Some graphs related to the small Mathieu groups*

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Abstract

Two different constructions are given of a rank 8 arc-transitive graph with 165 vertices and valency 8, whose automorphism group is M_{11}. One involves 3-subsets of an 11-set while the other involves 4-subsets of a 12-set, and the constructions are linked with the Witt designs on 11, 12 and 24 points. Four different constructions are given of a rank 9 arc-transitive graph with 55 vertices and valency 6 whose automorphism group is $\mathrm{PSL}(2,11)$. This graph occurs as a subgraph of the M_{11} graph, and two of the constructions involve 2-subsets of an 11-set while the remaining two involve 3-subsets of an 11-set. The $\mathrm{PSL}(2,11)$ and M_{11} graphs occur as the second and third members of a tower of graphs defined on a conjugacy class of involutions of the simple groups A_5, $\mathrm{PSL}(2,11)$, M_{11} and M_{12} with two involutions adjacent if they generate a special S_3. The first graph in the tower is the line graph of the Petersen graph while the fourth is the Johnson graph $J(12,4)$. The graphs also arise as collineation graphs of rank two truncations of various residues of certain P-geometries.

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1 Introduction

When studying transitive decompositions of Johnson graphs [7] we came across two interesting graphs: one, Γ, associated with M_{11} and the other, Π, associated with $\mathrm{PSL}(2,11)$. The graphs arise as collinearity graphs of known partial linear spaces related to the Petersen graph but are interesting in their own right as they have several alternative definitions related to geometrical structures preserved by the group. This paper outlines these various constructions as well as showing how the graphs can be placed in a uniform framework by considering involutions which generate an S_3.

1.1 Designs and geometries

A t-(v,k,λ) design is a collection of k-subsets (called *blocks*) of a v-set such that each t-subset is contained in λ blocks. The usual designs associated with the Mathieu groups are the Witt designs. The largest of these is a 5-$(24,8,1)$ design $\mathcal{W}_{24}$ whose blocks are referred to as *octads* and which has full automorphism group M_{24}. The Witt design $\mathcal{W}_{12}$ associated with M_{12} is a 5-$(12,6,1)$ design (blocks referred to as *hexads*) and the Witt design $\mathcal{W}_{11}$ associated with M_{11} is a 4-$(11,5,1)$ design (blocks referred to as *pentads*).

The group M_{11} has a 3-transitive action on a set Y of size 12 and each involution fixes 4 points. The collection of 4-subsets which are fixed point sets of involutions forms a 3-$(12,4,3)$ design $\mathcal{B}$ (see for example [11]). Let $y \in Y$ and let $\overline{\mathcal{B}}$ be the set of 3-subsets of $Y\backslash\{y\}$ whose union with $\{y\}$ is a block of $\mathcal{B}$. Then $\overline{\mathcal{B}}$ is a 2-$(11,3,3)$ design known as the Petersen design and the blocks are the sets of fixed points of involutions of $\mathrm{PSL}(2,11)$ in its 2-transitive action on 11 points. Repeating this contraction process again we obtain a 1-$(10,2,3)$ design whose point set is the vertex set of the Petersen graph and blocks are the edges. These three designs give rise to diagram geometries for the groups M_{11}, $\mathrm{PSL}(2,11)$ and A_5, respectively. The elements of the first geometry are the points, 2-subsets, 3-subsets and 4-subsets in $\mathcal{B}$ of a 12-set admitting the 3-transitive action of M_{11}. The elements of the second geometry are the points, 2-subsets and 3-subsets in $\overline{\mathcal{B}}$ of an 11-set admitting the 2-transitive action of $\mathrm{PSL}(2,11)$. Finally the geometry for A_5 has as elements the vertices and edges of the Petersen graph. For each geometry incidence is given by inclusion and each geometry occurs as a residue of the preceding geometry in the sequence. These geometries are listed as numbers 89, 88 and 84 of [2] and were characterized in [13]. Their diagrams are given in Figure 1.

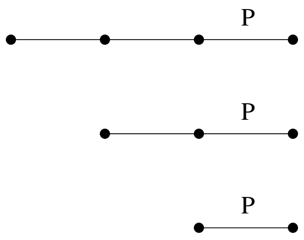

Figure 1: Diagrams for geometries

1.2 Graphs

The graph Γ can be defined as having vertices the 165 blocks of the 3-$(12, 4, 3)$ design $\mathcal{B}$ mentioned above, and two vertices are adjacent if their intersection is a 3-subset. This graph is then the collinearity graph of the partial linear space given by the rank two truncation of the M_{11} geometry onto elements of the last two types with points taken to be the elements of the last type (geometry 2.6 of [6]). Properties of Γ are recorded in Theorem 2.5. The graph Π is the graph with vertices the 55 blocks of $\overline{\mathcal{B}}$ such that two vertices are adjacent if their intersection is a 2-subset. Again, this is the collinearity graph of the partial linear space given by the rank two truncation of the $\mathrm{PSL}(2, 11)$ geometry onto elements of the last two types (geometry 6.1.2 of [3]). The properties of Π are recorded in Theorems 3.2 and 3.13. The collinearity graph for the partial linear space arising from the Petersen graph is the line graph of the Petersen graph.

Since $165 = \binom{11}{3}$ and the stabiliser in M_{11} of a vertex in Γ is the stabiliser in M_{11} of a 3-subset of an 11-set, it is natural to ask if Γ has a definition in terms of 3-subsets. Similiarly, since $55 = \binom{11}{2}$ and the stabiliser in $\mathrm{PSL}(2, 11)$ of a vertex in Π is the stabiliser in $\mathrm{PSL}(2, 11)$ of a 2-subset of an 11-set, it would appear that there should also be a 2-subset definition of Π. Indeed in both cases there is and we have the following theorems.

Theorem 1.1. *The graph* Γ *can be defined in the following ways.*

- $V\Gamma$ *is the set of blocks of* $\mathcal{B}$ *such that two vertices are adjacent if and only if they intersect in a* 3*-set.*
- $V\Gamma$ *is the set of* 3*-subsets of an* 11*-set forming the point set of* $\mathcal{W}_{11}$ *such that two vertices are adjacent if and only if they are disjoint and the complement of their union is a pentad.*

Theorem 1.2. *The graph* Π *can be identified in the following four ways.*

vertices	**adjacency**
blocks of the Petersen design	*intersection has size two*
blocks of the Petersen design	*disjoint and complement of union is a pentad*
2-*subsets of an* 11-*set*	*meet in one point and union is a block of the Petersen design*
2-*subsets of an* 11-*set*	*disjoint and union contains no blocks of the Petersen design.*

The analysis for Theorem 1.1 is in Section 2 while the analysis for Theorem 1.2 is in Section 3.

1.3 Involutions

As already noted the blocks of $\mathcal{B}$ are the sets of fixed points of involutions of M_{11} while the blocks of $\overline{\mathcal{B}}$ are the sets of fixed points of involutions of $\mathrm{PSL}(2, 11)$. This suggests a purely group theoretic definition of Γ and Π independent of the permutation representation. Such a definition gives rise to a tower of four graphs with the smallest being the line graph of the Petersen graph and the largest being the Johnson graph $J(12, 4)$.

