

# Factorisations of sporadic simple groups

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## Abstract

We determine all factorisations  $G = AB$  where the socle of  $G$  is a sporadic simple group.

*Key words:* group factorisations, sporadic simple groups

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## 1 Introduction

A group  $G$  is said to have a *factorisation* if there exist proper subgroups  $A$  and  $B$  such that  $G = AB = \{ab \mid a \in A, b \in B\}$ . We call the factorisation *exact* if  $A \cap B = 1$ . Group factorisations play an important role in permutation group theory, for if  $G$  is a transitive permutation group on  $\Omega$ , then a subgroup  $A$  of  $G$  is transitive if and only if  $G = AG_\alpha$ , where  $G_\alpha$  is the stabiliser in  $G$  of  $\alpha \in \Omega$ . Moreover,  $A$  is regular if and only if the factorisation is exact.

We say that a group  $G$  is *almost simple* if there exists a finite nonabelian simple group  $T$  such that  $T \leq G \leq \text{Aut}(T)$ . This is an important class of permutation groups as often the O’Nan-Scott Theorems for primitive and quasiprimitive groups (see for example [10]) enable many problems to be reduced to the almost simple case. All factorisations  $G = AB$  where  $G$  is an almost simple group and both  $A$  and  $B$  are maximal subgroups of  $G$  were determined in [9]. This classification was instrumental in the classification of all maximal subgroups of the alternating and symmetric groups given in [8]. Recently, all

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exact factorisations of almost simple groups for which one factor is a maximal subgroup have been determined [7]. This led to the determination of all finite primitive groups with a regular almost simple subgroup [7, Theorem 1.4] and a classification of all vertex-primitive Cayley digraphs of nonabelian simple groups [7, Theorem 1.6]. In this paper we determine all factorisations of almost simple groups  $G$  for which  $T$  is a sporadic simple group. First we state the case where  $G = T$ .

**Theorem 1.1** *Let  $T$  be a sporadic simple group and suppose that  $T = AB$ . Then  $T, A$  and  $B$  are given by one of the rows of Tables 1, 2 or 3.*

We do not list all proper subgroups of  $M_{12}$  or  $M_{24}$  which are transitive in the usual actions on 12 and 24 points respectively. We merely state the number of conjugacy classes of such subgroups and determine the isomorphism types of the regular ones. A full list can be obtained almost instantly using the Magma [2] command `Subgroups(G:IsTransitive)`.

If  $T < G \leq \text{Aut}(T)$  where  $T$  is a sporadic simple group then  $G = T.2$  and  $G = TB$  for any subgroup  $B$  not contained in  $T$ . We disregard such factorisations as they are in some sense trivial. We also obtain factorisations of  $G$  for each factorisation  $T = AB$  such that  $A$  has an overgroup in  $G$  which is not contained in  $T$  (see Lemma 2.1(5)). Hence to determine all factorisations of  $G$  we only need to consider those for which  $T \neq (A \cap T)(B \cap T)$ .

**Theorem 1.2** *Let  $G$  be an almost simple group with  $T < G \leq \text{Aut}(T)$  and  $T$  a sporadic simple group. If  $G = AB$  but  $T \neq (A \cap T)(B \cap T)$  then  $G, A$  and  $B$  are given by one of the rows of Table 4.*

There are many interesting geometrical structures whose full automorphism group is an almost simple sporadic group. From our results we can easily read off a determination of all groups of automorphisms of these geometries which are transitive or regular on various objects, such as points, lines, flags, vertices, edges. For example, we see that the only proper subgroup of  $M_{24}$  which is transitive on the set of octads of the associated Witt design is  $\text{PSL}(2, 23)$ . The fourth column of Tables 1, 2, 3 and 4 indicates whether  $A$  or  $B$  are stabilisers of some geometric object. Our comments are not intended to be exhaustive, merely illuminating.

One consequence of Theorems 1.1 and 1.2 is that we have determined all factorisations  $G = AB$  where  $A \cap B = 1$ , that is, all exact factorisations. Exact factorisations are noted in the fourth column of Tables 1, 2, 3 and 4, and have been collected in Table 5. All exact factorisations of almost simple groups where  $A$  is maximal in  $G$  were determined in [7]. In the case where the socle of  $G$  is sporadic, sometimes only the existence of an exact factorisation was determined theoretically. In these cases the structure and numbers of conjugacy classes of factorisations were found computationally by the author,

see [7, p 3] The regular subgroups of  $\text{Aut}(J_2)$  and  $\text{Aut}(HS)$  in their actions on 100 points were determined in [5, p 107] and [6, Proposition 11]. We provide a complete list of the exact factorisations of all almost simple sporadic groups.

