Some locally 3–arc transitive graphs constructed from triality ⋆

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Abstract

Two new infinite families of locally 3–arc transitive graphs are constructed which have $\mathrm{Aut}(\mathrm{P}\Omega^+(8,q))$ as their automorphism group.

Key words: locally s–arc transitive graphs, orthogonal groups, triality.

1 Introduction

The hyperbolic quadric associated with an 8-dimensional vector space has a rich geometry which admits a triality between the totally singular 1–spaces and the two classes of maximal totally singular subspaces. This triality has been used by Tits to construct generalised hexagons [18] and is associated with the graph automorphism of order three of the orthogonal groups $\mathrm{P}\Omega^+(8,q)$. In turn, this graph automorphism gives rise to the simple groups ${}^3D_4(q)$ discovered by Steinberg [16]. We exploit this geometry to construct two new infinite families of locally 3–arc transitive graphs which have $\mathrm{Aut}(\mathrm{P}\Omega^+(8,q))$ as their automorphism group.

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A graph Γ is *biregular of valency* $\{k_1, k_2\}$ if it is bipartite and vertices in the i^{th} part of the bipartition have valency k_i, for $i = 1, 2$. Given a vertex v in Γ, we denote the set of vertices adjacent to v by $\Gamma(v)$. Then for a group G of automorphisms of Γ, the permutation group induced by the vertex stabiliser G_v on $\Gamma(v)$ is denoted by $G_v^{\Gamma(v)}$. Given an intransitive normal subgroup N of G, the *quotient graph* Γ_N is the graph whose vertices are the orbits of N and two N–orbits B_1 and B_2 are adjacent if there exists a vertex in B_1 which is adjacent to a vertex in B_2. An *s–arc* in Γ is an $(s+1)$–tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that each v_i is adjacent to v_{i+1} and $v_{i-1} \neq v_{i+1}$. We call Γ *locally s–arc transitive* if for each vertex v of Γ, the stabiliser of v in the automorphism group of Γ induces a transitive action on the set of s–arcs beginning at v.

Theorem 1.1 *Let* $T = \mathrm{P}\Omega^+(8, q)$*, where* $q = p^f$ *is a prime power, and let* $G = \mathrm{Aut}(T)$*. Then there is a connected biregular graph* $\Gamma = \mathcal{G}(q)$*, of valency* $\{3, q^2 + q + 1\}$ *and with automorphism group* G *such that*

(1) Γ *is locally* 3*–arc transitive but not locally* 4*–arc transitive,*
(2) $\Gamma_T \cong K_{1,3}$*,*
(3) given a vertex v *of valency* 3 *and an adjacent vertex* w *of valency* q^2+q+1*, we have* $G_v^{\Gamma(v)} \cong S_3$ *and* $G_w^{\Gamma(w)} \cong \mathrm{P\Gamma L}(3, q)$*, and*
(4) $G_v = G_1 : S_3$*,* $G_w = G_2 : C_2$ *and* $G_{vw} = G_1 : C_2$*, where*

$$G_1 \cong [q^{11}] : ((C_{q-1}^3 \circ \mathrm{GL}(2, q))) : C_f),$$

$$G_2 \cong [q^9] : ((C_{q-1}^2 \circ \mathrm{GL}(3, q)) : C_f).$$

By a group $A \circ B$, we mean the group $(A \times B)/H$ where H is some normal subgroup contained in the centre of $A \times B$. For more precise information about the structure of the vertex stabilisers of these graphs, and in particular the subgroups G_1, G_2 of part (4), see Section 2 (especially the discussion preceding the displayed equations (2.1) and (2.2)).

Theorem 1.2 *Let* $T = \mathrm{P}\Omega^+(8, p)$*, where* p *is an odd prime, and let* $G = \mathrm{Aut}(T)$*. Then there is a connected biregular graph* $\Gamma = \mathcal{H}(p)$*, of valency* $\{4, 7\}$ *and with automorphism group* G *such that*

(1) Γ *is locally* 3*–arc transitive but not locally* 4*–arc transitive,*
(2) $\Gamma_T \cong K_{1,4}$*,*
(3) given a vertex v *of valency* 4 *and an adjacent vertex* w *of valency* 7*, we have* $G_v^{\Gamma(v)} \cong S_4$ *and* $G_w^{\Gamma(w)} \cong \mathrm{SL}(3, 2)$*, and*
(4) $G_v \cong (2^{3+6} : S_4).S_4$*,* $G_w \cong 2^{3+6} : (\mathrm{SL}(3, 2) \times S_3)$ *and* $G_{vw} \cong 2^{3+6} : (S_4 \times S_3)$*.*

Both $\mathcal{G}(q)$ and $\mathcal{H}(p)$ have geometric interpretations, with $\mathcal{G}(q)$ associated with the polar space of the hyperbolic quadric and $\mathcal{H}(p)$ associated with a geometry whose diagram is an extended D_4, that is, a central node with four neighbours, (see Remark 4.5). The construction for $\mathcal{H}(p)$ can also be applied to produce a graph $\mathcal{H}(q)$ corresponding to $\mathrm{P}\Omega^+(8, q)$ for any odd q, but if q is not prime, then $\mathcal{H}(q)$ is not connected, see Remark 4.6. Our proofs to determine the full automorphism groups of $\mathcal{G}(q)$ and $\mathcal{H}(p)$ use various results that rely on the finite simple group classification.

Let Γ be an undirected graph with vertex set $V\Gamma$. Given $G \leqslant \mathrm{Aut}(\Gamma)$, we say that Γ is *locally* (G, s)*–arc transitive* if Γ contains an s–arc and, given any two s–arcs α and β starting at the same vertex v, there is an element of G_v which maps α to β. Thus Γ is locally s–arc transitive if Γ is locally (G, s)–arc transitive for some $G \leqslant \mathrm{Aut}(\Gamma)$. If G also acts transitively on $V\Gamma$ then G acts transitively on the set of all s–arcs of Γ and we say that Γ is (G, s)*–arc transitive*. The problem of finding locally s–arc transitive graphs with large values of s has received much interest. Stellmacher [17], following earlier work of Tutte [19], [20] and Weiss [21] in the vertex transitive case, proved that, for a locally (G, s)–arc transitive graph with all vertices having valency at least three, $s \leq 9$. This bound is sharp as the maximum value is attained by the incidence graphs of the generalised octagons associated with the simple groups ${}^2F_4(2^n)$.

Suppose in this paragraph that all vertices in Γ have valency at least two. Then if Γ is locally (G, s)–arc transitive, it is also locally $(G, s-1)$–arc transitive. Also Γ is locally $(G, 2)$–arc transitive if and only if, for all vertices v, $G_v^{\Gamma(v)}$ is 2–transitive (see for example [6, Lemma 3.2]). If G is vertex intransitive, then Γ is a bipartite graph and the two parts Δ_1 and Δ_2 of the bipartition are orbits of G. Thus bipartite locally (G, s)–arc transitive graphs are biregular.

In [6] a program for studying locally (G, s)–arc transitive graphs with $s \geq 2$ was initiated which focused on the 'global' action of G. Previously, the local action of G_v on $\Gamma(v)$, and the structure of the subgroup G_v, had been the main theme of investigations. Let Γ be a bipartite graph with bipartite halves Δ_1 and Δ_2 and having a group G of automorphisms whose orbits are Δ_1 and Δ_2. If G has a nontrivial normal subgroup N intransitive on at least one of Δ_1, Δ_2, we can form the quotient graph Γ_N. It was shown in [6] that if Γ is a locally (G, s)–arc transitive graph and N is intransitive on both Δ_1 and Δ_2, then Γ_N is locally $(G/N, s)$–arc transitive. If N is intransitive on only one of the G–orbits, say on Δ_i, then Γ_N is the star $K_{1,n}$, where n is the number of orbits of N on Δ_i. This suggests that to study locally (G, s)–arc transitive graphs attention should focus on the case where G acts faithfully on both orbits and quasiprimitively on at least one (where a permutation group is *quasiprimitive* if all nontrivial normal subgroups are transitive).

