Limits of vertex-transitive graphs

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Abstract. The class of all connected vertex-transitive graphs with finite valency forms a metric space under a natural combinatorially defined metric. We prove some basic properties of this metric space and discuss the structure of graphs which are limit points of the subset consisting of all finite graphs that admit a vertex-primitive group of automorphisms. A description of these limit graphs would provide a useful description of the possible local structures of generic finite graphs that admit a vertex-primitive automorphism group.

1. Introduction

Let $\mathcal{G}$ be the set of all connected vertex-transitive graphs with finite valency. For each $\Gamma \in \mathcal{G}$ we define the usual metric d_Γ on the vertex set $V(\Gamma)$ of Γ, namely $d_\Gamma(x, y)$ denotes the length of a shortest path between the two vertices x and y. Given $x \in V(\Gamma)$ and a non-negative integer i we define the *ball of radius i with centre x* by $B_\Gamma(x, i) = \{y \in V(\Gamma) \mid d(y, x) \le i\}$. Note that Γ induces a subgraph on $B_\Gamma(x, i)$, which we will also denote by $B_\Gamma(x, i)$. This allows us to define a metric ρ on $\mathcal{G}$ in the following manner: If Γ_1 and Γ_2 are isomorphic then we define $\rho(\Gamma_1, \Gamma_2) = 0$. Otherwise, $\rho(\Gamma_1, \Gamma_2) = 1/2^i$, where i is the largest non-negative integer such that $B_{\Gamma_1}(x, i) \cong B_{\Gamma_2}(y, i)$ for some vertices $x \in V(\Gamma_1)$ and $y \in V(\Gamma_2)$. Since Γ_1 and Γ_2 are vertex-transitive this value of i is independent of the choices of x and y. Note that for any two graphs Γ_1 and Γ_2 of different valencies we have $\rho(\Gamma_1, \Gamma_2) = 1/2^0 = 1$. There are also, of course, non-isomorphic vertex-transitive graphs Γ_1, Γ_2 of the same valency with $\rho(\Gamma_1, \Gamma_2) = 1$, for example, the complete graph $\Gamma_1 = K_4$ and the complete bipartite graph $\Gamma_2 = K_{3,3}$. The next smallest ρ-value occurs for example, with Γ_1 the complete bipartite graph $K_{3,3}$ and Γ_2 the cube Q_3. We have $B_{\Gamma_1}(x, 1) \cong B_{\Gamma_2}(y, 1)$ for any $x \in V(\Gamma_1)$ and $y \in V(\Gamma_2)$. However, $B_{\Gamma_1}(x, 2) \not\cong B_{\Gamma_2}(y, 2)$ and so $\rho(\Gamma_1, \Gamma_2) = 1/2$. See Figure 1.

We wish to study the limits of convergent sequences in the metric space $(\mathcal{G}, \rho)$. Note that if Γ is a limit of the sequence $(\Gamma_i)_{i\ge 0}$ then we do not necessarily have that each Γ_i is a subgraph of Γ. All we know is that the restrictions of Γ and the Γ_i, for i sufficiently large, to balls of any given radius are isomorphic. For example,

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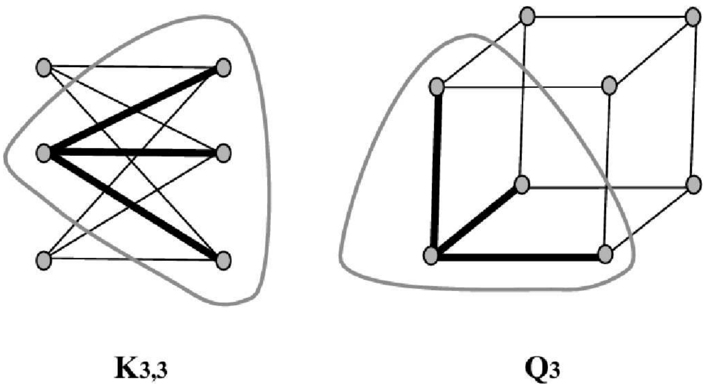

FIGURE 1. $K_{3,3}$ and Q_3

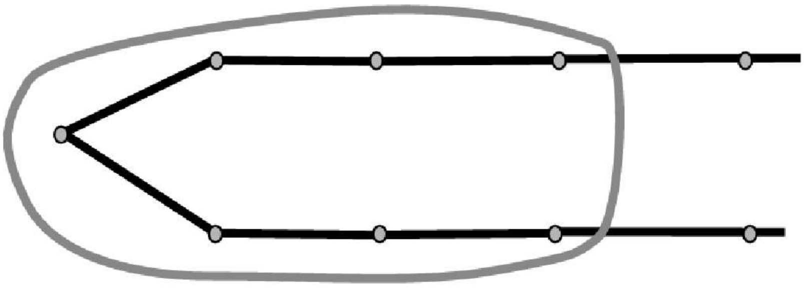

FIGURE 2. C_n for large n

for each $i \geq 3$, let C_i be the cycle of length i with the vertices labelled by the integers modulo i. Let n, m be positive integers such that $n < m$. Then for all $r \leq \lfloor \frac{n-2}{2} \rfloor$ we have $B_{C_n}(0,r) \cong B_{C_m}(0,r) \cong P_{2r}$, where P_{2r} is the path of length $2r$ but $B_{C_n}(0, \lfloor \frac{n}{2} \rfloor) = C_n \ncong B_{C_m}(0, \lfloor \frac{n}{2} \rfloor)$, see Figure 2. Thus

$$\rho(C_n, C_m) = \frac{1}{2^{\lfloor \frac{n-2}{2} \rfloor}}$$

and the sequence $(C_i)_{i \geq 3}$ converges to the infinite path.

This concept of a limit of a sequence of graphs, as well as a dependent concept of a limit of a sequence of automorphisms (see Section 2), was formulated and used in [**10**]. These concepts are related to the concepts of limits of metric spaces and groups in [**6**] and [**5**, Section 6]. They differ, for example, from the Fraïssé limit [**3**] discussed in [**1**, Section 5.6], and from the theory of homogeneous structures as expounded notably by Hrushovski [**7**].