**Theorem 1.3** *Let  $G$  be an almost simple group with exact factorisation  $G = AB$ . Then  $G, A$  and  $B$  are given in Table 5. Moreover, if  $G \neq \text{Aut}(M_{12})$  then either  $A$  or  $B$  is maximal in  $G$ .*

Theorem 1.3 has the following corollary for Cayley graphs. Given a group  $G$  and a subset  $S$  of  $G$  such that  $1 \notin S$  and  $S = S^{-1} = \{s^{-1} \mid s \in S\}$ , the *Cayley graph*  $\text{Cay}(G, S)$  of  $G$  with respect to  $S$  is the graph with vertex set  $G$  and edges  $\{x, y\}$  such that  $xy^{-1} \in S$ . Note that the action of  $G$  on itself by right multiplication is a regular group of automorphisms. Moreover, a graph  $\Gamma$  is a Cayley graph if and only if  $\text{Aut}(\Gamma)$  contains a regular subgroup.

**Corollary 1.4** *Let  $\Gamma = \text{Cay}(X, S)$  and let  $X < G \leq \text{Aut}(\Gamma)$  such that  $G$  is an almost simple group whose socle is a sporadic simple group. Then  $(G, X, G_\alpha)$  is given by one of the rows of Table 5 where  $\{X, G_\alpha\} = \{A, B\}$ .*

Exact factorisations  $G = AB$  with  $A, B$  selfnormalising perfect subgroups of  $G$  can be used to construct biperfect semisimple Hopf algebras. This is outlined in [4], where the exact factorisation  $M_{24} = \text{PSL}(2, 23)(2^4 \rtimes A_7)$  is used to construct an explicit example. Theorem 1.3 shows that there are no other suitable factorisations with  $G$  a sporadic simple group.

We note a few important lemmas concerning factorisations in Section 2 and determine all the factorisations in Section 3. All tables appear in Section 4. Our notation follows [3], except for split extensions and some simple groups.

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## 2 Preliminaries

First we collect together some useful facts concerning factorisations.

**Lemma 2.1** *Let  $G$  be a group such that  $G = AB$  and let  $N \triangleleft G$ .*

- (1)  $|G| = |A||B|/|A \cap B|$ .
- (2) If  $A \leq N$  then  $N = A(B \cap N)$ .
- (3)  $G/N = (AN/N)(BN/N)$ .
- (4) Let  $C$  and  $D$  be proper subgroups of  $G$  such that  $A \leq C$  and  $B \leq D$ . Then  $G = CD$ . Moreover, if  $A \neq C$  then  $C = A(C \cap D)$ . Conversely, if  $G = CD$  and  $A < C$  such that  $C = A(C \cap D)$  then  $G = AD$ .

(5) Suppose that  $N = CD$ . If there exists  $E \leq G$  containing  $C$  such that  $|E : E \cap N| = |G : N|$  then  $G = ED$ .

**PROOF.** (1) is well known. (2), (3) and (4) follow from [1, Lemmas 1.1.1 and 1.1.2(ii)]. For (5) comparing orders yields  $G = EN$  and then the result follows.

We also have the following useful lemma.

**Lemma 2.2** *Let  $G$  be a group with factorisation  $G = AB$ . Suppose that  $G$  has a unique conjugacy class of elements of order  $n$ , and both  $A$  and  $B$  contain elements of order  $n$ . Then  $A \cap B$  contains an element of order  $n$ .*

**PROOF.** Let  $a \in A$  and  $b \in B$  such that both  $a$  and  $b$  have order  $n$ . Then there exists  $g \in G$  such that  $a^g = b$ . Moreover, since  $G = AB$ , there exists  $a_1 \in A$  and  $b_1 \in B$  such that  $g = a_1 b_1$ . Thus  $b_1^{-1} a_1^{-1} a a_1 b_1 = b$  and so  $a_1^{-1} a a_1 = b_1 b b_1^{-1}$ . Hence  $a_1^{-1} a a_1 \in A \cap B$  and has order  $n$ .