Theorem 1.3. *There is a tower of graphs defined on a conjugacy class of involutions of* $A_5, \mathrm{PSL}(2, 11), M_{11}$ *and* M_{12} *with two involutions adjacent if they generate an* S_3 *from a certain conjugacy class such that the graphs are the line graph of the Petersen graph,* Π, Γ *and* $J(12, 4)$.

We define the tower and prove Theorem 1.3 in Section 4. The groups $\mathrm{PSL}(2, 11)$, M_{11} and M_{12} contain more than one class of subgroups S_3 and we specify the appropriate class in Construction 4.5 (for Π), 4.3 (for Γ) and 4.1 (for $J(12, 4)$).

The definition of the graphs in the tower is in the spirit of the investigations by Fischer [10] of groups generated by a conjugacy class of *3-transpositions*, that is a conjugacy class of involutions such that any two either commute or generate an S_3.

It was noted in Subsection 1.2 that Γ and Π were collinearity graphs of certain partial linear spaces. The S_3-involution definition of these graphs then gives rise to an alternative definition of the partial linear spaces, that is, the partial linear space with points the involutions of M_{11} and $\mathrm{PSL}(2, 11)$ respectively and the lines the sets of three involutions which are the involutions of an S_3 in the relevant conjugacy class.

2 The M_{11} graph

We begin by discussing some of the properites of the Witt design $\mathcal{W}_{24}$ which can be found for example in [8, Section 6]. First we give some properties of the octads.

(a) Two distinct octads meet in 0, 2, or 4 points.

(b) Every 4-set is contained in 5 octads. Moreover, the symmetric difference $O_1 \ominus O_2$ of any two octads O_1, O_2 intersecting in a 4-subset is also an octad [8, Lemma 6.8A].

(c) If O_1 and O_2 are two disjoint octads, then the complement of $O_1 \cup O_2$ is also an octad [12, p 78].

The symmetric difference of two octads which intersect in a 2-subset is called a *dodecad*. The complement of a dodecad is also a dodecad.

(d) An octad intersects a dodecad in exactly 2, 4, or 6 points (consequence of [8, Lemma 6.8C]).

Let D be a dodecad. We obtain the 12-point Witt design $\mathcal{W}_{12}$ on D by taking as hexads the intersections of size 6 of octads with D. The stabiliser in M_{24} of D is isomorphic to M_{12}.

(e) Complements of hexads are hexads and their corresponding octads have two points in common (consequence of [8, Lemma 6.8C(ii)]).

Let $\alpha \in D$. Then we obtain $\mathcal{W}_{11}$ on $D \setminus \{\alpha\}$ by taking as pentads the 5-sets which together with α form a hexad of $\mathcal{W}_{12}$. Moreover, $(M_{24})_{D,\alpha} \cong M_{11}$ and has the usual action of M_{11} on $D \setminus \{\alpha\}$ while acting 3-transitively on the complement D^* of D.

The octads of $\mathcal{W}_{24}$ are the fixed point sets of involutions in the conjugacy class $2A$ of M_{24}. For each octad O of $\mathcal{W}_{24}$ which intersects D in four points, there is a unique involution of M_{12} whose fixed points are the elements of O. The group M_{11} has a unique class of involutions and it follows that they fix three points in $D \setminus \{\alpha\}$ and four points of D^*.

We have the following lemma.

Lemma 2.1. *Let D and D^* be two complementary dodecads in $\mathcal{W}_{24}$ and let α be a point in D. Then for all 3-subsets w of $D \backslash \{\alpha\}$, there exists a unique octad O_w such that $O_w \cap D = w \cup \{\alpha\}$.*

Proof. Let w be a 3-subset in $D\backslash\{\alpha\}$. By property (b), there are five octads containing w and α. Since an octad intersects D in exactly 2, 4, or 6 points (property (d)), and since there is a unique octad through any 5 points, there are four octads containing w and α, and intersecting D in 6 points. Thus there is a unique octad, say O_w, which intersects D in 4 points. $\square$

If follows from Lemma 2.1 that

$$\mathcal{B} = \{O_w \cap D^* \mid w \text{ a 3-subset of } D \setminus \{\alpha\}\}$$

is the set of fixed point subsets of involutions in M_{11} acting on the 12-set D^* and is the set of blocks of the 3-$(12, 4, 3)$ design mentioned in the introduction.

We now define two graphs.

Definition 2.2. Let Γ_1 be the graph whose vertices are the blocks of $\mathcal{B}$ with two blocks being adjacent if and only if they meet in a 3-subset.

Definition 2.3. Let Γ_2 be the graph whose vertices are the 3-subsets of an 11-set forming the point set of $\mathcal{W}_{11}$, two 3-sets being adjacent if and only if the complement of their union is a pentad.

To show that Γ_1 and Γ_2 are isomorphic we first need to set up a framework.

Let D and D^* be two complementary dodecads in $\mathcal{W}_{24}$ and let α be a point in D. Set $X = D\backslash\{\alpha\}$ and $Y = D^*$. For a set W and integer $k \leq |W|$ define $\binom{W}{k}$ to be the set of all k-subsets of W. Define

$$\begin{array}{rccc} i: & \binom{X}{3} & \rightarrow & \mathcal{B} \\ & w & \mapsto & Y \cap O_w \end{array} \tag{1}$$

Theorem 2.4. *The map i defined in (1) induces an isomorphism from Γ_2 onto Γ_1.*

Proof. By Lemma 2.1, the map i is well-defined. Let $w_1, w_2 \in \binom{X}{3}$ and suppose $Y \cap O_{w_1} = Y \cap O_{w_2}$. Then O_{w_1} and O_{w_2} have 5 points in common and hence are equal. Thus i is one-to-one and by the definition of $\mathcal{B}$ is onto.

Let w_1 and w_2 be two 3-subsets of X corresponding to adjacent vertices of Γ_2, that is $X \setminus (w_1 \cup w_2)$ is a pentad, in other words, w_1 and w_2 are disjoint and, by (e) and (h), $w_1 \cup w_2$ is a hexad of D, contained in an octad O meeting D in 6 points. The octads O_{w_1} and O are distinct and meet in at least the 3-subset w_1. Thus by property (a) they meet in 4 points, which means that $i(w_1)$ contains exactly one point of O. Moreover by (b), the symmetric difference $O \ominus O_{w_1}$ is an octad containing w_2 and α and intersecting Y in 4 points. By Lemma 2.1, $O \ominus O_{w_1}$ is equal to O_{w_2}. Thus $i(w_1)$ and $i(w_2)$ meet in 3 points, and so they are adjacent vertices of Γ_1.

Table 1: Orbit descriptions for Γ_2

$w \in$	$\lvert w \cap w_0 \rvert$	Extra condition
A	0	$X \setminus (w \cup w_0)$ is a pentad.
B	1	$w = \{x, a, b\}$ with $x \in w_0$, the pentads P_a and P_b containing respectively $w_0 \cup \{a\}$ and $w_0 \cup \{b\}$ are distinct, and $(P_a \cup P_b) \setminus (w_0 \setminus \{x\})$ is not a pentad.
C	0	$w \cup w_0$ contains a pentad with 3 points in w_0.
D	2	
E	0	$w \cup w_0$ contains a pentad with 3 points in w.
F	1	$w \cup w_0$ is a pentad.
G	1	$w = \{x, a, b\}$ with $x \in w_0$, the pentads P_a and P_b containing respectively $w_0 \cup \{a\}$ and $w_0 \cup \{b\}$ are distinct, and $(P_a \cup P_b) \setminus (w_0 \setminus \{x\})$ is a pentad.