The quasiprimitive permutation groups were classified in [14] in an 'O'Nan–Scott like' theorem, and were described in [15] as being of one of eight types. The possible quasiprimitive types for the action of a locally $(G, 2)$–arc transitive graph on one of its orbits were determined in [6]. If G acts quasiprimitively on only Δ_1 then there are five possible types for this quasiprimitive action. For four of them, a nice characterisation of the possible graphs was given in [7]. The fifth type is the almost simple case, that is, there exists some nonabelian simple group T such that $T \leqslant G \leqslant \mathrm{Aut}(T)$ and T acts transitively on Δ_1. As G does not act quasiprimitively on Δ_2, T acts intransitively on Δ_2. It was shown in [7, Theorem 1.4] that for such a graph to exist $\mathrm{Out}(T)$ must have a 2–transitive representation of degree equal to the valency of the vertices in Δ_1. This result drew our attention to the groups $T = \mathrm{P}\Omega^+(8,q)$ as here $\mathrm{Out}(T)$ contains S_3 or S_4. Studying these groups led us to our constructions of two new families of locally 3–arc transitive graphs in Sections 3 and 4. The second construction is the only one we currently know of a locally $(G,2)$–arc transitive graph where G is an almost simple group whose socle T acts transitively on Δ_1 but intransitively on Δ_2 such that the vertices in Δ_1 have valency greater than 3.

Problem 1.3 *Classify the biregular, locally (G,s)–arc transitive graphs of valency $\{3,k\}$ or $\{4,k\}$, where G is an almost simple group with socle $T = \mathrm{P}\Omega^+(8,q)$ such that T acts transitively on Δ_1 and intransitively on Δ_2.*

Note that by [6, Lemma 5.6], such graphs are not locally $(G,4)$–arc transitive.

Given two groups G and H, we denote the split extension of G by H, by $G:H$ while $G.H$ denotes some extension of G by H, not necessarily split. Furthermore, $[n]$ denotes a group of order n, p^d denotes an elementary abelian group of order p^d, while p^{d+r} denotes a p–group P with centre $Z = p^d$ such that $P/Z = p^r$. For a subgroup L of G, we denote the normaliser of L in G by $N_G(L)$ and the centraliser of L in G by $C_G(L)$. Also $O_2(G)$ denotes the largest normal 2–subgroup of G.

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2 Some geometry

In this section we provide a brief outline of the geometry associated with the 8–dimensional orthogonal groups and introduce some notation. For more details, the reader may refer to [1], [3], [10], or [11]. The second and third references have a wealth of information in the 8–dimensional case. Let V be an 8–dimensional vector space over the field $F = \mathrm{GF}(q)$ equipped with a nondegenerate quadratic form Q, that is, $Q : V \to F$ is a function such that

$$Q(\lambda v) = \lambda^2 Q(v) \text{ for all } \lambda \in F \text{ and } v \in V$$

and

$$\begin{aligned} B : V \times V &\to F \\ (v, w) &\mapsto Q(v + w) - Q(v) - Q(w) \end{aligned}$$

is a nondegenerate bilinear form. A subspace W of V is called *totally singular* if $Q(v) = 0$ for all vectors $v \in W$. We let Q have *maximal Witt index*, that is, the maximal totally singular subspaces of V have dimension 4. Then V has a basis $\{e_1, \ldots, e_4, f_1, \ldots, f_4\}$, known as the *standard basis*, such that $\langle e_1, \ldots, e_4\rangle$ and $\langle f_1, \ldots, f_4\rangle$ are totally singular 4–spaces and $B(e_i, f_j) = \delta_{ij}$ for all $i, j \in \{1, \ldots, 4\}$.

The group $\Gamma\mathrm{L}(8, q)$ is the group of all invertible semilinear transformations of V under composition, that is, all $g : V \mapsto V$ such that g preserves addition and there exists $\alpha \in \mathrm{Aut}(F)$ such that, for all $\lambda \in F$ and $v \in V$,

$$(\lambda v)^g = \lambda^\alpha v^g.$$

The subgroup of all invertible linear transformations is denoted by $\mathrm{GL}(8, q)$. Let

$$\begin{aligned} \Gamma\mathrm{O}^+(8, q) &= \{g \in \Gamma\mathrm{L}(8, q) : \exists \lambda \in F, \alpha \in \mathrm{Aut}(F) \text{ s.t.} \\ &\qquad\qquad Q(v^g) = \lambda Q(v)^\alpha \ \forall v \in V\}, \\ \mathrm{GO}^+(8, q) &= \{g \in \mathrm{GL}(8, q) : \exists \lambda \in F \text{ s.t. } Q(v^g) = \lambda Q(v) \ \forall v \in V\}, \\ \mathrm{O}^+(8, q) &= \{g \in \mathrm{GL}(8, q) : Q(v^g) = Q(v) \ \forall v \in V\}, \\ \mathrm{SO}^+(8, q) &= \{g \in \mathrm{O}^+(8, q) : \det(g) = 1\} \end{aligned}$$

and let $\Omega^+(8, q)$ be the commutator subgroup of $\mathrm{SO}^+(8, q)$, that is the subgroup generated by all commutators $h^{-1}g^{-1}hg$. Let Z be the group of all scalar transformations in $\mathrm{GL}(8, q)$ and for each group X in our sequence let $PX = XZ/Z \cong X/(X \cap Z)$. Then $T = \mathrm{P}\Omega^+(8, q)$ is a finite nonabelian simple group and

$$|T| = \frac{1}{(2, q-1)^2} q^{12}(q^2 - 1)(q^4 - 1)^2(q^6 - 1).$$

Now $Z \leqslant \mathrm{GO}^+(8, q)$ and we have from [11, (2.7.2)], that if q is even, then $\mathrm{GO}^+(8, q) = \mathrm{O}^+(8, q) \times Z$. If q is odd, let μ be a primitive element of $\mathrm{GF}(q)$. Then the linear transformation

$$\delta = \begin{pmatrix} \mu I & 0 \\ 0 & I \end{pmatrix}$$

written with respect to the standard basis, lies in $\mathrm{GO}^+(8, q)$. (Here I is the 4×4 identity matrix.) Furthermore, by [11, (2.7.2)], $\mathrm{GO}^+(8, q) = \mathrm{O}^+(8, q) : \langle \delta \rangle$, when q is odd. If $\mathrm{GF}(q)$ has characteristic p, let ϕ be the semilinear transformation

$$\phi : \sum_{i=1}^{4}(\lambda_i e_i + \xi_i f_i) \mapsto \sum_{i=1}^{4}(\lambda_i^p e_i + \xi_i^p f_i).$$

Then by [11, (2.7.3)], $\Gamma\mathrm{O}^+(8,q) = \mathrm{GO}^+(8,q) : \langle \phi \rangle$.

We denote the set of totally singular 1–spaces of V by $\mathcal{P}$, the set of totally singular 2–spaces by $\mathcal{L}$, the set of totally singular 3–spaces by $\mathcal{T}$ and the set of totally singular 4–spaces by $\mathcal{S}$. The group T acts transitively on $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{T}$, and has two orbits $\mathcal{S}_1$ and $\mathcal{S}_2$ on $\mathcal{S}$. Two totally singular 4–spaces lie in the same T–orbit if and only if their intersection has even dimension. The two T–orbits on 4–spaces are fused together under $\mathrm{PO}^+(8,q)$. For a subspace X, T_X denotes the setwise stabiliser of X in T. Each $X \in \mathcal{T}$ lies in precisely two totally singular 4–spaces, say S and R, one lies in $\mathcal{S}_1$ and the other in $\mathcal{S}_2$. Thus $T_X = T_S \cap T_R$.