### 3 The proof

Our general method is as follows: Suppose that  $G = AB$ . Let  $C$  be a maximal subgroup of  $G$  containing  $A$  and  $D$  be a maximal subgroup containing  $B$ . Then by Lemma 2.1(4),  $G = CD$  and is determined in [9]. Thus  $T = M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_2, HS, He, Ru, Suz, Fi_{22}$  or  $Co_1$ . We then look at the subgroups of  $C$  and  $D$  to determine  $A$  and  $B$ . Note that if  $T < G = \text{Aut}(T)$  then we only look for factorisations  $G = AB$  such that  $T \neq (A \cap T)(B \cap T)$  and so by Lemma 2.1(2) neither  $A$  nor  $B$  are contained in  $T$ . Thus we may assume that for our maximal factorisation  $G = CD$ , neither  $C$  nor  $D$  is equal to  $T$ .

$G = M_{11}$ . By [9],  $C = \text{PSL}(2, 11)$ , and  $D = M_{10}$  or  $M_{9.2}$ . Since 11 does not divide the order of either possibility for  $D$  it follows that 11 divides  $|A|$  and so  $A = C_{11}, C_{11} \rtimes C_5$  or  $A = C = \text{PSL}(2, 11)$ .

Since  $M_{10}$  is the stabiliser in  $M_{11}$  of a point in the 4-transitive action of  $M_{11}$  on 11 points it follows that  $M_{11} = C_{11} M_{10}$  is an exact factorisation. Thus we also have  $M_{11} = (C_{11} \rtimes C_5) M_{10}$ . Furthermore, as  $M_{10}$  has no subgroups of index 5,  $M_{11} \neq (C_{11} \rtimes C_5) B$  for any  $B < M_{10}$ . By Lemma 2.1(1),  $M_{11} \neq C_{11}(M_{9.2})$ . Also,  $(55, |M_{9.2}|) = 1$ , so  $M_{9.2} \cap (C_{11} \rtimes C_5) = 1$ . Hence by comparing orders we have  $M_{11} = (C_{11} \rtimes C_5)(M_{9.2})$  is an exact factorisation.

Next suppose  $A = C = \text{PSL}(2, 11)$ , the stabiliser of a point in the action of  $M_{11}$  on 12 points. Hence  $G = AB$  if and only if  $B$  is transitive on 12 points, and so we have that  $B = M_{10}, M_{9.2}, \text{AGL}(1, 9)$  or  $M_9$ .

$G = M_{12}$ . By [9],  $C = M_{11}$  or  $\text{PSL}(2, 11)$ . Suppose first that  $C = M_{11}$  and note that there are two conjugacy classes of such subgroups. By [9], for either choice of  $C$  there are seven choices of conjugacy classes for  $D$ , these being the other class of  $M_{11}$ 's, one of the two classes of  $M_{10.2}$ 's, one of the two classes of  $M_9.S_3$ 's,  $\text{PSL}(2, 11)$ ,  $C_2 \times S_5$ ,  $C_4^2 \rtimes D_{12}$  and  $A_4 \times S_3$ . If  $A < C$  then  $M_{11} = C = A(C \cap D)$ . We have already determined all the factorisations of  $M_{11}$  and so  $\{A, C \cap D\}$  is one of  $\{C_{11}, M_{10}\}$ ,  $\{C_{11} \rtimes C_5, M_{10}\}$ ,  $\{C_{11} \rtimes C_5, M_{9.2}\}$  or  $\{\text{PSL}(2, 11), H\}$  where  $H \in \{M_{10}, M_{9.2}, \text{AGL}(1, 9), M_9\}$ . Now none of the 7 possibilities for  $D$  intersect  $C$  in an  $M_{10}, M_{9.2}, M_9, C_{11}$  or an  $\text{AGL}(1, 9)$  (note that  $C \cap D$  has index 12 in  $D$ ). Thus  $(D, C \cap D, A)$  is one of  $(\text{PSL}(2, 11), C_{11} \rtimes C_5, M_{9.2})$ ,  $(\text{PSL}(2, 11), C_{11} \rtimes C_5, M_{10})$  or  $(M_{11}, \text{PSL}(2, 11), H)$ . By Lemma 2.1(4), in each case we obtain a factorisation  $G = AD$ . Comparing orders, the factorisation  $G = \text{PSL}(2, 11)(M_{9.2})$  is exact. If  $G = M_{10}B$  with  $B < \text{PSL}(2, 11)$  then  $|B| \geq 12.11$ . However,  $\text{PSL}(2, 11)$  contains no such proper subgroups. Finally, if  $G = HB$  with  $B < D$ , then we also have  $G = CB$  and  $M_{11} = D = B(C \cap D)$ . Since  $C \cap D = \text{PSL}(2, 11)$ , by the factorisations of  $M_{11}$  we have  $H, B \in \{M_{10}, M_{9.2}, \text{AGL}(1, 9), M_9\}$ . However, 11 must divide the order of either  $H$  or  $B$  and so  $G \neq HB$  with  $B < D$ . It remains to check factorisations  $G = AB$  where  $A = C = M_{11}$ . In this case  $B$  is any proper subgroup of  $M_{12}$  transitive on the 12 elements of  $[G : A]$ . A Magma [2] calculation shows that there are 28 choices of conjugacy classes for  $B$ , three of which are regular. The three classes of regular subgroups are  $A_4, D_{12}$  and  $C_6 \times C_2$ .