Conversely, let v_1 and v_2 be two 4-subsets of Y corresponding to adjacent vertices of Γ_1, that is, they are elements of $\mathcal{B}$ which meet in a 3-subset. Then $O_1 = v_1 \cup \{\alpha\} \cup i^{-1}(v_1)$ and $O_2 = v_2 \cup \{\alpha\} \cup i^{-1}(v_2)$ are octads containing the 4-subset $\{\alpha\} \cup (v_1 \cap v_2)$. Since $v_1 \neq v_2$, the octads O_1, O_2 are distinct and so by property (a) we have $|O_1 \cap O_2| = 4$. Hence $i^{-1}(v_1)$ and $i^{-1}(v_2)$ are disjoint. Moreover, by (b) the symmetric difference of O_1 and O_2 is an octad, which intersects D in $i^{-1}(v_1) \cup i^{-1}(v_2)$. Hence $i^{-1}(v_1) \cup i^{-1}(v_2)$ is a hexad in D, and so by properties (e) and (h), its complement in X is a pentad. Thus $i^{-1}(v_1)$ and $i^{-1}(v_2)$ are adjacent vertices of Γ_2. $\square$

Theorem 2.4 proves Theorem 1.1 as the two graphs are Γ_1 and Γ_2.

Figure 2 gives the distance diagram of the graph $\Gamma \cong \Gamma_2$ according to the orbits of a vertex stabiliser as determined using MAGMA[4]. Each orbit of G_{w_0} on $V\Gamma \backslash \{w_0\}$ is denoted by a circle containing the number of vertices in the orbit. An edge from an orbit S to an orbit T with number a attached at the end connected to S means that each vertex in S is adjacent to a vertices in T. The remaining number next to the orbit S is the number of vertices of S adjacent to a fixed vertex of S. If this number is zero then we use a $-$. Table 1 describes the different orbits in terms of the fixed 3-subset w_0 of the graph Γ_2.

There is one other graph on 165 vertices with automorphism group M_{11} in the literature [9]. This is a half-arc-transitive graph of valency 48 where the set of neighbours of s_0 is $C \cup E$. It can be determined via MAGMA [4] that two vertices are adjacent in Γ if and only if in the valency 48 graph they

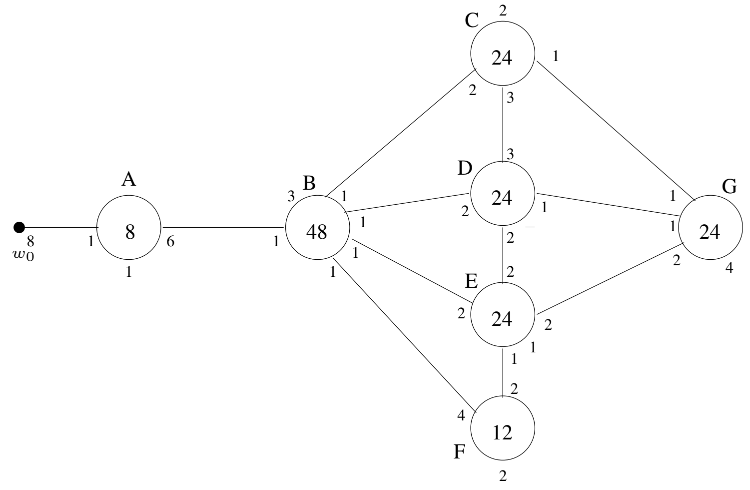

Figure 2: Distance diagram of Γ

are at distance two and there are precisely 9 vertices adjacent to both.

We now determine some properties of Γ.

Theorem 2.5. *The graph* Γ *has* 165 *vertices, valency* 8 *and diameter 4. Its maximal cliques are of size* 3 *and two cliques meet in at most one vertex. Its full automorphism group is* M_{11}*, which is arc-transitive and vertex-primitive, with vertex stabiliser* $M_8 \rtimes S_3 \cong 2{\cdot}S_4$*, and arc stabiliser* S_3*. Moreover, it is a rank* 8 *graph.*

Proof. The number of vertices is $\binom{11}{3} = 165$. Using the 4-subset definition of Γ, the vertices of Γ are the blocks of the 3-$(12, 4, 3)$ design $\mathcal{B}$. A vertex v contains four 3-subsets and each 3-subset is contained in 2 blocks other than v. Hence Γ has valency 8. It can be seen from the diagram of Figure 2 that each neighbour of a vertex w_0 is adjacent to exactly one other neighbour of w_0. Hence the maximal cliques have size 3 and two such cliques meet in at most one vertex. It also follows from the diagram that Γ has diameter 4.

From the distance diagram for Γ given in Figure 2, it follows that if an automorphism g of Γ fixes w_0 then it also fixes setwise A, B and G as these are the vertices are distance 1, 2 and 4 respectively. The automorphism g must also fix F setwise as these are the only vertices not adjacent to a vertex in G, and fix E setwise as these are the only vertices adjacent to two vertices in G. Since the vertices of D are the only vertices in $D \cup C$ which are adjacent to vertices in E, it also follows that g fixes D setwise. Hence the full automorphism group has rank 9 in its action on vertices. As D is the set of 3-subsets meeting w_0 in a 2-subset it follows that $\mathrm{Aut}(\Gamma) \leqslant \mathrm{Aut}(J(11, 3)) =$

S_{11}. Moreover, as g fixes A setwise it follows that $\mathrm{Aut}(\Gamma)$ preserves the Witt design $\mathcal{W}_{11}$ and so $\mathrm{Aut}(\Gamma) = M_{11}$.

Using the 3-subset definition again, the stabiliser of a vertex w in A is $M_8 \rtimes S_3 \cong 2 \cdot S_4$, the stabiliser in M_{11} of a 3-subset. By [5, p 18], the stabiliser in M_{11} of a pentad is S_5 and this acts 3-transitively (as $\mathrm{PGL}(2,5)$) on the 6 points not in the pentad. Hence the stabiliser in A of w acts transitively on the set of 8 pentads having an empty intersection with the 3-subset w. Thus A_w acts transitively on the set of neighbours of w and so A is arc-transitive on Γ_2 with arc stabiliser S_3. $\square$

3 The $\mathrm{PSL}(2,11)$ graph

Let D and D^* be complementary dodecads in $\mathcal{W}_{24}$ and let $\alpha \in D$ and $\beta \in D^*$. We recall that the stabiliser in M_{24} of D and α is M_{11} and acts 3-transitively on D^*. Furthermore, the stabiliser in M_{11} of β is $\mathrm{PSL}(2,11)$ which acts 2-transitively on $D \backslash \{\alpha\}$ and $D^* \backslash \{\beta\}$.