Given a subset X of $\mathcal{G}$ we are interested in the set $\lim(X)$ of all limits x of infinite sequences whose elements lie in $X \backslash \{x\}$. The elements of $\lim(X)$ provide us with insight into the local structure of the elements of X. Studying $\lim(X)$ also allows us to deduce useful information about the graphs in X, see for example

[**10**]. We are particularly interested in the case where X is the set of all finite vertex-primitive graphs. Our investigation was motivated in part by the following problem posed in the Kourovka Notebook.

PROBLEM 1. [**9**, Problem 12.89] *Describe the limit graphs for the subset* $\mathcal{FP}$ *of* $(\mathcal{G}, \rho)$*, where* $\mathcal{FP}$ *consists of all finite vertex-primitive graphs.*

In Section 2 we prove some basic properties of the metric space $(\mathcal{G}, \rho)$ (which first appeared in [**10**] without detailed proofs) to help lay the foundations for future work. In particular we prove that, if $(\Gamma_i)_{i\geq 0}$ is a sequence of graphs in $\mathcal{G}$ which converges to a graph Γ, then

- Γ is vertex transitive (Corollary 2.3);
- if each Γ_i is edge-transitive then Γ is edge-transitive (Proposition 2.6);
- if each Γ_i is s-arc transitive then Γ is s-arc transitive (Proposition 2.6).

If each of the Γ_i is vertex-primitive then Γ is not necessarily vertex-primitive. For example, the sequence $(C_p)_{p \text{ prime}}$ is a sequence of vertex-primitive graphs which converges to the infinite path, which is vertex-imprimitive. However, Proposition 2.6 implies that any block of imprimitivity of $V(\Gamma)$ is infinite.

The possible structures of finite primitive groups are given by the O'Nan-Scott Theorem. It transpires that only three types play a crucial role when studying $\lim(\mathcal{FP})$. Let G be a finite primitive permutation group acting on a set Ω. If G has an abelian normal subgroup which acts regularly on Ω we say that G is of type HA. Such a group is isomorphic to a subgroup of the affine group $\mathrm{AGL}(d,p)$ which contains all translations. If G is an almost simple group, that is, there exists a finite nonabelian simple group T such that $T \leqslant G \leqslant \mathrm{Aut}(T)$, then G is said to be of type AS. If G has a minimal normal subgroup $N \cong T^k$, for some nonabelian simple group T and positive integer $k \geq 2$, such that the stabiliser in N of a point is nontrivial and has no composition factor isomorphic to T, then G is said to be of type PA. Such a group is isomorphic to a subgroup of $H \,\mathrm{wr}\, S_k$ in its product action on Δ^k, where H is a primitive group of type AS on Δ with socle T. For each primitive type X, we denote by $\mathcal{FP}_{\mathrm{X}}$ the set of all vertex-primitive graphs admitting a vertex-primitive subgroup of automorphisms of type X. Note that $\mathcal{FP}_{\mathrm{X}}$ and $\mathcal{FP}_{\mathrm{Y}}$ are not necessarily disjoint for different primitive types X and Y. For example $K_{25} \in \mathcal{FP}_{\mathrm{AS}} \cap \mathcal{FP}_{\mathrm{HA}} \cap \mathcal{FP}_{\mathrm{PA}}$ since $\mathrm{Aut}(K_{25})$ contains primitive subgroups, S_{25}, $\mathrm{AGL}(5,2)$ and $S_5 \,\mathrm{wr}\, S_2$, of each of these three types. One of the main results from [**4**, Theorems 1.1 and 1.2] demonstrates the special role of these three primitive types.

THEOREM 1.1. (1) $\lim(\mathcal{FP}) = \lim(\mathcal{FP}_{\mathrm{HA}}) \cup \lim(\mathcal{FP}_{\mathrm{AS}}) \cup \lim(\mathcal{FP}_{\mathrm{PA}})$, *and moreover,* $\lim(\mathcal{FP} \backslash (\mathcal{FP}_{\mathrm{AS}} \cup \mathcal{FP}_{\mathrm{HA}} \cup \mathcal{FP}_{\mathrm{PA}})) = \varnothing$.

(2) $\lim(\mathcal{FP}_{\mathrm{HA}})$ *is disjoint from* $\lim(\mathcal{FP}_{\mathrm{AS}}) \cup \lim(\mathcal{FP}_{\mathrm{PA}})$*, and each graph in* $\lim(\mathcal{FP}_{\mathrm{HA}})$ *is a Cayley graph of a free abelian group of finite rank.*

We give examples of graphs in $\lim(\mathcal{FP}_{\mathrm{HA}})$, $\lim(\mathcal{FP}_{\mathrm{AS}})$ and $\lim(\mathcal{FP}_{\mathrm{PA}})$ in Section 3. In Section 4 we formulate some results concerning $\lim(\mathcal{FP}_{\mathrm{AS}})$ and $\lim(\mathcal{FP}_{\mathrm{PA}})$ from [**4**], and mention several other open problems about limit graphs.

2. Basic properties of $(\mathcal{G}, \rho)$

In this section we explore many of the basic properties of the metric space $(\mathcal{G}, \rho)$. We also develop the concept of automorphisms of a limit graph which arise from automorphisms of the graphs in a convergent sequence. Many of the results in this section first appeared in [**10**] without proof, and are extensively used in [**4**], so we take this opportunity to flesh out the details.

To prove that $(\mathcal{G}, \rho)$ is complete we first study a larger metric space $(\mathcal{G}^*, \rho^*)$. Here $\mathcal{G}^*$ is the set of all pairs (Γ, x) where Γ is a connected graph of finite valency and x is a distinguished vertex of Γ. The metric ρ^* is defined by

$$\rho^*((\Gamma_1, x), (\Gamma_2, y)) = \begin{cases} 0 & \text{if there is an isomorphism } \varphi : \Gamma_1 \to \Gamma_2 \\ & \text{such that } \varphi(x) = y \\ \dfrac{1}{2^i} & \text{if no such } \varphi \text{ exists and } i \text{ is the largest integer for which } B_{\Gamma_1}(x, i) \cong B_{\Gamma_2}(y, i) \end{cases}$$

If in $(\mathcal{G}^*, \rho^*)$ the sequence $((\Gamma_i, x_i))_{i \geq 0}$ converges to (Γ, x), then for all $i \geq 0$ there exists $\varphi_i : V(\Gamma) \to V(\Gamma_i)$ such that $\varphi_i(x) = x_i$ and, if $r \geq 0$, then for all sufficiently large i the restriction of φ_i to $B_\Gamma(x, r)$ induces an isomorphism with $B_{\Gamma_i}(x_i, r)$. We say that $((\Gamma_i, x_i))_{i \geq 0}$ is $(\varphi_i)_{i \geq 0}$-*convergent* to (Γ, x).