Suppose next that  $C = \text{PSL}(2, 11)$ . Then  $D = M_{10.2}$  or  $M_9.S_3$ , and there are two conjugacy classes for each choice. Since neither possibility for  $D$  has order divisible by 11 it follows that  $A = C_{11}, C_{11} \rtimes C_5$  or  $\text{PSL}(2, 11)$ . Since  $|G : D| \geq 66$ , the first two are too small for a factorisation. Hence  $A = C = \text{PSL}(2, 11)$  and so  $|B|$  is divisible by  $|G : A| = 144$ . If  $D$  is one of the choices for  $M_9.S_3$ , then the only proper subgroup of  $D$  with order divisible by 144 is  $B = M_{9.2}$ . Since  $B$  is the stabiliser in  $D$  of a point in an action of  $G$  on 12 points we have already seen that  $G = CB$  is an exact factorisation. Since there are two classes of possibilities for  $D = M_9.S_3$  there are 2 classes of possibilities for  $B = M_{9.2}$ .

Next we consider the case where  $D = M_{10.2}$ . Then  $G$  acts on a set  $\Omega$  of size 12 such that  $D$  is the stabiliser of a 2-set. Note that there are 2 conjugacy classes of such  $D$ , corresponding to the two actions of  $M_{12}$  on 12 points. Since  $D$  contains  $A_6$  as an index 4 normal subgroup, it follows that  $B \cap A_6$  has order divisible by  $144/4 = 36$ . Hence either  $A_6 < B$  or  $B \cap A_6 = C_3^2 \times C_4$ . The latter case implies that  $B = M_{9.2}$  which we have already encountered as a subgroup of  $M_9.S_3$ . Hence if  $B < M_{10.2}$ , we are left to consider  $B = M_{10}, S_6$

or  $\text{PGL}(2, 9)$ . We have already obtained the factorisation  $G = \text{PSL}(2, 11)M_{10}$ . Note that there are two choices for  $M_{10}$ , one for each choice of  $D$ . For the remaining two possibilities for  $B$ , note that  $D = B(C \cap D)$ . Now  $|C \cap D| = 10$ , but  $C = \text{PSL}(2, 11)$  does not contain an element of order 10 and so  $C \cap D = D_{10}$ . Since  $C \cap D \leq \text{PGL}(2, 9)$  we do not obtain a factorisation when  $B = \text{PGL}(2, 9)$ . Since the stabiliser in  $C$  of a point of  $\Omega$  has order 55, all elements of order two in  $C \cap D$  are fixed point free on  $\Omega$ . However, all elements of order two in a copy of  $S_6$  in  $D$  fix a point of  $\Omega$  ( $S_6$  has two orbits of length 6 and a fixed point free element of order two on one acts as a transposition on the other). Hence  $S_6 \cap (C \cap D) = C_5$  and we have a factorisation  $M_{10}.2 = S_6(C \cap D)$ . Thus by Lemma 2.1(4), we have a factorisation  $M_{12} = \text{PSL}(2, 11)S_6$ . There are two conjugacy classes from which to choose the  $S_6$ , corresponding to the two conjugacy classes of  $M_{10}.2$ .