Let $(D^*, \mathcal{B})$ be the 3-$(12,4,3)$ design given by the sets of fixed points of involutions of M_{11}. Let $\overline{\mathcal{B}}$ be the set of 3-subsets v of $D^* \backslash \{\beta\}$ such that $v \cup \{\beta\}$ is a block of $\mathcal{B}$. Then $|\overline{\mathcal{B}}| = 55$ and $(D^* \backslash \{\beta\}, \overline{\mathcal{B}})$ is a 2-$(11,3,3)$ design. This is the Petersen design mentioned in the introduction. Obviously, $\overline{\mathcal{B}}$ is the set of fixed point subsets of involutions in $\mathrm{PSL}(2,11)$ acting on the 11-set $D^* \backslash \{\beta\}$. We note that $\overline{\mathcal{B}}$ is the unique orbit of length 55 of $\mathrm{PSL}(2,11)$ on the set of 3-subsets of $D^* \backslash \{\beta\}$. The other orbit has length 110.

We now give our first definition of the graph Π.

Definition 3.1. Let X be a set of size 11 forming the point set of a 2-$(11,3,3)$ Petersen design. Let Π_1 be the graph with vertex set the set of blocks of the design such that two blocks are adjacent if they have two points in common.

The graph Π_1 is the collinearity graph of the geometry 6.1.2 of [3], from which we have reproduced the distance diagram in Figure 3. Note that regarding $\overline{\mathcal{B}}$ as a subset of $\mathcal{B}$, we see that Π_1 is a subgraph of Γ_1.

Theorem 3.2. *The graph* Π_1 *has* 55 *vertices, valency* 6 *and diameter 3. Its maximal cliques are of size* 3 *and two cliques meet in at most one vertex. The group* $L = \mathrm{PSL}(2,11)$ *is an automorphism group of* Π_1*, which is arc-transitive and vertex-primitive with vertex stabiliser* D_{12} *and arc stabiliser* C_2*. Moreover,* Π_1 *is a Cayley graph for the group* $C_{11} \rtimes C_5$*.*

Proof. Let v be a vertex of Π_1. Then v contains three 2-subsets and since the vertex set of Π_1 is the block set of a 2-$(11,3,3)$ design, each 2-subset of v is contained in 3 blocks. Hence v is adjacent to 6 other vertices. Moreover,

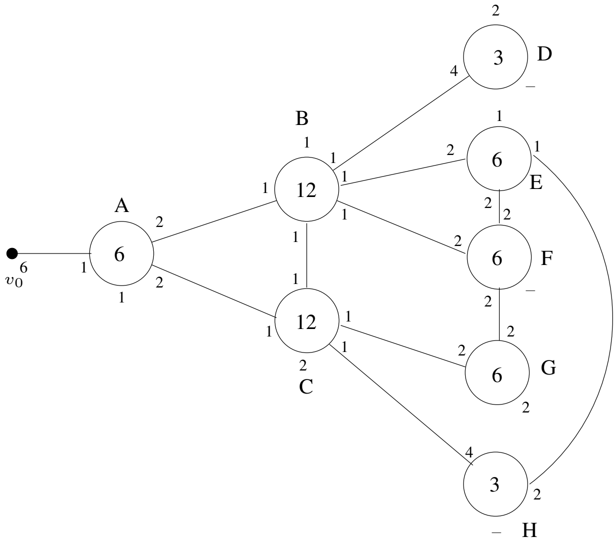

Figure 3: Distance diagram of Π_1

we see from Figure 3, that Π_1 has diameter 3, and the maximal cliques have size 3 and meet in at most one vertex. Since $L = \mathrm{PSL}(2, 11)$ preserves the design, $L \leqslant \mathrm{Aut}(\Pi_1)$.

If v is a vertex, then $L_v = D_{12}$. Let w be a vertex adjacent to v. Let $\overline{v} = v \cup \{\beta\}$ and $\overline{w} = x \cup \{\beta\}$ and regard Π_1 as a subgraph of Γ_1 with the vertices $\overline{v}, \overline{w}$ of Γ_1 corresponding to the vertices v, w, respectively, of Π_1. By Theorem 2.5, $(M_{11})_{\overline{v}\overline{w}} = S_3$ and since in the action of M_{11} on 12 points, elements of order three only have 3 fixed points, $(M_{11})_{\overline{v}\overline{w}}$ induces S_3 on the three points of $\overline{v} \cap \overline{w} = \{\beta\} \cup (v \cap w)$. Since L is the stabiliser in M_{11} of β, it follows that $L_{vw} = C_2$ and hence L is arc-transitive on Π_1.

Note that $\mathrm{PSL}(2, 11)$ has a subgroup $C_{11} \rtimes C_5$ which intersects trivially with D_{12} and so by comparing orders we have $\mathrm{PSL}(2, 11) = D_{12}(C_{11} \rtimes C_5)$. Hence $C_{11} \rtimes C_5$ acts regularly on the vertex set of Π_1 and so Π_1 is a Cayley graph of $C_{11} \rtimes C_5$. $\square$

The map i defined in (1) defines an isomorphism from Γ_2 to Γ_1, hence the preimage of Π_1 in Γ_2 is isomorphic to Π_1. Note that the preimages of the vertices of Π_1 are 3-sets w in $D \setminus \{\alpha\}$ such that $\beta \in i(w)$. If we take these 3-sets as blocks, we find an isomorphic 2-$(11, 3, 3)$ design, since this block set and $\overline{\mathcal{B}}$ are interchanged by elements of $\mathrm{PGL}(2, 11) \setminus \mathrm{PSL}(2, 11)$. This gives the following definition of a graph isomorphic to Π_1.

Definition 3.3. Let X be an 11-set forming the point set of a 2-$(11, 3, 3)$ Petersen design preserved by a group $L = \mathrm{PSL}(2, 11)$ and also forming the point set of $\mathcal{W}_{11}$ preserved by L. Let Π_2 be the graph with vertex set the set of blocks of the Petersen design such that blocks v_1 and v_2 are joined by an edge if $v_1 \cap v_2 = \varnothing$ and $X \backslash (v_1 \cup v_2)$ is a pentad.

Theorem 3.4. $\Pi_1 \cong \Pi_2$.

Proof. The isomorphism is given by the map i from (1). $\square$

By Theorem 3.2, the stabiliser of a vertex in $L = \mathrm{PSL}(2, 11)$ is D_{12} which has orbits of size 2, 3 and 6 on the 11-set. We will now describe the link between the blocks of the Petersen design and the pairs of an 11-set.

