What was the number?

Solution. Let the number be $1000w + 100x + 10y + z$. Then

$$4(1000w + 100x + 10y + z) = 1000x + 100w + 10z + y.$$

Rearranging we get:

$$\begin{aligned}3900w - 600x &= 6z - 39y \\ \text{i.e. } 1300w - 200x &= 2z - 13y\end{aligned}\tag{15}$$

Now $0 \leq y \leq 9$ and $0 \leq z \leq 9$. So

$$\begin{aligned}2.0 - 13.9 &\leq 2z - 13y \leq 2.9 - 13.0 \\ \text{i.e. } -117 &\leq 2z - 13y \leq 18.\end{aligned}$$

Now $100 \mid 1300w - 200x$. So 100 divides the RHS of (15). Hence $2z - 13y$ is 0 or -100 . Suppose that $2z - 13y = 0$. Then

$$13w - 2x = 0,$$

and hence

$$\begin{aligned}w &= 2t \\x &= 13t,\end{aligned}$$

for some integer t such that $0 < w \leq 9$ (w is the left-most digit and so is necessarily nonzero) and $0 \leq x \leq 9$, which is impossible.

Thus

$$2z - 13y = -100 \tag{16}$$

and hence by (15)

$$\begin{aligned}1300w - 200x &= -100 \\13w - 2x &= -1.\end{aligned}$$

Observe that

$$13.1 - 2.7 = -1 \tag{17}$$

So

$$\begin{aligned}w &= 1 + 13t \\x &= 7 + 2t\end{aligned}$$

for some integer t . The only possibility is $t = 0$. Hence $w = 1$, $x = 7$. We could use (17) again, or observe that

$$\begin{aligned}13.8 - 2.2 &= 100 \\ \text{i.e. } 2.2 - 13.8 &= -100.\end{aligned}$$

Hence, to satisfy the *Diophantine Equation* (16),

$$\begin{aligned}y &= 8 + 2s \\z &= 2 + 13s\end{aligned}$$

for some integer s , such that $0 \leq y \leq 9$ and $0 \leq z \leq 9$. The only possibility is $s = 0$. Hence $y = 8$, $z = 2$.

So the number was 1782.