$G = \text{Aut}(M_{12})$ . By [9], all maximal factorisations of  $G$  either arise from maximal factorisations of  $M_{12}$  or have one factor equal to  $M_{12}$ . Since the maximal factorisations of  $M_{12}$  have either  $M_{11}$ ,  $M_{10}.2$  or  $M_9.S_3$  as one of the factors, and these are each selfnormalising in  $\text{Aut}(M_{12})$ , it follows that if  $G = CD$  is a maximal factorisation then either  $C$  or  $D$  is equal to  $T$ . Thus if  $G = AB$  then either  $A$  or  $B$  is contained in  $T$  and hence by Lemma 2.1(2), the factorisation arises from a factorisation of  $T$ . Moreover, if  $G = AB$  is an exact factorisation, then  $T = (A \cap T)(B \cap T)$  is an exact factorisation. Hence either  $A \cap T = M_{11}$  and  $B \cap T$  is one of  $A_4$ ,  $D_{12}$  or  $C_6 \times C_2$ , or  $A \cap T = \text{PSL}(2, 11)$  and  $B \cap T = M_9.2$ . In the first case,  $M_{11}$  is selfnormalising in  $G$  and so  $A = M_{11}$  and  $B$  is a regular subgroup in the action of  $G$  on 24 points. A Magma [2] calculation shows that there are 5 choices of conjugacy classes for  $B$  given as in Table 5. In the second case  $M_9.2$  is selfnormalising in  $G$ , and so  $B = M_9.2$  and  $A = \text{PGL}(2, 11)$  gives an exact factorisation.

$G = \text{Aut}(M_{22})$ . By [9] we have  $C = \text{PGL}(2, 11)$  and  $D = \text{P}\Sigma\text{L}(3, 4)$ . Moreover,  $M_{22}$  has no factorisations. Since  $G = AD$  is a factorisation and  $|G : D| = 22$ , it follows that 22 divides  $|A|$ . Hence  $A = C_{11} \rtimes C_2$ ,  $C_{11} \rtimes C_{10}$ ,  $\text{PSL}(2, 11)$  or  $\text{PGL}(2, 11)$ . To investigate the case where  $A = C_{11} \rtimes C_2$  we need to look at the geometry of the Witt design for  $H = M_{24}$ . Let  $\{\alpha, \beta\}$  be a 2-set of the set  $\Omega$  of 24 points. Then  $G = H_{\{\alpha, \beta\}}$ . Let  $\Gamma_1$  and  $\Gamma_2$  be two disjoint dodecads with  $\alpha \in \Gamma_1$  and  $\beta \in \Gamma_2$ . Then

$$H_{\{\Gamma_1, \Gamma_2\}} = \text{Aut}(M_{12}), \quad H_{\Gamma_1, \Gamma_2} = M_{12}, \quad H_{\Gamma_1, \Gamma_2, \alpha} = M_{11},$$

$$H_{\Gamma_1, \Gamma_2, \alpha, \beta} = \text{PSL}(2, 11) \text{ and } H_{\{\Gamma_1, \Gamma_2\}, \{\alpha, \beta\}} = \text{PGL}(2, 11)$$

with  $\text{PGL}(2, 11)$  acting transitively on the 22 points of  $\Omega \setminus \{\alpha, \beta\}$ . Moreover, an element  $x$  of order 11 in  $\text{PSL}(2, 11)$  acts as a product of two 11-cycles, (the elements of these being  $\Gamma_1 \setminus \{\alpha\}$  and  $\Gamma_2 \setminus \{\beta\}$ ) which are interchanged by an outer automorphism  $\tau$  of  $\text{PSL}(2, 11)$  which normalises  $\langle x \rangle$ . Hence  $\langle x \rangle \rtimes$

$\langle \tau \rangle \cong C_{11} \rtimes C_2$  acts regularly on the 22 points and so we have an exact factorisation  $G = (C_{11} \rtimes C_2)D$ . It then follows that  $G = (C_{11} \rtimes C_{10})D$  is also a factorisation. Since  $D$  does not have an index 5 subgroup this is the only factorisation with  $A = C_{11} \rtimes C_{10}$ . Next suppose that  $A = \text{PSL}(2, 11)$ . Then  $A \cap D = A_5 = C \cap D$ . Hence by Lemma 2.1(1),  $G \neq AD$ . It remains to consider  $A = C = \text{PGL}(2, 11)$  and  $B < \text{P}\Sigma\text{L}(3, 4)$ . Then  $B$  has order divisible by  $2^5 \cdot 3 \cdot 7 = |G : C|$ . The only maximal subgroups of  $D = \text{P}\Sigma\text{L}(3, 4)$  with order divisible by 7 are  $\text{PSL}(2, 7) \times C_2$  and  $\text{PSL}(3, 4)$ . The first does not have order divisible by  $2^5$  and so  $B = \text{PSL}(3, 4)$ . However,  $C \cap D = A_5 < \text{PSL}(3, 4)$  and so  $D \neq B(C \cap D)$ . Hence by Lemma 2.1(4),  $G \neq CB$ .