Let $S = \langle e_1, e_2, e_3, e_4 \rangle$, a totally singular 4–space. Then S has $S' = \langle f_1, f_2, f_3, f_4 \rangle$ as a complementary totally singular 4–space and by [11, Lemma 4.1.9], the stabiliser, L, in $\mathrm{SO}^+(8,q)$ of both S and S' is the group of all matrices

$$\begin{pmatrix} B & 0 \\ 0 & (B^{-1})^T \end{pmatrix}$$

where B is any matrix in $\mathrm{GL}(4,q)$ and the matrices are written with respect to the standard basis. Such an element belongs to $\Omega^+(8,q)$ if and only if $\det(B)$ is a square ([11, Lemma 4.1.9]). Also let M be the subgroup consisting of all matrices of the form

$$\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$$

which lie in $\mathrm{SO}^+(8,q)$. Then by [11, Lemmas 2.1.8 and 4.1.12], C is any 4×4 matrix such that $C = -C^T$ and every entry of C on the diagonal is 0. Thus M is elementary abelian of order q^6 and by [11, Lemma 4.1.12], $M \leqslant \Omega^+(8,q)$. Also M is normalised by L, and we have

$$\begin{pmatrix} B^{-1} & 0 \\ 0 & B^T \end{pmatrix} \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & (B^{-1})^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ B^TCB & I \end{pmatrix}$$

and $\mathrm{SO}^+(8,q)_S = M : L$. Let R be the totally singular subspace $\langle e_1, e_2, e_3, f_4 \rangle$. Then $\dim(S \cap R) = 3$ and so S and R lie in distinct T–orbits. Also $\mathrm{SO}^+(8,q)_S \cap$

$SO^+(8,q)_R = M : (L_S \cap L_R)$, and $L_S \cap L_R$ consists of all matrices of the form

$$\left(\begin{array}{cc|cc} B & 0_{3\times 1} & \multicolumn{2}{c}{\multirow{2}{*}{$0_{4\times 4}$}} \\ v & \alpha & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$0_{4\times 4}$}} & (B^{-1})^T & w^T \\ & & 0_{1\times 3} & \alpha^{-1} \end{array}\right)$$

where $B \in GL(3,q)$, $v \in GF(q)^3$, $\alpha \in GF(q)\backslash\{0\}$ and $w = -\alpha^{-1}vB^{-1}$. Thus

$$SO^+(8,q)_S \cap SO^+(8,q)_R \cong [q^9] : (C_{q-1} \times GL(3,q)).$$

Now Z fixes both S and R and so when q is even

$$GO^+(8,q)_S \cap GO^+(8,q)_R \cong [q^9] : (C_{q-1} \times GL(3,q) \times Z).$$

When q is odd, we see that δ fixes both S and R, normalises M and centralises $L_S \cap L_R$. Thus in this case

$$GO^+(8,q)_S \cap GO^+(8,q)_R \cong [q^9] : (C_{q-1} \times GL(3,q) \times \langle\delta\rangle).$$

The element ϕ also fixes S and R, and so

$$\Gamma O^+(8,q)_S \cap \Gamma O^+(8,q)_R = (GO^+(8,q)_S \cap GO^+(8,q)_R) : \langle\phi\rangle.$$

Hence letting $H = P\Gamma O^+(8,q)$, we see that, when q is even,

$$H_S \cap H_R \cong [q^9] : ((C_{q-1} \times GL(3,q)) : C_f)$$

while when q is odd,

$$H_S \cap H_R \cong ([q^9] : ((C_{q-1}^2 \times GL(3,q)) : C_f))/Z.$$

We will denote the subgroup $H_S \cap H_R$ in both cases by

$$H_S \cap H_R \cong [q^9] : ((C_{q-1}^2 \circ GL(3,q)) : C_f). \tag{2.1}$$

Note that the subgroup $H_S \cap H_R$ is the group G_2 of Theorem 1.1(4). Furthermore, if we let U be a 1–space contained in $S \cap R$ then we have

$$\begin{aligned} H_S \cap H_R \cap H_U &\cong [q^9] : ((C_{q-1}^2 \circ ([q^2] : (C_{q-1} \times GL(2,q)))) : C_f) \\ &\cong [q^{11}] : ((C_{q-1}^3 \circ GL(2,q)) : C_f). \end{aligned} \tag{2.2}$$

This is the subgroup G_2 of Theorem 1.1(4).

The following lemma follows from (2.1).

Lemma 2.1 *Let $H = \mathrm{P\Gamma O}^+(8,q)$ and let S and R be totally singular 4–spaces such that $\dim(S \cap R) = 3$. Then $H_S \cap H_R$ induces the semilinear group $\Gamma\mathrm{L}(3,q)$ on $S \cap R$.*

For our second construction we will need further information about the structure of $T_{S\cap R} = T_S \cap T_R$. Now $L_S \cap L_R$ has a normal subgroup N of order q^3 consisting of all such matrices with $B = I$ and $\alpha = 1$, and so $\mathrm{SO}^+(8,q)_S \cap \mathrm{SO}^+(8,q)_R$ has a normal subgroup $Q = M : N$ of order q^9. The centre P of Q consists of all matrices of the form

$$\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$$

where

$$C = \begin{pmatrix} 0 & a & b & 0 \\ -a & 0 & c & 0 \\ -b & -c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The action of $\mathrm{SO}^+(8,q)_S \cap \mathrm{SO}^+(8,q)_R$ on P is equivalent to the action of $\mathrm{GL}(3,q)$ on the 3×3 matrices C for which $C^T = -C$ and every diagonal entry of C is 0, such that, for each $A \in \mathrm{GL}(3,q)$,

$$A : \begin{pmatrix} 0 & a & b \\ -a & 0 & d \\ -b & -d & 0 \end{pmatrix} \mapsto A^T \begin{pmatrix} 0 & a & b \\ -a & 0 & d \\ -b & -d & 0 \end{pmatrix} A.$$

This action is dual to the action of $\mathrm{SO}^+(8,q)_S \cap \mathrm{SO}^+(8,q)_R$ on the 3–space $S \cap R$, that is, if $g \in \mathrm{SO}^+(8,q)_S \cap \mathrm{SO}^+(8,q)_R$ fixes a 1–space of $S \cap R$ then it normalises a subgroup of order q^2 in P. We have the following lemma in the case where $q = 2$, which we use in Section 4.

Lemma 2.2 *Let $T = \mathrm{P\Omega}^+(8,2)$, and S and R be totally singular 4–spaces which intersect in a 3–space. Then $T_{S\cap R} \cong 2^{3+6} : \mathrm{SL}(3,2)$ and has two conjugacy classes of subgroups of the form $2^{3+6} : S_4$. Furthermore, subgroups from different classes are not isomorphic.*

PROOF. Since T acts transitively on the set of totally singular 3–spaces, and $T_S \cap T_R = T_{S\cap R}$, we have already seen that $T_{S\cap R} \cong 2^{3+6} : \mathrm{SL}(3,2)$ and the largest normal 2–subgroup of $T_{S\cap R}$ is Q. Also the centre, P, of Q has order 2^3 and the action of $T_{S\cap R}$ on P is the dual of the action of $T_{S\cap R}$ on $S \cap R$. The group $\mathrm{SL}(3,2)$ has two conjugacy classes of subgroups isomorphic to S_4, (the stabilisers of 1–spaces and the stabilisers of hyperplanes), and hence $T_{S\cap R}$ has two conjugacy classes of subgroups containing Q and of the form $2^{3+6} : S_4$. In one, the group S_4 centralises an involution of P and so a group in this class has a normal subgroup of order 2. In the other conjugacy class, S_4 has orbits of length 3 and 4 on the involutions of P. Let H be a subgroup in this second class and suppose that Y is a normal subgroup of H of order 2. Then $Y \leqslant O_2(H) = Q.2^2$. Since H acts irreducibly on $O_2(H)/Q$ it follows that $Y \leqslant Q$. Thus Y is a normal subgroup of Q of order 2 and so lies in the centre, P, of Q. However, H has orbits of lengths 3 and 4 on the involutions of P and so no such Y exists. Thus subgroups of the form $2^{3+6} : S_4$ in different T–conjugacy classes are not isomorphic.