Proposition 2.1. *Let d be a positive integer. Then every infinite sequence $((\Gamma_i, x_i))_{i \geq 0}$ in $(\mathcal{G}^*, \rho^*)$, such that each Γ_i has valency at most d, has a convergent subsequence.*

Proof. The result (as well as many other results in this section) is proved by standard compactness arguments. First we construct an infinite increasing sequence $(i_j)_{j \geq 0}$ such that, for each non-negative integer r and each $k \geq r$, we have $B_{\Gamma_{i_k}}(x_{i_k}, r) \cong B_{\Gamma_{i_r}}(x_{i_r}, r)$. Let $i_0 = 0$ and note that for all (Γ_i, x_i), we have $B_{\Gamma_i}(x_i, 0) \cong B_{\Gamma_0}(x_0, 0)$. Next suppose that, for some $s \geq 0$, we have chosen the first $s+1$ terms in our sequence and the following property holds: there is an infinite set Λ_s of indices such that for all $i \in \Lambda_s$, $B_{\Gamma_i}(x_i, s) \cong B_{\Gamma_s}(x_s, s)$. (Note that this condition holds for $s = 0$.) As $|B_{\Gamma_i}(x_i, s+1)| \leq 1 + d + \cdots + d^{s+1}$ for each graph Γ_i of valency at most d, there are only finitely many possibilities for the induced subgraph $B_{\Gamma_i}(x_i, s+1)$. It follows that there is an infinite subset $\Lambda'_s \subseteq \Lambda_s$ such that for all $i, i' \in \Lambda'_s$, $B_{\Gamma_i}(x_i, s+1) \cong B_{\Gamma_{i'}}(x_{i'}, s+1)$. We choose i_{s+1} to be the smallest integer greater than i_s, with $i_{s+1} \in \Lambda'_s$. Applying this process indefinitely constructs the required sequence $(i_j)_{j \geq 0}$.

Let $L_0 = \{x_{i_0}\}$ and, for each $r > 0$, let $L_r = B_{\Gamma_{i_r}}(x_{i_r}, r) \backslash B_{\Gamma_{i_r}}(x_{i_r}, r-1)$. We note that for all $k \geq r$ we have $B_{\Gamma_{i_k}}(x_{i_k}, r) \cong B_{\Gamma_{i_r}}(x_{i_r}, r)$. For each $r > 0$, let $\psi_r : V(\Gamma_{i_{r-1}}) \to V(\Gamma_{i_r})$ such that the restriction to $B_{\Gamma_{i_{r-1}}}(x_{i_{r-1}}, r-1)$ induces a graph isomorphism onto $B_{\Gamma_{i_r}}(x_{i_r}, r-1)$ that maps $x_{i_{r-1}}$ to x_{i_r}. Define E_r to consist of all pairs $\{y, z\}$ such that $y \in L_r$, $z \in L_{r'}$ for some $r' = r-1$ or r, and the edge set $E(\Gamma_{i_r})$ contains $\{y, z'\}$ where

$$z' = \begin{cases} z & \text{if } r' = r \\ \psi_r(z) & \text{if } r' = r-1 \end{cases}$$

Then let Γ be the graph with vertex set $\bigcup L_r$ and edge set $\bigcup E_r$ and let $x = x_{i_0}$. For each $r \geq 0$ define $\varphi_r : V(\Gamma) \to V(\Gamma_{i_r})$ to be any map such that the restriction of φ_r to L_r is the identity map and, the restriction of φ_r to L_{r-1} (which is a subset of $V(\Gamma_{r-1})$) is equal to the restriction of ψ_r to L_{r-1}. Then the restriction of φ_r to $B_\Gamma(x, r)$ is an isomorphism onto $B_{\Gamma_{i_r}}(x_{i_r}, r)$ and $((\Gamma_{i_j}, x_{i_j}))_{i_j \geq 0}$ is $(\varphi_{i_j})_{j \geq 0}$-convergent to (Γ, x). □

We wish to show that if each graph in a convergent sequence $((\Gamma_i, x_i))_{i\geq 0}$ is vertex-transitive then so is the limit. To do this we need to first introduce the concept of a limit automorphism. Suppose that $((\Gamma_i, x_i))_{i\geq 0}$ is $(\varphi_i)_{i\geq 0}$-convergent to (Γ, x). A sequence $(h_i)_{i\geq 0}$ of automorphisms $h_i \in \mathrm{Aut}(\Gamma_i)$ is said to $(\varphi_i)_{i\geq 0}$-converge to $h \in \mathrm{Aut}(\Gamma)$ if for all $y \in V(\Gamma)$ we have $\varphi_i h(y) = h_i\varphi_i(y)$ for all sufficiently large i. We call h a *limit automorphism*, or more exactly, a $(\varphi_i)_{i\geq 0}$-*limit automorphism.* If $G_i \leqslant \mathrm{Aut}(\Gamma_i)$ for each i, then the subgroup of $\mathrm{Aut}(\Gamma)$ generated by all limit automorphisms obtained from sequences $(h_i)_{i\geq 0}$ with $h_i \in G_i$, is called the *limit group* with respect to $(G_i)_{i\geq 0}$. We have the following proposition.