$G = M_{23}$ . By [9] we have  $C = C_{23} \rtimes C_{11}$  and  $D$  is one of  $M_{22}, \text{P}\Sigma\text{L}(3, 4)$  or  $2^4 \rtimes A_7$ . The latter two are exact factorisations and so we only need to consider  $B \leq D = M_{22}$ . Since  $|D|$  is not divisible by 23 it follows that  $A = C_{23}$  or  $C_{23} \rtimes C_{11}$ . Clearly  $G = C_{23}M_{22}$  is an exact factorisation and so we are left to deal with the case where  $A = C_{23} \rtimes C_{11}$  and  $B < M_{22}$ . However,  $M_{22}$  does not have an index 11 subgroup and so  $G \neq AB$  for any  $B < M_{22}$ .

$G = M_{24}$ . By [9],  $M_{23}$  and  $\text{PSL}(2, 23)$  are the only two possibilities for  $C$ . Suppose first that  $C = M_{23}$ . Then  $D$  is one of  $M_{12}.2, 2^6 \rtimes (3 \cdot S_6), \text{PSL}(2, 23), 2^6 \rtimes (\text{PSL}(3, 2) \times S_3)$  or  $\text{PSL}(2, 7)$ . If  $A \neq C$  then  $C = A(C \cap D)$  and by our list of factorisations of  $M_{23}$  we have that  $\{A, C \cap D\}$  is one of  $\{C_{23}, M_{22}\}$  or  $\{C_{23} \rtimes C_{11}, X\}$  with  $X$  one of  $M_{22}, \text{P}\Sigma\text{L}(3, 4)$  or  $2^4 \rtimes A_7$ . Thus  $D = \text{PSL}(2, 23)$  and  $C \cap D = C_{23} \rtimes C_{11}$ . Lemma 2.1(4) then implies that we get three factorisations  $G = X \text{PSL}(2, 23)$ . Since 23 does not divide any  $|X|$  and all proper subgroups of  $\text{PSL}(2, 23)$  divisible by 23 have order at most  $23 \cdot 11$ , it follows that there are no factorisation  $G = XB$  with  $X < \text{PSL}(2, 23)$ . Next suppose that  $A = C = M_{23}$ . Then  $B$  is any proper subgroup which is transitive on the 24 points. By a Magma [2] calculation there are 173 conjugacy classes of such subgroups, of which 8 are regular. The eight classes of regular subgroups are as given in Table 5.

Suppose next that  $C = \text{PSL}(2, 23)$ . Then we have three maximal factorisations  $G = \text{PSL}(2, 23)D$  where  $D = M_{22}.2, 2^4 \rtimes A_8$  or  $\text{PSL}(3, 4).S_3$ . None of these three groups have order divisible by 23 and so  $A$  is  $C_{23}, C_{23} \rtimes C_{11}$  or  $\text{PSL}(2, 23)$ . Moreover, since none of the possibilities for  $D$  have order  $|G|/23$  it follows that  $A \neq C_{23}$ . Order arguments also eliminate the case  $A = C_{23} \rtimes C_{11}$ . Hence  $A = C = \text{PSL}(2, 23)$ . Now  $|G : \text{PSL}(2, 23)| = 40320$  and hence 40320 divides  $|B|$ . Suppose first that  $B < D = M_{22}.2$ . Looking at the subgroup structure of  $B$ , as given in [3], it follows that  $B = \text{P}\Sigma\text{L}(3, 4)$  or  $M_{22}$ . Both of these subgroups are contained in an  $M_{23}$  and so we have already seen that they give factorisations. Suppose next that  $B < D = 2^4 \rtimes A_8$ . Since 40320 divides  $|B|$  it follows that  $B$  is a subgroup of index at most 8. Hence the only possibility is  $B = 2^4 \rtimes A_7$ . Note that  $B$  is then the stabiliser in  $G$  of an octad-point incident

pair while in  $C$  the stabiliser of a point has order 23.11. Hence  $B \cap C = 1$  and by comparing orders we have that  $G = \text{PSL}(2, 23)(2^4.A_7)$  is an exact factorisation. This leaves  $B < D = \text{PSL}(3, 4).S_3$ . Since 40320 divides  $|B|$  it follows that  $B = \text{P}\Sigma\text{L}(3, 4) < M_{23}$ . We have already seen that this gives rise to a factorisation.