Consider $(D^*, \mathcal{B})$, a 3-$(12, 4, 3)$ design given by the sets of fixed points of involutions of M_{11}, where X has size 12. Let $\beta \in X$. We recall that the set of 3-subsets v of $D^* \backslash \{\beta\}$ such that $v \cup \{\beta\}$ is a block of $\mathcal{B}$ yields the block-set $\overline{\mathcal{B}}$ of the Petersen design $(D^* \backslash \{\beta\}, \overline{\mathcal{B}})$, which is a 2-$(11, 3, 3)$ design. Let $v \in \overline{\mathcal{B}}$. Since $(D^*, \mathcal{B})$ is a 3-$(12, 4, 3)$ design there are three blocks of $\mathcal{B}$ containing v, one of which is $v \cup \{\beta\}$. Let ϵ_1 and ϵ_2 be the two points such that $v \cup \{\epsilon_i\}$ are also blocks of $\mathcal{B}$. This allows us to define a map

$$\begin{array}{rccc} j: & \overline{\mathcal{B}} & \rightarrow & \binom{D^* \backslash \{\beta\}}{2} \\ & v & \mapsto & \{\epsilon_1, \epsilon_2\} \end{array} \tag{2}$$

Since L preserves $\mathcal{B}$, we must have that the pair stabilised by L_v is $j(v)$.

Let p be a pair in the 11-set $D^* \backslash \{\beta\}$. Then p is contained in 3 blocks s_1, s_2 and s_3 of the Petersen design, and $(s_1 \cup s_2 \cup s_3) \setminus p$ is a 3-set of $D^* \backslash \{\beta\}$. Since L preserves $\overline{\mathcal{B}}$, the block of the Petersen design stabilised by L_p must be this 3-set. We define the map

$$\begin{array}{rccc} b: & \binom{D^* \backslash \{\beta\}}{2} & \rightarrow & \overline{\mathcal{B}} \\ & p & \mapsto & (s_1 \cup s_2 \cup s_3) \setminus p \end{array} \tag{3}$$

Since $b(p)$ is the block stabilised by L_p and $j(b(p))$ is the 2-set stabilised by $L_{b(p)}$ it follows that b is the inverse of j.

The correspondence between 2-subsets and blocks suggests a 2-subset definition of Π_1.

Definition 3.5. Let X be an 11-set equipped with a 2-$(11, 3, 3)$ Petersen design. Let Π_3 be the graph with vertex set the set of all 2-subsets of X such that two vertices x_1, x_2 are adjacent if they are disjoint and $x_1 \cup x_2$ contains no blocks of the Petersen design.

We will need the following lemma.

Lemma 3.6. *A block of $\mathcal{B}$ does not contain two distinct blocks of $\overline{\mathcal{B}}$*

Proof. Suppose a block w of $\mathcal{B}$ contains distinct blocks v_1 and v_2 of $\overline{\mathcal{B}}$. Then $|v_1 \cap v_2| = 2$ and the three blocks w, $v_1 \cup \{\beta\}$ and $v_2 \cup \{\beta\}$ of $\mathcal{B}$ form a triangle of Γ_1. On the other hand, the third block of $\mathcal{B}$ containing v_1 also forms a triangle with w and $v_1 \cup \{\beta\}$. Since $v_2 \cup \{\beta\}$ does not contain v_1 this is a second triangle of Γ sharing an edge with the first, contradicting Theorem 2.5. □

We have the following isomorphism.

Lemma 3.7. *The map j defined in (2) induces an isomorphism from Π_1 to Π_3.*

Proof. We have already proved that j is a bijection. By Theorem 3.2, Π_1 has valency 6. We claim that Π_3 also has valency 6. Let $p = \{\gamma, \delta\}$ be a vertex of Π_3 and let $\{\xi_1, \xi_2, \xi_3\} = b(p)$. Among the 36 pairs disjoint from p, the pairs that are non-adjacent to p are those that either form a block of $\overline{\mathcal{B}}$ with γ or δ, or that contain a point in $b(p)$. There are 21 pairs in the second case, 3 contained in $b(p)$ and 18 meeting it in a single point. Let us count the pairs forming a block with γ or δ. It is easy to count that γ is contained in 15 blocks, among which 3 also contain δ, and similarly interchanging γ and δ. So there are 24 blocks containing either γ or δ but not both. Let $p_1 = j(\{\gamma, \delta, \xi_1\})$, $p_2 = j(\{\gamma, \delta, \xi_2\})$, $p_3 = j(\{\gamma, \delta, \xi_3\})$. These three pairs form a block with γ and a block with δ, as j is the inverse of b. Hence there are 21 pairs in the first case. Finally we count the number of pairs which are in the intersection, that is, which form a block of $\mathcal{B}$ with γ or with δ and contain ξ_1, ξ_2 or ξ_3. By Lemma 3.6, such a pair cannot be contained in $b(p)$, as otherwise the block $\{\xi_1, \xi_2, \xi_3, \gamma\}$ (or $\{\xi_1, \xi_2, \xi_3, \delta\}$) of $\overline{\mathcal{B}}$ contains two blocks of $\mathcal{B}$. Therefore such a pair meets $\{\xi_1, \xi_2, \xi_3, \}$ in exactly one point. There are two blocks containing a given point of p and a given point of $b(p)$ but not containing p, and so there are $2.3.2 = 12$ pairs in both the first and second cases. Thus there are $21 + 21 - 12 = 30$ vertices non-adjacent to p, and so there are 6 vertices adjacent to p.

It remains to show that j maps adjacent vertices of Π_1 to adjacent vertices of Π_3. Let $v_1, v_2 \in \overline{\mathcal{B}}$ such that $|v_1 \cap v_2| = 2$, that is, v_1, v_2 are adjacent vertices in Π_1. Suppose that $j(v_1)$ and $j(v_2)$ have a point ϵ in common. Then $v_1 \cup \{\beta\}, v_2 \cup \{\beta\}, v_2 \cup \{\epsilon\}$ and $v_1 \cup \{\epsilon\}$ are blocks of $\mathcal{B}$ forming a 4-cycle in Γ_1. However, looking at the distance diagram for Γ_1 in Figure 2, we see that Γ_1 does not contain 4-cycles. Thus $j(v_1) \cap j(v_2) = \varnothing$. The points which together with $j(v_1)$ form a block of $\overline{\mathcal{B}}$ are those in $b(j(v_1)) = v_1$. Suppose $j(v_2) \cap v_1$ is not empty. Then this intersection has size 1 (because $j(v_2) \cap v_2 = \varnothing$), and $v_1 \cup v_2 \in \mathcal{B}$, which contradicts Lemma 3.6. Therefore, $j(v_2) \cap v_1 = \varnothing$, and so no block of $\overline{\mathcal{B}}$ contains $j(v_1)$ and one point of $j(v_2)$.

Similarly no block of $\overline{\mathcal{B}}$ contains $j(v_2)$ and one point of $j(v_1)$. Thus $v_1 \cup v_2$ contains no block of $\overline{\mathcal{B}}$, and hence $j(v_1)$ and $j(v_2)$ are adjacent vertices of Π_3. $\square$

We also have the following definition of a graph on the set of 2-subsets of an 11-set.

Definition 3.8. Let X be an 11-set forming the point set of a 2-$(11,3,3)$ Petersen design. Let Π_4 be the graph with vertex set the set of all 2-subsets of X such that two vertices are adjacent if they have one point in common and their union is a block of the Petersen design.