Proposition 2.2. *Let $((\Gamma_i, x_i))_{i\geq 0}$ be a sequence in $(\mathcal{G}^*, \rho^*)$ that $(\varphi_i)_{i\geq 0}$-converges to (Γ, x) and let r be a positive integer. If for each $i \geq 0$, $g_i \in \mathrm{Aut}(\Gamma_i)$ is such that $d_{\Gamma_i}(x_i, g_i(x_i)) \leq r$ then there exists an increasing sequence $(i_j)_{j\geq 0}$ such that $(g_{i_j})_{j\geq 0}$ is $(\varphi_{i_j})_{j\geq 0}$-convergent to an automorphism of Γ.*

Proof. By passing to a subsequence if necessary, we may suppose that for each non-negative integer d and each $k \geq d$, the map φ_k induces an isomorphism from $B_\Gamma(x, d)$ to $B_{\Gamma_k}(x_k, d)$. We now construct a subsequence $(i_j)_{j\geq 0}$ and an automorphism g of Γ such that, for each $d \geq 0$,

$$\varphi_{i_j}(g(y)) = g_{i_j}(\varphi_{i_j}(y)) \text{ for all } y \in B_\Gamma(x, d) \text{ and for all } j \geq d + r. \tag{2.1}$$

Suppose that for some $d \geq 0$, we already have a subsequence $(i_j)_{j\geq 0}$ and a map $g : B_\Gamma(x, d-1) \to V\Gamma$ which satisfy (2.1) for all $d' < d$. (In the case $d = 0$, $g(y)$ is not defined for any vertex $y \in V(\Gamma)$ and (2.1) holds vacuously for $d' < d$.) Note that, since (2.1) holds for $d - 1$, we have that, for all $j \geq d - 1 + r$, the restriction of $\varphi_{i_j}^{-1} \circ g_{i_j} \circ \varphi_{i_j}$ to $B_\Gamma(x, d-1)$ is the map g. Now let $B = B_\Gamma(x, d)$ and replace the sequence $((\Gamma_i, x_i))_{i\geq 0}$ by the subsequence $((\Gamma_{i_j}, x_{i_j}))_{j\geq 0}$. Then we have that, for all $i \geq d - 1 + r$, the restriction of $\varphi_i^{-1} \circ g_i \circ \varphi_i$ to $B_\Gamma(x, d-1)$ is the map g. Then for all $i \geq d + r$, φ_i induces an isomorphism from $B_\Gamma(x, d+r)$ to $B_{\Gamma_i}(x_i, d+r)$ and in particular $\varphi_i(B) = B_{\Gamma_i}(x_i, d)$. Then as $d_{\Gamma_i}(x_i, g(x_i)) \leq r$ we have that $g_i(\varphi_i(B)) \subseteq B_{\Gamma_i}(x_i, d+r)$. Thus $\varphi_i^{-1}(g_i(\varphi_i(B))) \subseteq B_\Gamma(x, d+r)$. As the number of possible maps from B to $B_\Gamma(x, d+r)$ is finite, there is an infinite subsequence $(i_j)_{j\geq d+r}$ such that the restrictions to B of $\varphi_{i_j}^{-1} \circ g_{i_j} \circ \varphi_{i_j}$ are the same for all $j \geq d + r$. For $y \in B$, we define $g(y) := \varphi_{i_{d+r}}^{-1}(g_{i_{d+r}}(\varphi_{i_{d+r}}(y)))$. This definition is compatible with the previous definition of g on $B_\Gamma(x, d-1)$, and for all $j \geq d + r$ we have $\varphi_{i_j}(g(y)) = g_{i_j}(\varphi_{i_j}(y))$ for all $y \in B_\Gamma(x, d)$. Thus $(g_{i_j})_{j\geq 0}$ is $(\varphi_{i_j})_{j\geq 0}$-convergent to g. □

Corollary 2.3. *If $((\Gamma_i, x_i))_{i\geq 0}$ is an infinite sequence in $(\mathcal{G}^*, \rho^*)$ which converges to (Γ, x) and each Γ_i is vertex-transitive, then Γ is vertex-transitive.*

Proof. Let $y \in V(\Gamma)$ and $r = d_\Gamma(x, y)$. Suppose that $((\Gamma_i, x_i))_{i\geq 0}$ is $(\varphi_i)_{i\geq 0}$-convergent to (Γ, x). Let $y_i = \varphi_i(y)$. Now for all sufficiently large i, φ_i induces an isomorphism from $B_\Gamma(x, r+1)$ to $B_{\Gamma_i}(x_i, r+1)$ and $\varphi_i(x) = x_i$, so $d_{\Gamma_i}(x_i, y_i) = r$. As each Γ_i is vertex-transitive, there exists an automorphism g_i of Γ_i mapping x_i to y_i. Then $d_{\Gamma_i}(x_i, g_i(x_i)) \leq r$ and so by Proposition 2.2, there exists a subsequence $(i_j)_{j\geq 0}$ such that $(g_{i_j})_{j\geq 0}$ is $(\varphi_{i_j})_{j\geq 0}$-convergent to an automorphism g of $\mathrm{Aut}(\Gamma)$. By the definition of a limit automorphism, for sufficiently large i, $\varphi_i(g(x)) = g_i(\varphi_i(x))$, which in turn is equal to $g_i(x_i) = y_i = \varphi_i(y)$, and hence $g(x) = y$. Thus Γ is vertex-transitive. □

Recall that $\mathcal{G}$ is the set of all connected vertex-transitive graphs of finite valency and forms a metric space when equipped with ρ. We note that for each $\Gamma_1, \Gamma_2 \in \mathcal{G}$ we have $\rho(\Gamma_1, \Gamma_2) = \rho^*((\Gamma_1, x), (\Gamma_2, y))$ for any choice of $x \in V(\Gamma_1)$ and $y \in V(\Gamma_2)$. We can write $\mathcal{G} = \dot{\bigcup}_{d\geq 0} \mathcal{G}_d$ where each $\mathcal{G}_d$ is the set of all connected vertex-transitive graphs of valency d. The next result uses some standard topological concepts that we briefly summarise. A metric space is *compact* if every infinite sequence has a convergent subsequence and is *locally compact* if every point has a compact neighbourhood. Hence Proposition 2.1 and Corollary 2.3 imply that each $\mathcal{G}_d$ is compact and $\mathcal{G}$ is locally compact. A sequence $(\Gamma_i)_{i\geq 0}$ is a *Cauchy sequence* if for all $\epsilon > 0$ there is a positive integer N such that $\rho(\Gamma_i, \Gamma_j) < \epsilon$ for all $i, j \geq N$. A metric space is *complete* if every Cauchy sequence converges. A subset of a metric space is *connected* if it cannot be expressed as a union of two disjoint, nonempty subsets that are open in its induced topology. Also a metric space is *totally disconnected* if for all points x, the only connected subset containing x is singleton set $\{x\}$.