$G = J_2$ . By [9], we have  $C = \text{PSU}(3, 3)$  and  $D = A_5 \times D_{10}$ . Now  $|G : D| = 1008$  and so 1008 divides  $|A|$ . Hence  $A = C$  and  $|G : A| = 100$  divides  $|B|$ . Thus if  $B \neq D$  then  $B = D_{10} \times D_{10}$  or  $A_5 \times C_5$ . Suppose first that  $B = D_{10} \times D_{10}$ . Then  $B$  intersects each conjugacy class of involutions of  $D$  nontrivially. Since  $|A \cap D| = 6$  it follows that  $|A \cap B| = 2$  and so  $G \neq AB$ . According to [3, p 42], the involutions in the  $A_5$  are from the  $2B$  class, and inspecting the character table shows that this class intersects  $A$  trivially. Hence if  $B = A_5 \times C_5$  we have  $|B \cap C| = 3$  and so  $G = \text{PSU}(3, 3)(A_5 \times C_5)$  is a factorisation.

$G = \text{Aut}(J_2)$ . By [9] we have  $C = G_2(2)$  and  $D$  is either  $5^2 \times (C_4 \times S_3)$  or  $(A_5 \times D_{10}) \cdot 2$ . Note that  $C \cap T = \text{PSU}(3, 3)$ . By [9], in the first case  $T \neq (C \cap T)(D \cap T)$  while in the second case  $T = (T \cap C)(D \cap T)$ . The two possibilities for  $D$  have index 2016 and 1008 respectively in  $G$ , so 1008 divides  $|A|$ . Since  $A \leq C$  it follows that either  $A = C$  or  $A = \text{PSU}(3, 3)$ . However,  $\text{PSU}(3, 3) \leq T$  and since we are only looking for factorisations not giving factorisations of  $T$ , Lemma 2.1(2) implies that  $A = C = G_2(2)$ . Then  $|G : A| = 100$  and so 100 divides  $|B|$ . Hence  $G = G_2(2)B$  for any proper subgroup  $B$  acting transitively on 100 points. A Magma [2] calculation shows that there are 10 conjugacy classes of such subgroups, two of which are regular, one is  $J_2$  and 3 of which give factorisations of  $J_2$ . Hence there are six factorisations  $G = G_2(2)B$  which do not give rise to factorisations of  $J_2$ . The two regular subgroups are the 2 nonisomorphic subgroups of shape  $5^2 \times C_4$  in  $5^2 \times (C_4 \times S_3)$ , which are the groups 11 and 12 in the Magma catalogue of groups of order 100. All subgroups  $B$  which do not give factorisations of  $J_2$  lie in  $5^2 \times (C_4 \times S_3)$  and contain one of the regular subgroups.

$G = HS$ . By [9] we have  $C = M_{22}$  and  $D$  is either  $\text{PSU}(3, 5).2$  or  $(C_5 \times C_4) \times A_5$ . There are two possible conjugacy classes for the first choice of  $D$ . For all choices of  $D$ , 176 divides  $|G : D|$  and so 176 divides  $|A|$ . Hence  $A = C$  and  $|G : A| = 100$  divides  $|B|$ . Suppose first that  $B < \text{PSU}(3, 5).2 = D$  and let  $D'$  be the socle of  $D$ . By [9, p 119],  $A \cap D = A_7 \leq D'$  and so by order arguments  $B \neq \text{PSU}(3, 5)$ . Looking at the subgroup structure of  $D$  it follows that  $B \leq 5_+^{1+2} \times C_8 \times C_2$ , the intersection of two subgroups isomorphic to  $\text{PSU}(3, 5).2$  from different conjugacy classes in  $G$ . A Magma [2] calculation shows that  $B = 5_+^{1+2} \times C_8 \times C_2$  gives a factorisation, while no further subgroups do.

Suppose next that  $B < (C_5 \times C_4) \times A_5$ . Since 100 divides  $|B|$  it follows that

$C_5 \times C_5 < B$ . By a Magma [2] calculation, there are two such groups  $B$  for which  $G = M_{22}B$ . These are  $B = (C_5 \times C_2) \times A_5$  and  $C_5 \times A_5$ . Neither of these factorisations are exact.

$G = \text{Aut}(HS)$ . By [9] we have  $C = M_{22}.2$  and  $D$  is either  $(C_5 \times C_4) \times S_5$  or  $5_+^{1+2} \rtimes S$  where  $S$  is the Sylow 2-subgroup of  $\text{GL}(2, 5)$ . The two choices for  $D$  have index 36960 and 22176 respectively in  $G$ . Since  $A$  is not contained in  $T$  it follows that  $A = C = M_{22}.2$ . Hence the factorisations are all of the form  $G = (M_{22}.2)B$  with  $B$  a proper subgroup acting transitively on 100 points. A Magma [2] calculation shows that there are 23 conjugacy classes of such subgroups. Only 10 do not give factorisations of  $HS$  and four of these are regular. The factorisations that do not give rise to factorisations of  $HS$  have  $B \leq 5_+^{1+2} \rtimes S$ . The isomorphism types of the regular subgroups are given in Table 5 and these are groups 9, 10, 11 and 12 in the Magma catalogue of groups of order 100.