The group T can also be interpreted as a Chevalley group of type D_4 over GF(q) (see [2]). Let $\{r_1, r_2, r_3, r_4\}$ be a fundamental system of roots with Dynkin diagram given in Figure 1.

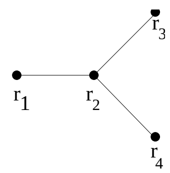

Fig. 1. Dynkin diagram of type D_4

For any subset $\{i, j, k\}$ of $\{1, 2, 3, 4\}$, let $P_{i,j,k}$ be the parabolic subgroup of T associated with $\{r_i, r_j, r_k\}$. The Dynkin diagram has symmetry group S_3 and by [8, p 78], there is a corresponding subgroup $A \leqslant \mathrm{Aut}(T)$ such that $A = \langle \tau, \sigma\rangle \cong S_3$ where τ and σ induce the symmetries (r_1, r_3, r_4) and (r_3, r_4) respectively, of the Dynkin diagram. Then $P^{\tau}_{1,2,3} = P_{2,3,4}$, $P^{\tau}_{2,3,4} = P_{1,2,4}$, $P^{\tau}_{1,2,4} = P_{1,2,3}$ and $P^{\tau}_{1,3,4} = P_{1,3,4}$ while $P^{\sigma}_{1,2,3} = P_{1,2,4}$, $P^{\sigma}_{1,2,4} = P_{1,2,3}$, $P^{\sigma}_{2,3,4} = P_{2,3,4}$ and $P^{\sigma}_{1,3,4} = P_{1,3,4}$. Furthermore, (see for example [10]) $\mathrm{Aut}(T) = \langle \mathrm{P\Gamma O}^+(8, q), \tau\rangle$ and $|\,\mathrm{Aut}(T) : \mathrm{P\Gamma O}^+(8, q)| = 3$. Any automorphism of T which induces a symmetry of order three of the Dynkin diagram is called a *triality automorphism* and does not lie in $\mathrm{P\Gamma O}^+(8, q)$. Also

$$\mathrm{Out}(T) = \mathrm{Aut}(T)/T \cong \begin{cases} S_3 \times C_f & \text{for } q = 2^f \\ S_4 \times C_f & \text{for } q = p^f, p \text{ odd.} \end{cases}$$

We can interpret our parabolic subgroups geometrically in such a way that

$$P_{1,2,3} = T_{R_0}, P_{1,2,4} = T_{S_0}, P_{2,3,4} = T_{U_0} \text{ and } P_{1,3,4} = T_{W_0}, \qquad (2.3)$$

where $S_0 \in \mathcal{S}_1$, $R_0 \in \mathcal{S}_2$, $U_0 \in \mathcal{P}$, and $W_0 \in \mathcal{L}$, such that $U_0 < W_0 < S_0 \cap R_0$. Following [10], we can then define an action of $\mathrm{Aut}(T)$ on $\mathcal{C} = \mathcal{P} \cup \mathcal{L} \cup \mathcal{S}_1 \cup \mathcal{S}_2$

such that, for each $X \in \mathcal{C}$ and $a \in \mathrm{Aut}(T)$, we define X^a so that $T_{X^a} = (T_X)^a$. The action of $\mathrm{P\Gamma O}^+(8,q)$ is the natural action while $U_0^\tau = S_0$, $S_0^\tau = R_0$, $R_0^\tau = U_0$ and $W_0^\tau = W_0$, and so $\mathcal{P}^\tau = \mathcal{S}_1$, $\mathcal{S}_1^\tau = \mathcal{S}_2$, $\mathcal{S}_2^\tau = \mathcal{P}$ and $\mathcal{L}^\tau = \mathcal{L}$. Moreover, $S_0^\sigma = R_0$ and $R_0^\sigma = S_0$, and σ fixes setwise each 1–dimensional subspace of $S_0 \cap R_0$. This action preserves incidence of elements of $\mathcal{C}$, where we define two subspaces to be incident either if one is contained in the other, or if they are both totally singular 4–spaces which intersect in a 3–space.

3 The First Family

We can now define our first family of graphs.

Construction 3.1 *Let V be an 8–dimensional vector space over $\mathrm{GF}(q)$ equipped with a quadratic form Q of maximal Witt index. Let $\mathcal{P}$ be the set of totally singular 1–spaces of V, and $\mathcal{S}_1$ and $\mathcal{S}_2$ be the two orbits of $\mathrm{P\Omega}^+(8,q)$ on the set of totally singular 4–spaces. Let*

$$\begin{aligned}
\Delta_1 &= \{\{U,S,R\} : U \in \mathcal{P}, S \in \mathcal{S}_1, R \in \mathcal{S}_2, \dim(S\cap R) = 3 \textit{ and } U < S \cap R\},\\
B_1 &= \{\{U,S\} : U \in \mathcal{P}, S \in \mathcal{S}_1 \textit{ and } U < S\},\\
B_2 &= \{\{S,R\} : S \in \mathcal{S}_1, R \in \mathcal{S}_2 \textit{ and } \dim(S\cap R) = 3\},\\
B_3 &= \{\{U,R\} : U \in \mathcal{P}, R \in \mathcal{S}_2 \textit{ and } U < R\},
\end{aligned}$$

and let $\Delta_2 = B_1 \cup B_2 \cup B_3$. Define $\mathcal{G}(q)$ to be the bipartite graph with vertex set $\Delta_1 \cup \Delta_2$, such that two vertices $\{U,S,R\}$ and $\{X,Y\}$ are adjacent if and only if $\{X,Y\} \subseteq \{U,S,R\}$.

From [1], we see that there are $2(1+q)(1+q^2)(1+q^3)$ totally singular 4–spaces in V and that these are divided equally between $\mathcal{S}_1$ and $\mathcal{S}_2$. Each totally singular 3–space lies in a unique totally singular 4–space in $\mathcal{S}_1$ and each totally singular 4–space contains $(q^4-1)/(q-1)$ totally singular 3–spaces. Hence there are $(1+q)(1+q^2)(1+q^3)(q^4-1)/(q-1)$ totally singular 3–spaces in V. Thus

$$|\Delta_1| = \frac{(q^3-1)(q^4-1)(1+q)(1+q^2)(1+q^3)}{(q-1)^2}$$

and

$$|\Delta_2| = \frac{3(q^4-1)(1+q)(1+q^2)(1+q^3)}{q-1}.$$

Every vertex in Δ_1 is adjacent to precisely three vertices in Δ_2 while every vertex in Δ_2 is adjacent to $(q^3-1)/(q-1) = q^2+q+1$ vertices in Δ_1. Thus $\mathcal{G}(q)$ is biregular of valency $\{3, q^2+q+1\}$.

Now T acts transitively on Δ_1 and by [10, see Table 1, line 4] the stabiliser of a vertex in Δ_1 is a maximal subgroup of $\mathrm{Aut}(T)$. Thus $\mathrm{Aut}(T)$ acts primitively on Δ_1. On the other hand, T has three orbits B_1, B_2 and B_3 on Δ_2. Recall from Section 2 that $\mathrm{Aut}(T)$ has a subgroup $A = \langle \tau, \sigma \rangle \cong S_3$ that permutes transitively the three T–orbits B_1, B_2 and B_3. As $\mathrm{Aut}(T)$ preserves incidence among $\mathcal{P} \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{L}$, it follows that $G = \mathrm{Aut}(T) \leqslant \mathrm{Aut}(\mathcal{G}(q))$.