THEOREM 2.4. *$(\mathcal{G}, \rho)$ is a locally compact, complete and totally disconnected metric space. Moreover, $\mathcal{G}$ has the same cardinality as the continuum.*

PROOF. Let $(\Gamma_i)_{i\geq 0}$ be an infinite Cauchy sequence in $(\mathcal{G}, \rho)$. Then for $0 < \epsilon < 1$, there is a positive integer N such that $\rho(\Gamma_i, \Gamma_j) < \epsilon$ for all $i, j \geq N$. In particular, there exists d such that for all $i \geq N$ we have $\Gamma_i \in \mathcal{G}_d$. As $\mathcal{G}_d$ is compact, it is also complete and so $(\Gamma_i)_{i\geq 0}$ converges. Thus $(\mathcal{G}, \rho)$ is complete. As the metric ρ is rational valued it follows that $\mathcal{G}$ is totally disconnected.

The set of all graphs with countable vertex sets corresponds to the set of symmetric countably infinite $\{0, 1\}$-matrices, and so has the same cardinality as the continuum. Each connected graph with finite valency has a countable (that is, finite or countably infinite) vertex set and so the cardinality of $\mathcal{G}$ is at most the cardinality of the continuum. Since there is a set of pairwise nonisomorphic Cayley graphs of finitely generated groups, which has the same cardinality as the continuum [**5**, Theorem B], it follows that $\mathcal{G}$ also has the same cardinality as the continuum. □

We now prove the following.

PROPOSITION 2.5. *Let $((\Gamma_i, x_i))_{i\geq 0}$ be a sequence in $\mathcal{G}^*$ that $(\varphi_i)_{i\geq 0}$-converges to (Γ, x). For each $i \geq 0$, let $G_i \leqslant \mathrm{Aut}(\Gamma_i)$, and let G be the limit group with respect to $(G_i)_{i\geq 0}$. Then, for any non-negative integers r_1, r_2, there exists a non-negative integer $i(r_1, r_2)$ such that, for any $i \geq i(r_1, r_2)$ and any $g' \in G_i$ with $d_{\Gamma_i}(x_i, g'(x_i)) \leq r_1$, there exists $g \in G$ such that $\varphi_i(g(y)) = g'(\varphi_i(y))$ for all $y \in B_\Gamma(x, r_2)$.*

PROOF. Let $r_1, r_2 \geq 0$ for which the proposition does not hold. Then there exist an infinite sequence $(n_i)_{i\geq 0}$ and elements $g_{n_i} \in G_{n_i}$ with $d_{\Gamma_{n_i}}(x_{n_i}, g_{n_i}(x_{n_i})) \leq r_1$ such that there is no $g \in G$ with $\varphi_{n_i}(g(y)) = g_{n_i}(\varphi_{n_i}(y))$ for all $y \in B_\Gamma(x, r_2)$. Then $(g_{n_i})_{i\geq 0}$ is an infinite sequence such that $d_{\Gamma_i}(x_{n_i}, g_{n_i}(x_{n_i})) \leq r_1$ and so by Proposition 2.2 there exists an infinite subsequence $(i_j)_{j\geq 0}$ such that $(g_{n_{i_j}})_{j\geq 0}$ is $(\varphi_{n_{i_j}})_{j\geq 0}$-convergent to an automorphism g of Γ. By definition, $g \in G$, and for all $y \in V(\Gamma)$ we have that $\varphi_{n_{i_j}}(g(y)) = g_{n_{i_j}}(\varphi_{n_{i_j}}(y))$ for all $j \geq j(y)$. Let J be the maximum of $j(y)$ over all $y \in B_\Gamma(x, r_2)$. Then, for all $j \geq J$ and all $y \in B_\Gamma(x, r_2)$, $\varphi_{n_{i_j}}(g(y)) = g_{n_{i_j}}(\varphi_{n_{i_j}}(y))$, which is a contradiction. Thus the proposition holds for all $r_1, r_2 \geq 0$. □

Proposition 2.5 allows us to prove the following statements about the action of a limit group. An *s-arc* in a graph is an $(s+1)$-tuple $(v_0, v_1, \dots, v_s)$ of vertices such any two consecutive vertices are adjacent while $v_i \neq v_{i+2}$, for each $i = 0, \dots, s-2$. For $G \leqslant \mathrm{Aut}(\Gamma)$, we say that G is *s-arc transitive* if G acts transitively on the set of s-arcs in Γ.

Proposition 2.6. *Let $((\Gamma_i, x_i))_{i\geq 0}$ be a sequence in $\mathcal{G}^*$ that $(\varphi_i)_{i\geq 0}$-converges to (Γ, x), and let r be a non-negative integer. For each $i \geq 0$ let G_i be a vertex-transitive subgroup of* $\mathrm{Aut}(\Gamma_i)$ *and let G be the limit group with respect to $(G_i)_{i\geq 0}$. Then:*

1. *If X is a finite block of imprimitivity of G in $V(\Gamma)$, then, for each sufficiently large i, $\varphi_i(X)$ is a block of imprimitivity of G_i in $V(\Gamma_i)$ of size $|X|$.*
2. *For all sufficiently large i, the restriction of φ_i to $B_\Gamma(x, r)$ induces a graph isomorphism from $B_\Gamma(x, r)$ to $B_{\Gamma_i}(x_i, r)$ and a permutation isomorphism from the group induced by $(G_i)_{x_i}$ on $B_{\Gamma_i}(x_i, r)$ to a subgroup of the group induced by G_x on $B_\Gamma(x, r)$.*
3. *If, for all sufficiently large i, the group G_i is s-arc-transitive, where $s \geq 0$, then G is s-arc-transitive. If, for all sufficiently large i, the group G_i is edge-transitive, then G is edge-transitive.*