Next we show that Π_4 is isomorphic to Π_1. To do this we first recall the following setup. Let D and D^* be two complementary dodecads in $S(5,8,24)$ and let $\alpha \in D$ and $\beta \in D^*$. Let $L \cong \mathrm{PSL}(2,11)$ be the stabiliser in M_{24} of D, α and β. Recall from (1) the map i from the set of 3-subsets of $D\backslash\{\alpha\}$ into the set of 4-subsets of D^* such that the image $\mathcal{B}$ of i is a 3-$(12,4,3)$ design and the set of blocks of $\mathcal{B}$ containing β yields a 2-$(11,3,3)$ Petersen design on $D^*\backslash\{\beta\}$ with blocks $\overline{\mathcal{B}}$.

Let $v \in \overline{\mathcal{B}}$ and $j(v) = \{\epsilon_1, \epsilon_2\}$. Since $v \cup \{\beta\}, v \cup \{\epsilon_1\}, v \cup \{\epsilon_2\}$ are mutually adjacent in Γ_1, Theorem 2.4 implies that we can partition D as $\{\alpha, \delta, \mu\} \cup w_1 \cup w_2 \cup w_3$ such that $i(w_1) = v \cup \{\beta\}$, $i(w_2) = v \cup \{\epsilon_1\}$ and $i(w_3) = v \cup \{\epsilon_2\}$, and for $i = 1,2,3$ the set $\{\delta, \mu\} \cup w_i$ is a pentad. Thus we can also define the map

$$\begin{array}{rccl} k: & \overline{\mathcal{B}} & \rightarrow & \binom{D\backslash\{\alpha\}}{2} \\ & v & \mapsto & \{\delta, \mu\} \end{array} \tag{4}$$

The maps j and k are linked in the following way.

Lemma 3.9. *Let $v \in \overline{\mathcal{B}}$ and let j, k be the maps defined in (2), (4) respectively. Then $\{\beta\} \cup v \cup j(v) \cup k(v)$ is an octad.*

Proof. Let $j(v) = \{\epsilon_1, \epsilon_2\}$. By the definition of the map i in (1) we have three octads $O_{w_1} = \{\alpha, \beta\} \cup v \cup w_1$, $O_{w_2} = \{\alpha, \epsilon_1\} \cup v \cup w_2$ and $O_{w_3} = \{\alpha, \epsilon_2\} \cup v \cup w_3$. By property (b), $O_{w_2} \ominus O_{w_3} = \{\epsilon_1, \epsilon_2\} \cup w_2 \cup w_3$ is an octad disjoint to O_{w_1}. Thus by property (c), $k(v) \cup (D^*\backslash(\{\beta, \epsilon_1, \epsilon_2\} \cup v))$ is also an octad and so by property (e), $k(v) \cup v \cup j(v) \cup \{\beta\}$ is an octad. $\square$

We now show that Π_1 and Π_4 are isomorphic.

Theorem 3.10. *The map k defined in (4) induces an isomorphism from Π_1 to Π_4.*

Proof. We first construct Π_1 by using the Petersen design $(D^*\backslash\{\beta\},\overline{\mathcal{B}})$. Thus the vertices of Π_1 are the blocks of $\overline{\mathcal{B}}$ and two vertices are adjacent if they intersect in a 2-set. Now $L = \mathrm{PSL}(2,11)$ preserves the set of octads containing $\{\alpha,\beta\}$ and meeting D in 4 points. There are 55 such octads and so the set $\mathcal{C} = \{3-set \quad w \subset D \setminus \{\alpha\} \mid \beta \in O_w\}$ is an orbit of length 55 of L on 3-subsets of $D\backslash\{\alpha\}$. Thus $(D\backslash\{\alpha\},\mathcal{C})$ is a 2-$(11,3,3)$ Petersen design and we can construct Π_4 from this design, that is the vertices of Π_4 are the 2-subsets of $D\backslash\{\alpha\}$ and two vertices are adjacent if they have one point in common and their union is a block of $\mathcal{C}$.

Since L preserves $\overline{\mathcal{B}}$ and $\mathcal{C}$, it also preserves the set of images of k. Then as L acts 2-transitively on $D\backslash\{\alpha\}$, it follows that k is onto. Moreover, as $|\overline{\mathcal{B}}| = |\mathcal{C}| = 55 = \binom{11}{2}$ it follows that k is a bijection.

We saw in Theorem 3.2 that Π_1 is of valency 6. Moreover, since each 2-set is contained in 3 blocks of $\mathcal{C}$, it follows that Π_4 also has valency 6. Thus we only need to show that if v_1 and v_2 are adjacent in Π_1 then $k(v_1)$ and $k(v_2)$ are adjacent in Π_4.

Let $v_1, v_2 \in \overline{\mathcal{B}}$ such that $|v_1 \cap v_2| = 2$, that is, v_1, v_2 are adjacent vertices in Π_1. We have shown in the proof of Lemma 3.7 that $j(v_1) \cap j(v_2) = \varnothing$. Moreover, by Lemma 3.9, $O_1 = v_1 \cup j(v_1) \cup k(v_1) \cup \{\beta\}$ and $O_2 = v_2 \cup j(v_2) \cup k(v_2) \cup \{\beta\}$ are octads containing the three points $(v_1 \cap v_2) \cup \{\beta\}$. Hence by property (a) they have four points in common and so $|k(v_1) \cap k(v_2)| = 1$.

Let $k(v_1) = \{\delta_1, \mu\}$ and $j(v_1) = \{\epsilon_1, \epsilon_2\}$, and let w_1, w_2, w_3 be 3-subsets of $D\backslash\{\alpha\}$ such that $O_{w_1} = \{\alpha,\beta\} \cup w_1 \cup v_1$, $O_{w_2} = \{\alpha, \epsilon_1\} \cup w_2 \cup v_1$ and $O_{w_3} = \{\alpha, \epsilon_2\} \cup w_3 \cup v_1$ are octads. Similarly, let $k(v_2) = \{\delta_2, \mu\}$ and $j(v_2) = \{\epsilon_1', \epsilon_2'\}$, and let w_1', w_2', w_3' be 3-subsets of $D\backslash\{\alpha\}$ such that $O_{w_1'} = \{\alpha,\beta\} \cup w_1' \cup v_2$, $O_{w_2'} = \{\alpha, \epsilon_1'\} \cup w_2' \cup v_2$ and $O_{w_3'} = \{\alpha, \epsilon_2'\} \cup w_3' \cup v_2$ are octads. Recall the octads $O_1 = v_1 \cup j(v_1) \cup k(v_1) \cup \{\beta\}$ and $O_2 = v_2 \cup j(v_2) \cup k(v_2) \cup \{\beta\}$, which have the four points $(v_1 \cap v_2) \cup \{\beta, \mu\}$ in common. Hence by property (b),

$$O_1 \ominus O_2 = \{\delta_1, \delta_2\} \cup (v_1 \ominus v_2) \cup \{\epsilon_1, \epsilon_2, \epsilon_1', \epsilon_2'\}$$

is an octad. Since $O_1 \ominus O_2$ contains one point of v_2 it has at least one point in common with $O_{w_1'}$ and so by property (a) it follows that $\delta_1 \in w_1'$. Similarly, comparing $O_1 \ominus O_2$ with O_{w_1} we see that $\delta_2 \in w_1$. Thus $O_1 \ominus O_{w_1} = w_1 \cup \{\alpha, \delta_1, \mu, \epsilon_1, \epsilon_2\}$ and $O_2 \ominus O_{w_1'} = w_1' \cup \{\alpha, \delta_2, \mu, \epsilon_1', \epsilon_2'\}$ are octads with the 4-set $w = \{\delta_1, \delta_2, \mu, \alpha\}$ in common. By property (b), the 4-set w is contained in 5 octads, which by Lemma 2.1 are $O_{\{\delta_1,\delta_2,\mu\}}$ and four octads R_1, R_2, R_3, R_4 meeting D in 6 points. The octads $R_1 := O_1 \ominus O_{w_1}$ and $R_2 := O_2 \ominus O_{w_1'}$ are two of these four octads. Moreover, the 2-sets $(R_i \cap D)\backslash w$ for $i = 1,2,3,4$, partition the set $D\backslash w$ of eight points. The octads R_1, R_2 account for the four points of $(w_1 \cup w_1')\backslash\{\delta_1, \delta_2\}$ and so for $i = 3$ or 4, the two points of