For a vertex v of $\Gamma = \mathcal{G}(q)$, we denote the set of vertices adjacent to v by $\Gamma(v)$. Let U, S and R be the subspaces U_0, S_0 and R_0 respectively given in (2.3). Then $v = \{U, S, R\} \in \Delta_1$, and

$$\Gamma(v) = \{\{U, S\}, \{U, R\}, \{S, R\}\} \subseteq \Delta_2.$$

As A induces S_3 on $\{U, S, R\}$ it follows that $A \leqslant G_v$ and so $G_v^{\Gamma(v)} \cong S_3$. In fact, $G_v = (G_S \cap G_R \cap G_U) : A$ and $G_S \cap G_R \cap G_U \leqslant \mathrm{P\Gamma O}^+(8, q)$. Since G is transitive on pairs of incident totally singular 4–spaces we have from (2.2) that

$$G_S \cap G_R \cap G_U \cong [q^{11}] : ((C_{q-1}^3 \circ \mathrm{GL}(2, q)) : C_f)$$

and so G_v is as given in Theorem 1.1(4).

Next let $w = \{S, R\}$. Then

$$\Gamma(w) = \{\{U', S, R\} : U' \in \mathcal{P} \text{ and } U' < S \cap R\}.$$

Now $\sigma \in G_w$, as σ interchanges S and R. Thus $G_w = (G_S \cap G_R) : \langle \sigma \rangle$. Furthermore, $G_S \cap G_R \leqslant \mathrm{P\Gamma O}^+(8, q)$ and so by (2.1)

$$G_S \cap G_R \cong [q^9] : ((C_{q-1}^2 \circ \mathrm{GL}(3, q)) : C_f).$$

Hence G_w is as given in Theorem 1.1(4). Also by Lemma 2.1, $G_w^{\Gamma(w)} \cong \mathrm{P\Gamma L}(3, q)$ which is a 2–transitive group. Thus $\mathcal{G}(q)$ is locally $(G, 2)$–arc transitive.

Now we consider 3–arcs. Let $u = \{U_2, S, R\} \in \Delta_1$ where $U_2 \in \mathcal{P} \backslash \{U\}$ and $U_2 < S \cap R$. Then $\Gamma(u) \backslash \{w\} = \{\{U_2, R\}, \{U_2, S\}\}$ where $w = \{S, R\}$. Now $\sigma \in G_{vwu}$ interchanges R and S and hence G_{vwu} acts transitively on $\Gamma(u) \backslash \{w\}$. Hence G_v acts transitively on the set of 3–arcs emerging from v. Since G acts transitively on Δ_1 it follows that G acts transitively on the set of 3–arcs starting in Δ_1. Moreover, each 3–arc starting in Δ_1 ends in Δ_2 and vice versa. Hence G acts transitively on the set of 3–arcs starting in Δ_2 and so G_w acts transitively on the set of 3–arcs starting at w. Thus $\mathcal{G}(q)$ is locally $(G, 3)$–arc transitive.

Now $T \lhd \mathrm{Aut}(T)$, T has three orbits on Δ_2, and T is transitive on Δ_1. Hence [6, Lemma 5.5] implies that the quotient graph of $\mathcal{G}(q)$ with respect to the orbits of T is $K_{1,3}$ and so, by [6, Lemma 5.6], $\mathcal{G}(q)$ is not locally $(\mathrm{Aut}(T), 4)$–arc transitive. To complete the proof of Theorem 1.1, we show that there are no further automorphisms of $\mathcal{G}(q)$.

Lemma 3.2 $\mathrm{Aut}(\mathcal{G}(q)) = \mathrm{Aut}(T)$.

PROOF. Now $\Gamma = \mathcal{G}(q)$ is locally $(\mathrm{Aut}(\Gamma), 3)$–arc transitive and is not a complete bipartite graph. Hence by [6, Lemma 5.3], $\mathrm{Aut}(\Gamma)$ acts faithfully on both Δ_1 and Δ_2. Let $n = |\Delta_1|$. As noted above, by [10, see Table 1, Line 4], $\mathrm{Aut}(T)$ is primitive on Δ_1, and it follows from [12] that the only group which may possibly lie between $\mathrm{Aut}(T)$ and S_n is A_n. Since neither A_{n-1} nor S_{n-1} has a permutation representation of degree three, we deduce that $\mathrm{Aut}(\Gamma)^{\Delta_1} = \mathrm{Aut}(T)$. The faithfulness of $\mathrm{Aut}(\Gamma)$ on Δ_1 then yields the result.

This completes the proof of Theorem 1.1.

4 The Second Family

For our second family of examples we first review the construction of an edge transitive graph from a given group and pair of subgroups. Let G be a group with subgroups L and R such that $L \cap R$ is core free in G. Let Δ_1 be the set $[G : L]$ of right cosets of L in G, and Δ_2 the set $[G : R]$ of right cosets of R in G. The graph $Cos(G, L, R)$ is the bipartite graph with vertex set the disjoint union $\Delta_1 \dot\cup \Delta_2$ such that two vertices Lx and Ry are adjacent if and only if $xy^{-1} \in LR$. Moreover G acts on the vertex set of Γ by right multiplication, G is edge transitive, and L and R are the stabilisers of the adjacent vertices L, R respectively. We collect the following results concerning coset graphs, see for example [6, Section 3.2]. A subgroup H of G is core free if $\cap_{g \in G} H^g = 1$.

Lemma 4.1 *Let $\Gamma = Cos(G, L, R)$ for some group G with subgroups L and R such that $L \cap R$ is core free. Let $\Delta_1 = [G : L]$ and $\Delta_2 = [G : R]$. Then*

(1) Γ *is connected if and only if* $\langle L, R\rangle = G$,
(2) $G \leqslant \mathrm{Aut}(\Gamma)$, Γ *is G–edge transitive, and Δ_1 and Δ_2 are G–orbits on vertices,*
(3) G acts faithfully on Δ_1 and Δ_2 if and only if both L and R are core free,
(4) Γ *is locally $(G, 2)$–arc transitive if and only if L acts 2–transitively on $[L : L \cap R]$ and R acts 2–transitively on $[R : L \cap R]$.*

Conversely, if Γ is a G–edge transitive but not G–vertex transitive graph, and v and w are adjacent vertices then, $\Gamma \cong Cos(G, G_v, G_w)$.

We also collect together the following results from [10, Propositions 2.3.8, 3.4.2, 3.4.3 and Table 1] concerning $\mathrm{P}\Omega^+(8, p)$.

Proposition 4.2 *Let $T = \mathrm{P}\Omega^+(8, p)$ for some odd prime p and let $G = \mathrm{Aut}(T)$. Then the following hold.*

(1) T has 4 conjugacy classes of maximal subgroups isomorphic to $\Omega^+(8, 2)$

permuted naturally by $G/T \cong S_4$. Furthermore, for such a subgroup H, $N_G(H) \cong \mathrm{Aut}(H)$.

(2) *T has 4 conjugacy classes of subgroups isomorphic to $2^{3+6} : \mathrm{SL}(3,2)$ permuted naturally by $G/T \cong S_4$. Furthermore, given such a subgroup K,*
   - (a) *K has a unique minimal normal subgroup P of order 2^3 whose involutions belong to the only T–conjugacy class of involutions which is fixed by a triality automorphism of T,*
   - (b) *K acts irreducibly on P inducing $\mathrm{SL}(3,2)$,*
   - (c) *$N_G(K) \cong K.S_3$, and*
   - (d) *$N_G(K)$ is a maximal subgroup of $T.S_3$.*

Before we give our second construction we need the following group theoretic lemma.