Proof. (1) By Corollary 2.3, G is vertex-transitive on $V(\Gamma)$ so we may assume that $x \in X$. Let r_2 be the smallest integer such that $X \subseteq B_\Gamma(x, r_2)$. Such an integer exists because X is finite and Γ is connected. Let $i(4r_2, r_2)$ be as defined in Proposition 2.5 with the added condition that for all $i \geq i(4r_2, r_2)$, φ_i defines an isomorphism from $B_\Gamma(x, r_2)$ to $B_{\Gamma_i}(x_i, r_2)$. Let $i \geq i(4r_2, r_2)$. Then $|\varphi_i(X)| = |X|$. Suppose that $g' \in G_i$ is such that $g'(\varphi_i(X)) \cap \varphi_i(X) \neq \varnothing$. Then there exists $y_i \in \varphi_i(X)$ such that $g'(y_i) \in \varphi_i(X)$ and as φ_i is a graph isomorphism from $B_\Gamma(x, r_2)$ to $B_{\Gamma_i}(x_i, r_2)$, it follows that $d_{\Gamma_i}(y_i, g'(y_i)) \leq 2r_2$. Also $d_{\Gamma_i}(x_i, y_i) \leq r_2$ and so $d_{\Gamma_i}(x_i, g'(x_i)) \leq d_{\Gamma_i}(x_i, y_i) + d_{\Gamma_i}(y_i, g'(y_i)) + d_{\Gamma_i}(g'(y_i), g'(x_i)) \leq 4r_2$. Thus by Proposition 2.5 there exists $g \in G$ such that $\varphi_i(g(y)) = g'(\varphi_i(y))$ for all $y \in B_\Gamma(x, r_2)$. Thus $\varphi_i(g(X)) \cap \varphi_i(X) = \varphi_i(g(X) \cap X) \neq \varnothing$ and hence $g(X) \cap X \neq \varnothing$. As X is a block of imprimitivity of G on $V(\Gamma)$, it follows that $g(X) = X$ and hence $\varphi_i(X) = \varphi_i(g(X)) = g'(\varphi_i(X))$. Hence $\varphi_i(X)$ is a block of imprimitivity for G_i.

(2) The first part follows from the definition of $(\varphi_i)_{i\geq 0}$-convergence and the second part follows from Proposition 2.5.

(3) Suppose first that each G_i is s-arc transitive. From part (2), for sufficiently large i, the group induced by $(G_i)_{x_i}$ on $B_{\Gamma_i}(x_i, s)$ is permutationally isomorphic to the group induced by G_x on $B_\Gamma(x, s)$. Hence G_x is transitive on the set of s-arcs starting at x. Furthermore, since each G_i acts transitively on $V(\Gamma_i)$, Corollary 2.3 implies that G acts transitively on $V(\Gamma)$ and so G is s-arc transitive. If each G_i is edge-transitive on Γ_i then Proposition 2.5 implies that G is edge-transitive on Γ. □

The Sims Conjecture states that there is a function f on the natural numbers such that, for all finite primitive permutation groups G, if the point stabiliser G_x has an orbit of length $d > 1$ then $|G_x| \leq f(d)$. This was proved in [**2**] and the proof required the Classification of Finite Simple Groups. The Sims Conjecture has the following important consequence in our context.

Proposition 2.7. *Let* $((\Gamma_i, x_i))_{i\geq 0}$ *be a sequence in* $\mathcal{G}^*$*, with each* Γ_i *finite, that* $(\varphi_i)_{i\geq 0}$*-converges to* (Γ, x)*. Suppose that, for each* $i \geq 0$*,* G_i *is a vertex-primitive subgroup of* $\mathrm{Aut}(\Gamma_i)$*. Then there exists an infinite subsequence* $(i_j)_{j\geq 0}$ *and a finite group* H *such that* $(G_{i_j})_{x_{i_j}} \cong H$ *for all* $j \geq 0$*.*

Proof. For i sufficiently large $B_{\Gamma_i}(x_i, 1) \cong B_\Gamma(x, 1)$, so we may assume that all the Γ_i have the same finite valency d. Thus by the Sims Conjecture, $|(G_i)_{x_i}|$ is bounded above by $f(d)$ and so there exists an infinite subsequence $(i_j)_{j\geq 0}$ such that $|(G_{i_j})_{x_{i_j}}|$ is constant. Then as there are only finitely many groups of a given order, passing to a subsequence if necessary, there exists a finite group H such that $(G_{i_j})_{x_{i_j}} \cong H$ for all $j \geq 0$. □

Proposition 2.7 implies that to find limits of convergent sequences of finite vertex-primitive graphs it is sufficient to find limits of convergent sequences of finite graphs admitting vertex-primitive groups of automorphisms with isomorphic vertex stabilisers. In particular, we need to find sequences $(G_i)_{i\geq 0}$ of primitive groups such that $|G_i|$ tends to ∞ while $|(G_i)_{x_i}|$ remains constant. This is used extensively in [**4**] to study the families of primitive groups which can occur as the automorphism groups of graphs in a convergent sequence.

A stronger form of the Sims Conjecture was proved in [**8**], where it was shown that for each finite connected graph Γ with a primitive automorphism group G, given $x \in V(\Gamma)$ the vertex stabiliser G_x acts faithfully on the ball of radius six centred at x.

3. Examples

In this section we give several examples of convergent sequences of vertex-primitive graphs and the graphs which occur as limits. These examples show that each of $\lim(\mathcal{FP}_{\mathrm{HA}})$, $\lim(\mathcal{FP}_{\mathrm{AS}})$ and $\lim(\mathcal{FP}_{\mathrm{PA}})$ contains infinitely many graphs. For a group G and a subset X closed under inverses, recall that $\mathrm{Cay}(G, X)$ is the graph with vertex set G and two elements $x, y \in G$ are adjacent if and only if $xy^{-1} \in X$.