$(R_i \cap D)\backslash w$ are in $w_2 \cup w_3$ but not in w_1'. If $\beta \in R_i$ for $i = 3, 4$, then R_i will have 5 points in common with either $O_1 \ominus O_{w_2} = w_2 \cup \{\alpha, \beta, \delta_1, \mu, \epsilon_2\}$ or $O_1 \ominus O_{w_3} = w_3 \cup \{\alpha, \beta, \delta_1, \mu, \epsilon_1\}$. Since these two octads do not contain δ_2, it follows that neither of them are R_i, and so by property (a), β is not in any of the R_i. Now R_1, R_2, R_3, R_4 define eight points of D^* and $i(\{\delta_1, \delta_2, \mu\})$ is the set of 4 points of D^* not in any R_i. For $i \neq j$ the sets $R_i \cap D^*$ and $R_j \cap D^*$ are disjoint and have size 2. Hence $R_1, \ldots, R_4$ provide us with 8 points of $D^*\backslash\{\beta\}$. Thus $\beta \in i(\{\delta_1, \delta_2, \mu\})$ and so $\{\delta_1, \delta_2, \mu\} \in \mathcal{C}$. Hence $\{\delta_1, \delta_2, \mu\}$ is a block of the Petersen design $(D\backslash\{\alpha\}, \mathcal{C})$ preserved by $\mathrm{PSL}(2, 11)$. Thus $k(v_1)$ is adjacent to $k(v_2)$ in Π_4 and so $\Pi_1 \cong \Pi_4$. □

Corollary 3.11. $\Pi_1 \cong \Pi_2 \cong \Pi_4 \cong \Pi_3 \cong \Pi$.

This completes the proof of Theorem 1.2.

We also have that Π is self-dual in the following sense:

Proposition 3.12. Π *is isomorphic to the graph whose vertices are the triangles of* Π *and two triangles are adjacent if they have a common vertex.*

Proof. We will show that Π_4 is isomorphic to the graph whose vertices are the triangles of Π_1 and two triangles are adjacent if they have a common vertex. Since both Π_1 and Π_4 are isomorphic to Π, this will yield the lemma.

Since the Petersen design is a 2-$(11, 3, 3)$ design, each 2-subset is contained in three blocks and these three blocks form a triangle in Π_1. Since no edge is contained in two triangles by Theorem 3.2, each triangle in Π_1 is described by a unique 2-subset, this being the 2-subset in common with each of the three blocks which are vertices in the triangle. Thus there is a bijection between the vertices of Π_4 and the triangles of Π_1. Moreover, two triangles of Π_1 have a vertex in common if and only if the union of the two 2-sets corresponding to the two triangles is the block of the Petersen design given by the common vertex. Hence the result follows. □

We finish this section by giving the full automorphism of Π.

Theorem 3.13. *The full automorphism group of* Π *is* $\mathrm{PSL}(2, 11)$ *and* Π *is a rank 9 graph.*

Proof. It follows from Corollary 3.11 that Figure 3 is the distance diagram for Π_4. It can be checked that $A, B, \ldots, H$ are the orbits of a vertex stabiliser in L. Table 2 (found with the help of Magma [4]) describes the different orbits in terms of the fixed 2-subset v_0 of the graph Π_4.

Let g be an automorphism of Π fixing v_0. Then g clearly fixes A setwise and must also fix D setwise as these are the only vertices at distance 3 from

Table 2: Orbit descriptions for Π_4

$v \in$	$\lvert v \cap v_0 \rvert$	Extra condition in Π_4
A	1	$v \cup v_0$ is a block of the Petersen design.
B	0	$\lvert v \cap b(v_0)\rvert = 1$ and $\lvert v_0 \cap b(v)\rvert = 1$.
C	1	$v \cup v_0$ is not a block of the Petersen design.
D	0	$\lvert v \cap b(v_0)\rvert = 2$ and $\lvert v_0 \cap b(v)\rvert = 0$.
E	0	$\lvert v \cap b(v_0)\rvert = 1$ and $\lvert v_0 \cap b(v)\rvert = 0$.
F	0	$\lvert v \cap b(v_0)\rvert = 0$ and $\lvert v_0 \cap b(v)\rvert = 0$.
G	0	$\lvert v \cap b(v_0)\rvert = 0$ and $\lvert v_0 \cap b(v)\rvert = 1$.
H	0	$\lvert v \cap b(v_0)\rvert = 0$ and $\lvert v_0 \cap b(v)\rvert = 2$.

v_0 adjacent to 4 vertices at distance 2 and to two vertices at distance 3 which are themselves adjacent to 4 vertices at distance 2. Since the vertices in B are the only vertices at distance 2 adjacent to a vertex of D, it follows that g fixes B and hence also C setwise. As $A \cup C$ is the set of 2-subsets meeting v_0 in a 1-subset it follows that $\mathrm{Aut}(\Pi_4) \leqslant \mathrm{Aut}(J(11,2)) = S_{11}$. Moreover, as g fixes A setwise it follows that $\mathrm{Aut}(\Pi)$ preserves the Petersen design, whose automorphism group is $\mathrm{PSL}(2,11)$ by [3, p182] and so $\mathrm{Aut}(\Pi) = \mathrm{PSL}(2,11)$. It follows that Π is a rank 9 graph. $\square$

4 A tower of graphs

The Johnson graph $J(12,4)$ can be defined in terms of one of the two conjugacy classes of involutions of M_{12}.

Construction 4.1. Let $G = M_{12}$. Let Δ_1 be the graph with vertex set the set of involutions of G from the conjugacy class $2B$ with 4 fixed points (in an M_{12} action of degree 12) [5, p 33] such that two involutions are joined by an edge if and only if they generate an S_3 which has 3 fixed points.

Lemma 4.2. $\Delta_1 \cong J(12,4)$.

Proof. Let Y be a set of size 12 such that G acts 5-transitively on Y. Since the class $2B$ has $495 = \binom{12}{4}$ elements, it follows that the map ϕ from the vertices of Δ_1 to the vertices of $J(12,4)$ which maps each involution g to $\mathrm{Fix}(g)$ is a bijection.