Lemma 4.3 *Let $T = \mathrm{P}\Omega^+(8,p)$ for some odd prime p and $G = \mathrm{Aut}(T)$. Then T contains subgroups K_1, K_2, K_3 and K_4 of the form $2^{3+6} : \mathrm{SL}(3,2)$ and $N_G(K_i) = K_i.S_3 \cong 2^{3+6} : (\mathrm{SL}(3,2) \times S_3)$ for each $i = 1,2,3,4$. Furthermore, if $L_0 = K_1 \cap K_2 \cap K_3 \cap K_4$, then*

(1) *$L_0 \cong 2^{3+6} : S_4$, and*
(2) *$N_G(L_0) = L_0.S_4$.*

PROOF. Let H be a maximal subgroup of T isomorphic to $\Omega^+(8,2)$. Then by Proposition 4.2.1, $N_G(H) = \mathrm{Aut}(H)$ and so $N_G(H) = H : A$ where $A = \langle \tau, \sigma \rangle \cong S_3$ is a group of graph isomorphisms of H as defined in Section 2 (note here τ and σ are defined for H, not for T). As H is maximal in T, $A \cap T = 1$, $T : A \cong T : S_3$ and τ is also a triality automorphism of T.

Let V be an 8–dimensional vector space over $\mathrm{GF}(2)$ upon which H acts naturally, and let U, S and R be the totally singular subspaces U_0, S_0 and R_0 of V given in (2.3), (again for H, not T). Let $K_1 = H_S \cap H_R$, $K_2 = H_R \cap H_U$ and $K_3 = H_U \cap H_S$. Then A permutes the set $\{K_1, K_2, K_3\}$ inducing the group S_3 and by Lemma 2.2, each $K_i \cong 2^{3+6} : \mathrm{SL}(3,2)$. Let $Q = O_2(K_1) = 2^{3+6}$ and P be the centre of Q. By Lemma 2.2, the action of K_1 on P is the dual of the action of K_1 on $S \cap R$. For any distinct $i, j \in \{1,2,3\}$, we have

$$K_1 \cap K_2 \cap K_3 = K_i \cap K_j = H_S \cap H_U \cap H_R.$$

Then as $H_S \cap H_U \cap H_R$ is the stabiliser in K_1 of a 1–space in $S \cap R$ we have $Q \leqslant K_1 \cap K_2 \cap K_3 \cong 2^{3+6} : S_4$ such that the S_4 normalises a subgroup Z of P of order 4. Furthermore, since $O_2(S_4)$ centralises each of the 3 involutions of Z and is regular on $P \backslash Z$, it follows that Z is equal to the centre of $O_2(K_1 \cap K_2 \cap K_3) = Q : O_2(S_4)$.

Now K_1 acts irreducibly on P inducing the group $\mathrm{SL}(3,2)$ and so all involutions in P lie in the same T–conjugacy class $\mathcal{C}$. Furthermore, Z is characteristic in $K_1 \cap K_2 \cap K_3$ and A normalises $K_1 \cap K_2 \cap K_3$. As A contains a triality automorphism of T, it follows that the class $\mathcal{C}$ is fixed by some triality automorphism of T. Thus by [10, Proposition 3.4.2], each K_i is one of the subgroups described in Proposition 4.2.2.

By Proposition 4.2.2, $N_G(K_i) = K_i.S_3$. We claim that in fact, $N_G(K_i) = 2^{3+6} : (\mathrm{SL}(3,2) \times S_3)$. Since all the K_i are conjugate in G it suffices to consider K_1. Since $\sigma \in A$ interchanges S and R we have $\sigma \in N_G(K_1)$. Thus it follows that $N_G(K_1) = 2^{3+6} : (\mathrm{SL}(3,2).S_3)$. Now the outer automorphism group of $\mathrm{SL}(3,2)$ is C_2 and any outer automorphism interchanges the stabilisers of 1–spaces with the stabilisers of 2–spaces. It follows that the S_3 must centralise $\mathrm{SL}(3,2)$, as otherwise it would interchange the centraliser of an involution of P with the normaliser of a subgroup of order 2^2. the two conjugacy Thus $N_G(K_1) = 2^{3+6} : (\mathrm{SL}(3,2) \times S_3)$.

By Proposition 4.2, T has 4 conjugacy classes of subgroups isomorphic to K_1 and $G/T \cong S_4$ permutes them naturally. Then as A permutes the set $\{K_1, K_2, K_3\}$ inducing S_3, either K_1, K_2 and K_3 lie in the same T–conjugacy class or in distinct T–conjugacy classes. We claim that the K_i lie in distinct T–conjugacy classes. Suppose to the contrary that K_1, K_2 and K_3 lie in the same T–conjugacy class $\mathcal{C}_1$. Then it follows that the 4 conjugacy classes of subgroups isomorphic to K_1 correspond to the 4 conjugacy classes of subgroups isomorphic to H and so each subgroup in $\mathcal{C}_1$ lies in some conjugate of H. By Proposition 4.2.2, K_1 is self normalising in T and so $|\mathcal{C}_1| = |T : K_1|$. However, H contains $|H : K_1|$ subgroups isomorphic to K_1 and there are $|T : H|$ conjugates of H in T. Therefore, each subgroup in $\mathcal{C}_1$ lies in precisely one conjugate of H and so $N_G(K_1) \leqslant N_G(H)$. By Proposition 4.2.2, K_1 is normalised by some triality automorphism τ' of T of order three. Then as $N_G(K_1) \leqslant N_G(H)$, it follows that τ' normalises H. However, this would imply that τ' was a triality automorphism of H normalising K_1 and we have already seen that any triality automorphism of H cyclically permutes the set $\{K_1, K_2, K_3\}$. Hence the K_i lie in distinct T–conjugacy classes.

Let $\mathcal{C}_i$ be the conjugacy class containing K_i for $i = 1, 2, 3$. By Proposition 4.2.2, there is one extra T–conjugacy class $\mathcal{C}_4$ of subgroups isomorphic to K_1. It remains to find a $K_4 \in \mathcal{C}_4$ such that for $L_0 = K_1 \cap K_2 \cap K_3 \cap K_4$ we have $N_G(L_0)/L_0 \cong S_4$. Let $\rho \in \mathrm{Aut}(T)$ such that K_3^ρ is not conjugate in T to K_3. Since G/T induces the group S_4 on the four conjugacy classes of subgroups isomorphic to K_3 we may choose ρ such that $K_1^\rho \in \mathcal{C}_1$, $K_2^\rho \in \mathcal{C}_2$ and $K_4 := K_3^\rho \in \mathcal{C}_4$. Furthermore, by adjusting ρ by some element of T if necessary, we may assume that $K_1^\rho = K_1$. Thus

$$K_1 \cap K_2 \cap K_3 \cong (K_1 \cap K_2 \cap K_3)^\rho = K_1 \cap K_2^\rho \cap K_4 \leqslant K_1.$$

However, by Lemma 2.2, K_1 contains only one conjugacy class of subgroups isomorphic to $K_1 \cap K_2 \cap K_3$ and so we may have chosen ρ, adjusting by some element of K_1 if necessary, such that

$$K_1 \cap K_2 \cap K_3 = (K_1 \cap K_2 \cap K_3)^\rho = K_1 \cap K_2^\rho \cap K_4.$$

Hence $K_1 \cap K_2 \cap K_3 \cap K_4 = K_1 \cap K_2 \cap K_3$ and is normalised by ρ. It is also normalised by A and so, letting $L_0 = K_1 \cap K_2 \cap K_3 \cap K_4$, we see that $N_G(L_0)/L_0 \cong S_4$.

We are now in a position to give our second construction.

Construction 4.4 *Let* $\mathcal{H}(p) = Cos(G, L, R)$ *where* $G = \mathrm{Aut}(T)$, $R = N_G(K_4)$ *and* $L = N_G(L_0)$, *where* K_4 *and* L_0 *are as given in Lemma* 4.3.