Example 3.1. *(Grids)* Let k be a positive integer, and let p be an odd prime. Let $V = \mathbb{Z}^k$, $V_p = \mathbb{Z}_p^k$, and $\psi_p : V \to V_p$ be the natural map that replaces each entry of $x = (x_1, \ldots, x_k) \in \mathbb{Z}^k$ by its value modulo p. Define

$$\begin{aligned} X = \{&(1,0,\ldots,0), (-1,0,\ldots,0),\\ &(0,1,0,\ldots,0), (0,-1,0,\ldots,0),\\ &\vdots\\ &(0,\ldots,0,1), (0,\ldots,0,-1)\}, \end{aligned}$$

a subset of V of size $2k$, and let $X_p = \psi_p(X)$, so $|X_p| = 2k$ also. Then the graph $\Gamma = \mathrm{Cay}(V, X)$ is the *k-dimensional grid*, of valency $2k$, and each graph $\Gamma_p = \mathrm{Cay}(V_p, X_p)$ also has valency $2k$. Let $x = (0, \ldots, 0)$ denote the zero vector in either V or V_p (where the latter is really $\psi_p(x)$). If H_p is the group of all $k \times k$ matrices over $\mathbb{Z}_p$ which have precisely one nonzero element in each row and column and this nonzero element is either 1 or -1, then $H_p \cong C_2 \,\mathrm{wr}\, S_k$ and $G_p := V_p \rtimes H_p \leqslant \mathrm{Aut}(\Gamma_p)$. The only proper, non-zero subspaces of V_p fixed by the group of permutation matrices are $\langle(1, \ldots, 1)\rangle$ and $\{(v_1, v_2, \ldots, v_k) : \sum_{i=1}^k v_i = 0\}$.

Neither of these subspaces is fixed by H_p and so H_p acts irreducibly on V_p. Thus G_p acts primitively on the vertex set of Γ_p and is of type HA. Moreover, the stabiliser in G_p of the vertex $(0,\dots,0)$ is H_p, and H_p is transitive on $\Gamma_p(x) = X_p$. Thus G_p is an edge-transitive subgroup of $\mathrm{Aut}(\Gamma_p)$.

Provided $r < (p-1)/2$, the induced subgraph $B_{\Gamma_p}(x,r)$ is isomorphic to the subgraph $B_\Gamma(x,r)$ of the k-dimensional grid Γ. Thus, for any increasing sequence $(p_i)_{i\geq 0}$ of odd primes, the sequence $(\Gamma_{p_i})_{i\geq 0}$ of graphs from $\mathcal{FP}_{\mathrm{HA}}$ converges to Γ and hence, for any $k \geq 1$, the k-dimensional grid lies in $\lim(\mathcal{FP}_{\mathrm{HA}})$.

Example 3.2. *(Trivalent tree)* For each prime $p \equiv \pm 1 \pmod{16}$ let $G_p = \mathrm{PSL}(2,p)$. Wong [**12**] constructed a graph Γ_p of valency 3 that admits a 4-arc transitive action of G_p, so that, for $x_p \in V(\Gamma_p)$, the stabiliser $(G_p)_{x_p} \cong S_4$. Let g_p be the girth of Γ_p, let Γ be the infinite trivalent tree and let $x \in V(\Gamma)$. A. Weiss [**11**] showed that the girths g_p are unbounded. Let $(p_i)_{i\geq 0}$ be an infinite increasing sequence of such primes. Then for each i we can define a map $\varphi_i : V(\Gamma) \to V(\Gamma_{p_i})$ which maps x to x_{p_i} such that the sequence $((\Gamma_{p_i}, x_{p_i}))_{i\geq 0}$ in $\mathcal{FP}_{\mathrm{AS}}$, is $(\varphi_i)_{i\geq 0}$-convergent to (Γ, x).

We now give an example where the limit graph is not a tree. First we need to introduce the concept of a coset graph. Let G be a group with a core-free subgroup H. Let $g \in G$ such that g does not normalise H and $g^{-1} \in HgH$. Then we define the *coset graph* $\Gamma = \mathrm{Cos}(G, H, HgH)$ to be the graph with vertex set $[G : H]$ and two cosets Hx and Hy are adjacent if and only if $xy^{-1} \in HgH$. Then G acts on $V\Gamma$ by right multiplication and $G \leqslant \mathrm{Aut}(\Gamma)$. Furthermore, Γ is connected if and only if $\langle H, g\rangle = G$ and the valency of Γ is $|H : H \cap H^g|$.

Example 3.3. For each prime $p \equiv \pm 1 \pmod{50}$ let $G_p = \mathrm{PSL}(2,p)$. Then G_p has a maximal subgroup $H_p \cong A_5$ and contains an element g_p of order 25 which lies in a maximal subgroup of G_p isomorphic to $D_{p\pm 1}$ such that $g_p^5 \in H_p$. Since $g_p \in D_{p\pm 1}$ and H_p contains a subgroup of $D_{p\pm 1}$ isomorphic to D_{10}, there exists an element $h_p \in H_p$ such that $h_p^{-1} g_p h_p = g_p^{-1}$. Thus $g_p^{-1} \in H_p g_p H_p$. Let $\Gamma_p = \mathrm{Cos}(G_p, H_p, H_p g_p H_p)$. Since H_p is a maximal subgroup of G_p, we have $\langle H_p, g_p\rangle = G_p$ and so Γ_p is connected. Now G_p is vertex-primitive and arc-transitive on Γ_p (acting by right multiplication). Furthermore, since $H_p^{g_p} \cap H_p \cong C_5$ we have that Γ_p has valency 12. Now $H_p, H_p g_p, H_p g_p^2, H_p g_p^3, H_p g_p^4, H_p g_p^5 = H_p$ is a cycle of length 5 in Γ_p, so Γ_p has girth at most 5. Let $x_p = H_p \in V(\Gamma_p)$, and let $(p_i)_{i\geq 0}$ be an infinite increasing sequence of primes $p_i \equiv \pm 1 \pmod{50}$. Then by Proposition 2.1, the sequence $((\Gamma_{p_i}, x_{p_i}))_{i\geq 0}$ has a convergent subsequence. Furthermore, the limit of this convergent subsequence has girth at most 5.