Let g_1 and g_2 be two adjacent vertices of Δ_1. Then $\langle g_1, g_2\rangle \cong S_3$ fixes 3 points. Thus $\lvert \mathrm{Fix}(g_1) \cap \mathrm{Fix}(g_2)\rvert = 3$ and so $\phi(g_1)$ and $\phi(g_2)$ are adjacent in $J(12,4)$.

Conversely, let v_1 and v_2 be two adjacent vertices of $J(12,4)$, that is, 4-subsets of Y intersecting in a 3-set. The pointwise stabiliser in G of this 3-set is $L \cong M_9 \cong C_3^2 \rtimes Q_8$, which has a unique Sylow 3-subgroup S and a unique conjugacy class of 9 subgroups isomorphic to Q_8 (the Sylow 2-subgroups). Let $g \in L$ be an involution. Then g is the unique involution of some Q_8 and so inverts each nontrivial element of S. Moreover, the 9 involutions of L are the elements sg, for $s \in S$. The two involutions fixing respectively v_1 and v_2 (that is $\phi^{-1}(v_1)$ and $\phi^{-1}(v_2)$) lie in L, so are s_1g and s_2g, for some $s_1, s_2 \in S$. Now $(s_1g)(s_2g) = s_1s_2^{-1}$, which has order three and so $\langle s_1g, s_2g\rangle \cong S_3$ and is contained in L, hence fixes three points. Thus $\phi^{-1}(v_1)$ and $\phi^{-1}(v_2)$ are adjacent in Δ_1 and the proof is complete. □

Let Y be a set of size 12 on which the group $G = M_{12}$ acts 5-transitively. Now G contains two classes of subgroups isomorphic to M_{11} and these are interchanged by an outer automorphism with one class being the stabilisers in G of a point of Y. Involutions in G lie in one of two conjugacy classes: fixed point free involutions and the class $2B$ where they each fix 4 points [5, p 33]. Since the class $2B$ is fixed setwise by outer automorphisms of M_{12} it follows that the involutions in any M_{11} lie in the class $2B$. Now M_{11} has 165 involutions and they form a single conjugacy class. Let H be a subgroup of G isomorphic to M_{11} which is not the stabiliser of a point, that is, is 3-transitive on Y. Then the stabiliser in H of a 3-subset of Y is isomorphic to S_3, and as M_{11} is 3-transitive on Y, the set of subgroups of H isomorphic to S_3 which fix 3 points forms a single conjugacy class. Since the normaliser of such an S_3 fixes the 3-set of fixed points setwise, each such S_3 subgroup has normaliser $S_3 \times S_3$.

This suggests a connection with Γ_1.

Construction 4.3. Let $H = M_{11}$. Let Δ_2 be the graph with vertex set the set of involutions of H such that two involutions are adjacent if and only if they generate an S_3 with normaliser $S_3 \times S_3$.

Lemma 4.4. $\Delta_2 \cong \Gamma_1$.

Proof. Since Δ_2 is the subgraph of Δ_1 induced by the involutions of H and Γ_1 is the image of Δ_2 under the isomorphism between Δ_1 and $J(12,4)$ it follows that $\Delta_2 \cong \Gamma_1$. □

We have just exhibited an embedding of the graph Δ_2 in $J(12,4)$. It is proved in [7] that in fact $J(12,4)$ decomposes into 12 pairwise disjoint copies of Δ_2 and these 12 copies are transitively permuted by M_{12}.

Next we look at a subgroup $K = \mathrm{PSL}(2,11)$ of $H = M_{11}$. This occurs as the stabiliser in H of a point $\beta \in Y$. Hence K contains a class of subgroups

S_3 which fix 3 points of Y, one of which is β. Since K is 2-transitive on the 11 points of $Y\backslash\{\beta\}$ it follows that all such subgroups S_3 are conjugate in K. Thus we can use this conjugacy class of S_3 subgroups to define a graph on the set of involutions of K, and this graph will be a subgraph of Δ_2.

Construction 4.5. Let Δ_3 be the graph whose vertex set is the set of involutions of $\mathrm{PSL}(2,11)$ such that two involutions are adjacent if and only if they generate an S_3 which fixes 2 points in the action on 11 points.

Lemma 4.6. $\Delta_3 \cong \Pi_1$

Proof. The graph Δ_3 is the subgraph of Δ_2 induced on the set of involutions of $\mathrm{PSL}(2,11)$. Then the isomorphism $\phi : V\Delta_1 \mapsto J(12,4)$ which maps an involution to its set of four fixed points in Y induces an isomorphism from Δ_3 to a subgraph of Γ_1. Since all involutions in $\mathrm{PSL}(2,11)$ fix the point β and the blocks of the Petersen design are the sets of fixed points of the involutions of $\mathrm{PSL}(2,11)$ on $Y\backslash\{\beta\}$ this provides an isomorphism from Δ_3 to Π_1. $\square$

Note that $\mathrm{PSL}(2,11)$ contains two classes of subgroups S_3 and these are fused in $\mathrm{PGL}(2,11)$. Hence, the graph formed on the set of involutions of $\mathrm{PSL}(2,11)$, with two involutions joined by an edge if they generate an S_3 in the class other than the one used to define Δ_3, is isomorphic to Δ_3.

Since the stabiliser in $\mathrm{PSL}(2,11)$ of an arc of Δ_3 is C_2 (Theorem 3.2) while the stabiliser of an arc of Δ_2 in M_{11} is S_3 (Theorem 2.5), it follows that if we spin the subgraph Δ_3 of Δ_2 under M_{11} we do not obtain a decomposition of Δ_3. Instead we obtain a cover of Δ_2 with each edge occurring in precisely 3 of the 12 subgraphs isomorphic to Δ_3, that is, we obtain a transitive 3-cover.

Inside K we can choose a subgroup $L \cong A_5$. Note that there are two conjugacy classes of such subgroups that are interchanged by $\mathrm{PGL}(2,11)$. Each such subgroup L contains 15 involutions and these form a single L-conjugacy class.

Construction 4.7. Let Δ_4 be the graph with vertex set the 15 involutions of A_5 such that two involutions are adjacent if and only if they generate an S_3.

Lemma 4.8. Δ_4 *is the line graph of the Petersen graph.*

Proof. When written in the usual permutation representation of A_5 on 5 points, the involutions of A_5 are of the form $(a,b)(c,d)$. Let τ be the map which takes each involution $(a,b)(c,d)$ in A_5 to the edge $\{\{a,b\},\{c,d\}\}$ of the Petersen graph. This is clearly a bijection. There are precisely four involutions of A_5 which generate an S_3 with $(a,b)(c,d)$. If e is the fifth point of the set then these involutions are $(a,b)(c,e)$, $(a,b)(d,e)$, $(a,e)(c,d)$ and $(b,e)(c,d)$. Under τ, these are mapped to the 4 edges incident with the edge $\{\{a,b\},\{c,d\}\}$ of the Petersen graph. Hence τ is an isomorphism. $\square$

Since the stabiliser in A_5 of an arc of Δ_4 is C_2 and this is the stabiliser in $\mathrm{PSL}(2,11)$ of an arc in Δ_3, it follows that Δ_3 decomposes into 11 copies of Δ_4 and these copies are transitively permuted by $\mathrm{PSL}(2,11)$.

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