Now $\mathcal{H}(p)$ is a bipartite graph with bipartite halves $\Delta_1 = [G : L]$ and $\Delta_2 = [G : R]$, and we have

$$|\Delta_1| = \frac{p^{12}(p^2-1)(p^4-1)^2(p^6-1)}{2^{14}.3}$$

and

$$|\Delta_2| = \frac{p^{12}(p^2-1)(p^4-1)^2(p^6-1)}{2^{14}.3.7}.$$

By Lemma 4.1, $G \leqslant \mathrm{Aut}(\Gamma)$. From Proposition 4.2, $N_G(K_4)$ is a maximal subgroup of $T.S_3$ and so $\langle L, R \rangle = G$. Hence Lemma 4.1 implies that $\mathcal{H}(p)$ is connected. From Lemma 4.3, we have

$$R = 2^{3+6} : (\mathrm{SL}(3,2) \times S_3)$$

and so $R \cap L = R \cap N_G(L_0) = 2^{3+6} : (S_4 \times S_3)$. Hence $\mathcal{H}(p)$ is biregular of valency $\{4, 7\}$. The action of L on $[L : L \cap R]$ is equivalent to the action of S_4 on four points while the action of R on $[R : L \cap R]$ is equivalent to the action of $\mathrm{SL}(3,2)$ on the seven 1–spaces of a 3–dimensional vector space over $\mathrm{GF}(2)$. As both of these actions are 2–transitive, Lemma 4.1 implies that $\mathcal{H}(p)$ is locally $(G, 2)$–arc transitive.

Let $\Gamma = \mathcal{H}(p)$. We now consider 3–arcs. Let v be the vertex of Δ_1 given by L and w the vertex of Δ_2 given by R. For $w_2 \in \Gamma(v) \backslash \{w\}$ we have that $G_{w_2 v w} = T_v.S_2 \cong 2^{3+6} : (S_4 \times S_2)$ which still acts transitively on the six vertices of $\Gamma(w) \backslash \{v\}$ inducing the group S_4. Hence G_{w_2} acts transitively on the set of 3–arcs starting at w_2. Using the fact that every 3–arc starting in Δ_2 ends in Δ_1 and vice versa, it follows that Γ is locally $(G, 3)$–arc transitive.

Now T acts transitively on Δ_1 and so the action of G on Δ_1 is quasiprimitive. However, T has four orbits on Δ_2 and so by [6, Lemma 5.5], $\Gamma_T = K_{1,4}$.

Furthermore, [6, Lemma 5.6] implies that Γ is not locally $(G,4)$–arc transitive. All that is left to complete the proof of Theorem 1.2 is to prove that $\mathrm{Aut}(\Gamma) = G$. We do this in the next section, see Proposition 5.1.

Remark 4.5 In [9], Kantor constructed a geometry which is almost a building whose diagram is the extended D_4, that is, a middle node with four neighbours. The maximal parabolic subgroups corresponding to the four outer nodes are isomorphic to $\mathrm{P}\Omega^+(8,2)$, and the K_i in Lemma 4.3 are the intersection of any three. Thus the graphs $\mathcal{H}(p)$ can be constructed in a similar way to the graphs $\mathcal{G}(q)$.

Remark 4.6 Note that for $T = \mathrm{P}\Omega^+(8,p^f)$, where $f > 1$, the subgroup H is no longer maximal in T as it is contained in the subgroup $\mathrm{P}\Omega^+(8,p)$. If we choose L_0 and K_4 inside H as before and take L and R to be their normalisers in $G = \mathrm{Aut}(T)$, we get $\langle L, R\rangle = \mathrm{Aut}(\mathrm{P}\Omega^+(8,p)) \times \langle \phi \rangle$ where ϕ is a field automorphism of T of order f which centralises $\mathrm{P}\Omega^+(8,p)$. Hence in this case, Lemma 4.1 implies that the graph $Cos(G,L,R)$ is not connected.

5 The full automorphism group of $\mathcal{H}(p)$

In this section we complete the proof of Theorem 1.2 by proving the following. Let p be an odd prime and $T = \mathrm{P}\Omega^+(8,p)$.

Proposition 5.1 $\mathrm{Aut}(\mathcal{H}(p)) = \mathrm{Aut}(T)$.

Let $\Gamma = \mathcal{H}(p)$, $G = \mathrm{Aut}(T)$, $A = \mathrm{Aut}(\Gamma)$ and let $v \in \Delta_1$ correspond to the subgroup L. For a finite group H, let $\Pi(H)$ be the set of primes dividing $|H|$. First we prove the following.

Lemma 5.2 $\{2,3\} \subseteq \Pi(A_v) \subseteq \{2,3,5\}$.

PROOF. Since $G_v \leqslant A_v$ it follows from Lemma 4.3 that 2 and 3 divide $|A_v|$. Suppose that A_v contains an element g of prime order greater than 5. Then as $|\Gamma(v)| = 4$, g fixes $\Gamma(v)$ pointwise. Next let $w \in \Gamma(v)$. Then g fixes $v \in \Gamma(w)$ and so as $|\Gamma(w)| = 7$, g also fixes $\Gamma(w)$ pointwise. Proceeding in this manner through the vertices in $V\Gamma$, we deduce from the connectivity of Γ that g fixes $V\Gamma$ pointwise, contradicting $g \neq 1$. Thus the result holds.

Now as Γ is not a complete bipartite graph, [6, Lemma 5.2], implies that A acts faithfully on each of the two bipartite halves Δ_1 and Δ_2 of Γ. This enables us to prove the following.

Lemma 5.3 *$C_A(T) = 1$ and $N_A(T) = G$.*

PROOF. Now $G_v = N_G(L_0)$ with $L_0 \leqslant T$ as in Lemma 4.3. Furthermore, $N_T(L_0) = L_0 = T_v$. Thus $|\operatorname{fix}_{\Delta_1}(L_0)| = |N_T(L_0) : L_0| = 1$, and so by [5, Theorem 4.2A], $C_{\mathrm{Sym}(\Delta_1)}(T) = 1$. As A acts faithfully on Δ_1, the result follows.

Now suppose that $A = \mathrm{Aut}(\Gamma) \neq G$. Then A contains a subgroup X which properly contains G as a maximal subgroup. Let N be a minimal normal subgroup of X. Then $N = S_1 \times \cdots \times S_k$ where each $S_i \cong S$ for some finite simple group S.

Lemma 5.4 *$N \cap G = 1$, and $X = N : G$.*

PROOF. Since $N \lhd X$, we have $N \cap G \lhd G$ and so either $N \cap G = 1$ or $T \leqslant N \cap G$. Suppose $T \leqslant N \cap G$. Since $T \neq 1$, there exists i such that $\pi_i(T) \neq 1$, where $\pi_i : N \to S_i$ is the i^{th} projection map. Since T is simple, $\pi_i(T) \cong T$ and so S_i has a subgroup isomorphic to T, whence N has a subgroup isomorphic to T^k. In particular S is a nonabelian simple group. Let r be a prime greater that 5 which divides $|T|$ and let r^s be the largest power of r dividing $|T|$. By Lemma 5.2, $\Pi(T_v) \subseteq \{2, 3, 5\}$, and so r^s is the largest power of r dividing $|\Delta_1|$. However r^{sk} divides $|N|$ and $r \nmid |N_v|$. Therefore r^{sk} divides $|N : N_v|$ which in turn divides $|\Delta_1|$. Thus $k = 1$ and so $T \leqslant N = S$. Since $|N| = |\Delta_1||N_v|$ and $\Pi(N_v) \subseteq \{2, 3, 5\}$, it follows that the only primes dividing $|N|$ are those dividing $|T|$. Hence, we see from [13, Corollary 5] that $N = T$. Thus $X \leqslant N_A(T) = G$, contradicting the fact that G is a proper subgroup of X. Hence $N \cap G = 1$. Since G is maximal in X it follows that $X = N : G$.