Example 3.4. Let $n > 1$, p be a prime, $H = S_n$ and V be an n-dimensional vector space over $\mathrm{GF}(p)$ with basis $\{e_1, e_2, \dots, e_n\}$ such that H permutes the basis. Assume that $p > n$. Let Q be the nondegenerate quadratic form given by $Q(\sum \lambda_i e_i) = \sum \lambda_i^2$. Then H preserves Q and fixes the nondegenerate subspace $W = \{\sum \lambda_i e_i : \sum \lambda_i = 0\}$ setwise while centralising $W^\perp = \langle e_1 + e_2 + \cdots + e_n\rangle$. Hence H is isomorphic to a subgroup $H_p \leqslant G_p = O(n-1,p)$, the $(n-1)$-dimensional orthogonal group on W. (It is not necessary for us to determine the type of the form.) Let $v = e_1 + e_2 + \cdots + e_{n-1} + (p-n+1)e_n \in W$. Then as $p > n$ we have $Q(v) = n(n-1) \neq 0$ and $(H_p)_v \leqslant (G_p)_{\langle v\rangle} = O(n-2,p) \times O(1,p)$. Moreover, $(H_p)_v$ is equal to the subgroup S_{n-1} fixing the vector e_n. As $O(1,p) \cong C_2$ we can choose $g_p \in O(1,p)$ of order two such that $H_p \cap (H_p)^{g_p} = (H_p)_v$ and $|H_p : H_p \cap (H_p)^{g_p}| = n$.

Consider a sequence of primes $(p_i)_{i\geq 0}$ with $p_i \to \infty$, the corresponding sequence of groups $(G_{p_i})_{i\geq 0}$ and subgroups $H_{p_i} \cong H$, and the sequence of coset graphs $\Gamma_i = \mathrm{Cos}(G_{p_i}, H_{p_i}, H_{p_i} g_{p_i} H_{p_i})$. Each Γ_i has valency $|H_{p_i} : H_{p_i} \cap (H_{p_i})^{g_{p_i}}| = n$ and admits G_{p_i} as a group of automorphisms. Since H_{p_i} is a maximal subgroup of G_{p_i} we have that G_{p_i} is a vertex-primitive group. By Proposition 2.1, there is a subsequence that converges to a graph $\Gamma \in \lim(\mathcal{FP}_{\mathrm{AS}})$ of valency n.

Example 3.5. Let $(\Delta_i)_{i\geq 0}$ be an infinite sequence of graphs in $\mathcal{FP}_{\mathrm{AS}}$ which converge to Δ. For each i, let $\Delta_i^{\square k}$ be the k^{th} Cartesian power of Δ_i, that is, the graph with vertex set $(V(\Delta_i))^k$ and edges all pairs $\{(x_1, x_2, \ldots, x_k), (y_1, y_2, \ldots, y_k)\}$ such that there exists j with $1 \leq j \leq k$, for which $x_n = y_n$ for all $n \neq j$ and $\{x_j, y_j\}$ is an edge in Δ_i. Note that if $G_i \leqslant \mathrm{Aut}(\Delta_i)$ is primitive of type AS, then $G_i \,\mathrm{wr}\, S_k \leqslant \mathrm{Aut}(\Delta_i^{\square k})$ and is primitive of type PA. Furthermore, $(\Delta_i^{\square k})_{i\geq 0}$ converges to $\Delta^{\square k}$.

4. Further work

We saw in Theorem 1.1 that each graph in $\lim(\mathcal{FP}_{\mathrm{HA}})$ is a Cayley graph of a free abelian group of finite rank. A natural problem to investigate is then:

Problem 2. *Determine which Cayley graphs of free abelian groups of finite rank lie in* $\lim(\mathcal{FP}_{\mathrm{HA}})$.

As yet we have no similarly general understanding of the graphs in $\lim(\mathcal{FP}_{\mathrm{AS}}) \cup \lim(\mathcal{FP}_{\mathrm{PA}})$.

Problem 3. *Give a useful description of the limit graphs in* $\lim(\mathcal{FP}_{\mathrm{AS}})$ *and* $\lim(\mathcal{FP}_{\mathrm{PA}})$. *In particular, determine if* $\lim(\mathcal{FP}_{\mathrm{AS}})$ *and* $\lim(\mathcal{FP}_{\mathrm{PA}})$ *are disjoint.*

An important subset of $\mathcal{FP}$ is the set $\mathcal{FP}^{\mathrm{e-trans}}$ of all edge-transitive vertex-primitive graphs. By Proposition 2.6, each graph in $\lim(\mathcal{FP}^{\mathrm{e-trans}})$ is edge-transitive and as discussed in [**4**], the graphs in $\lim(\mathcal{FP}^{\mathrm{e-trans}})$ in some sense can be used to construct all the graphs in $\lim(\mathcal{FP})$. Thus we have the following natural problem.

Problem 4. *Study the graphs in* $\lim(\mathcal{FP}^{\mathrm{e-trans}})$.

An important subclass $\mathcal{FP}^{min}$ of finite edge-transitive graphs comprises those of minimal valency, that is, those G-vertex-transitive graphs Γ whose valency is minimal amongst all connected graphs Σ with $V(\Sigma) = V(\Gamma)$ and $G \leqslant \mathrm{Aut}(\Sigma)$. In fact each of the limit graphs constructed in Example 3.4 lies in $\lim(\mathcal{FP}^{min}_{AS})$. Thus $\lim(\mathcal{FP}^{min}_{AS})$ is at least countably infinite. We ask more.

Problem 5. *Is* $\lim(\mathcal{FP}^{min}_{AS})$ *countable?*

Finally, it was shown in [**4**, Theorem 6.6] that a graph lies in $\lim(\mathcal{FP}^{min}_{PA})$ if and only if it is a Cartesian power of some graph in $\lim(\mathcal{FP}^{min}_{AS})$. This raises the following question.

Problem 6. *Is there a description of the graphs in* $\lim(\mathcal{FP}^{\mathrm{e-trans}}_{PA})$ *in terms of the graphs in* $\lim(\mathcal{FP}^{\mathrm{e-trans}}_{AS})$*?*

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