For an integer n and a prime r, the r*–part* of n is the highest power r^s dividing n.

Lemma 5.5 $\Pi(N) \subseteq \{2, 3, 5\}$.

PROOF. By Lemma 5.2, if r is a prime dividing $|X|$ and $r > 5$, then r does not divide $|X_v|$. Hence the r–part of $|X|$ divides $|\Delta_1|$. However, the r–part of $|T|$ divides $|X|$ and $|\Delta_1|$ divides $|T|$. Hence the r–part of $|X|$ equals the r–part of $|T|$. By Lemma 5.4, $N : T \leqslant X$ and so it follows that r does not divide $|N|$. Thus $\Pi(N) \subseteq \{2, 3, 5\}$.

Lemma 5.6 $\Pi(N_v) \subseteq \{2, 3\}$ *and, for $w \in \Delta_2$, $N_w^{\Gamma(w)} = 1$.*

PROOF. Let w be a vertex in Δ_2. Then $N_w^{\Gamma(w)} \lhd X_w^{\Gamma(w)}$. Now $|\Gamma(w)| = 7$ and $X_w^{\Gamma(w)}$ contains $G_w^{\Gamma(w)} = \mathrm{SL}(3,2)$. Thus $X_w^{\Gamma(w)} = \mathrm{SL}(3,2), A_7$ or S_7. Then as 7 does not divide $|N|$ it follows that $N_w^{\Gamma(w)} = 1$.

Now let g be an element of N_v of order 5. Then g fixes $\Gamma(v)$ pointwise. Let $w \in \Gamma(v)$. Then $g \in N_w$ and so also fixes $\Gamma(w)$ pointwise. It then follows from the connectivity of Γ that g fixes $V\Gamma$ pointwise, contradicting $g \neq 1$. Thus $\Pi(N_v) \subseteq \{2,3\}$.

Lemma 5.7 *N is abelian.*

PROOF. Suppose that N is nonabelian. Then by Lemma 5.5, S is a nonabelian simple group such that $\Pi(S) \subseteq \{2,3,5\}$. By Burnside's '$p^a q^b$ Theorem', $\Pi(S) = \{2,3,5\}$. As $X = N : G$ and N is a minimal normal subgroup of X, the action by conjugation of G on the k simple direct factors of N is transitive. As T is normal in G, all T–orbits on the S_i have the same length. Suppose that T fixes some S_i in this action, that is $T \leqslant N_X(S_i)$. Since $\mathrm{Out}(S_i)$ is soluble and $\Pi(S_i) = \{2,3,5\}$, it follows that $\mathrm{Aut}(S_i)$ has no subgroup isomorphic to T. Hence $T \leqslant C_X(S_i)$. However, this contradicts Lemma 5.3 and so T does not fix any simple direct factor of N. Hence T has a transitive action of some degree $t > 1$, with t dividing k. By [4], t is at least $\frac{(p^4-1)(p^3+1)}{p-1} > p^6$ and so $k > p^6$.

Now 5^k divides $|N|$ and by Lemma 5.6, $\Pi(N_v) \subseteq \{2,3\}$. Thus 5^k divides $|\Delta_1|$ and so 5^{p^6} divides $|\Delta_1|$. Hence 5^{p^6} divides $p^{12}(p^4-1)^2(p^2-1)(p^6-1)$. This is not possible and so N is abelian.

We can now complete the proof of Proposition 5.1.

PROOF. [Proof of Proposition 5.1] Suppose that $G \neq \mathrm{Aut}(\Gamma)$. Let X be a subgroup of $\mathrm{Aut}(\Gamma)$ which properly contains G as a maximal subgroup. Let N be a minimal normal subgroup of X. By Lemma 5.4, $N \cap G = 1$ and $X = N : G$. By Lemmas 5.5 and 5.7, $N = C_r^k$ where $r = 2, 3$ or 5, and G acts irreducibly on N. By Lemma 5.3, $C_A(T) = 1$, so T does not centralise N and in particular $k > 1$. Let $v \in \Delta_1$. Then $|\Gamma(v)| = 4$ and $G_v^{\Gamma(v)} = S_4$, so we have $N_v^{\Gamma(v)} \lhd X_v^{\Gamma(v)} = S_4$. Thus $N_v^{\Gamma(v)} = 1$ or C_2^2. By Lemma 5.6, $N_w^{\Gamma(w)} = 1$ for all $w \in \Delta_2$.

We claim that either $|N|$ divides $|\Delta_1|$, or $r = 2$ and $|N|/4$ divides $|\Delta_1|$. Suppose first that $N_v^{\Gamma(v)} = 1$. Since $N_w^{\Gamma(w)} = 1$ for all $w \in \Delta_2$, the connectivity of Γ implies that $N_v = 1$. Thus $|N|$ divides $|\Delta_1|$. Suppose now that $N_v^{\Gamma(v)} = C_2^2$. Then $N_v^{\Gamma(v)}$ is regular. If $g \in N_v$ fixes $\Gamma(v)$ pointwise then g fixes $\Gamma(w)$

pointwise for each $w \in \Gamma(v)$. Then for $u \in \Gamma(w)$, since g fixes $w \in \Gamma(u)$ and $N_u^{\Gamma(u)}$ is regular, it follows that g fixes $\Gamma(u)$ pointwise. Thus by the connectivity of Γ, g fixes $V\Gamma$ pointwise and so $g = 1$. Thus $N_v \cong N_v^{\Gamma(v)} = C_2^2$. Hence $|N|/4 = |N : N_v|$ divides $|\Delta_1|$. Thus the claim is proved, so we have that r^k divides $|\Delta_1|$ if $r = 3$ or 5, and 2^{k-2} divides $|\Delta_1|$ if $r = 2$.

By Lemma 5.3, T does not centralise N, and hence G acts faithfully on N and we have $G \leqslant \mathrm{GL}(k, r)$. The smallest degree of an irreducible representation of T over a field of characteristic $r = 2, 3$ or 5 is $p^2(p^3 - 1) > p^4$ if $p \neq r$ and 8 if $p = r$ (see for example [11, Theorem 5.3.9 and Proposition 5.4.13]). Thus $k > p^4$ if $r \neq p$ and $k \geq 8$ if $r = p$. Since none of 5^{p^4}, 3^{p^4} or 2^{p^4-2} divide $|\Delta_1|$ it follows that $r = p = 3$ or 5 and $k \geq 8$ (recall that p is odd). When $p = 3$ the largest power of 3 dividing $|\Delta_1|$ is 3^{11} while when $p = 5$ the largest power of 5 dividing $|\Delta_1|$ is 5^{12}. Hence $8 \leq k \leq 12$ and G is an irreducible subgroup of $\mathrm{GL}(k, p)$. Also, by Lemma 5.3, $C_N(T) = 1$. If T were reducible, then it would normalise and act irreducibly on some proper subgroup M of N of order p^t, for some t such that $8 \leq t < k$. By the irreducibility of G, there exists $g \in G$ such that $M^g \neq M$. Since $T \lhd G$, T also normalises M^g and hence normalises $M \cap M^g$. As T is irreducible on M, we have $M \cap M^g = 1$. Hence $t \geq 2k$, contradicting $8 \leq k \leq 12$. Thus T is also an irreducible subgroup of $\mathrm{GL}(k, p)$. Hence by [11, Theorem 5.4.11] and using its terminology, $k = 8$ and N is either the natural module or one of the two spin modules for T. However, a triality automorphism for T permutes these three T–modules (see [3, IV.2.4 and IV.3.1]) and so none of these three T–modules is also a G–module. Hence $X = G$ and so $G = \mathrm{Aut}(\Gamma)$.

This completes the proof of Theorem 1.2.

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