$G = He$ . From [9], we have  $C = \text{Sp}(4, 4).2$  and  $D = 7^2.\text{SL}(2, 7)$ . Since  $|G : D| = 244800$  it follows that 244800 divides  $|A|$ . Looking at the subgroup structure of  $C$  it follows that  $A = \text{Sp}(4, 4)$  or  $A = C$ . By [9, p 120],  $C \cap D$  is cyclic of order 8. Suppose first that  $A = \text{Sp}(4, 4)$ . Since  $A$  contains no elements of order 8 it follows that  $|A \cap D| = 4$ . Hence by Lemma 2.1(1) we have  $G = \text{Sp}(4, 4)(7^2.\text{SL}(2, 7))$ . Now  $|G : A| = 4116$  so 4116 divides  $|B|$ . However,  $D$  has no such proper subgroups and so  $B = D = 7^2.\text{SL}(2, 7)$ .

Suppose now that  $A = C = \text{Sp}(4, 4).2$ . Then  $|G : A| = 2058$ , which divides  $|B|$ . Thus, if  $B \neq D$  we have  $B = 7^2 \rtimes (C_2 \times (C_7 \rtimes C_3))$ . Moreover,  $D = B(C \cap D)$ . Now  $D$  contains a unique class of involutions, and both  $B$  and  $C \cap D$  also contain involutions. Hence by Lemma 2.2,  $B \cap C \cap D$  contains an involution. Thus  $|A \cap B| \geq 2$  and so  $|AB| < |G|$  Hence  $G \neq AB$  with  $B < D$ .

$G = \text{Aut}(He)$ . By [9]  $C = \text{Sp}(4, 4).4$  and  $D$  is either  $7^2.\text{SL}(2, 7).2$  or  $7_+^{1+2} \rtimes (S_3 \times C_6)$ . By [9], the first factorisation yields a factorisation of  $He$  while the second factorisation does not. Suppose first that  $B \leq 7_+^{1+2} \rtimes (S_3 \times C_6) = D$ . Now  $|G : D| = 652800$  and so 652800 divides  $|A|$ . Since  $A$  is not contained in  $T$  it follows that  $A = C$ . Thus  $|G : A| = 2058$  divides  $|B|$  and so  $B$  contains the normal 7-subgroup  $M$  of  $D$ . Let  $D = M \rtimes (\langle x, y \rangle \times \langle a, b \rangle)$  such that  $x^3 = y^2 = a^3 = b^2 = 1$ ,  $x^y = x^{-1}$  and  $ab = ba$ . Then  $D \cap T = M \rtimes (\langle x, y \rangle \times \langle a \rangle)$  and by [9, p 120],  $A \cap D = \langle x, y \rangle$ . Recall that  $B$  is not contained in  $T$ . Now  $D$  contains 3 conjugacy classes of elements of order 2, with representatives  $y, b$  and  $yb$ . It also contains three conjugacy classes of cyclic subgroups of order 3, with representatives  $\langle x \rangle, \langle a \rangle$  and  $\langle xa \rangle$ . Suppose first that  $|B| = 2058 = 7^3.3.2$ . Then for  $G = AB$  to be a factorisation, we require that  $B \cap A = 1$  and so neither  $x$  nor  $y$  is contained in  $B$ . Then since  $yb$  does not normalise  $\langle xa \rangle$ , it follows that there are three subgroups  $B$  of  $D$  of order  $7^3.3.2$  which give exact

factorisations. These are  $M \rtimes \langle a, b \rangle$ ,  $M \rtimes \langle a, yb \rangle$  and  $M \rtimes \langle xa, b \rangle$ . Next suppose that  $|B| = 7^3 \cdot 3^2 \cdot 2$ . Then  $B$  must meet  $A$  in a subgroup of order three. There are two conjugacy classes of involutions of  $D$  not in  $T$  and so we get two subgroups  $M \rtimes \langle x, b, a \rangle$  and  $M \rtimes \langle x, a, yb \rangle$  which give factorisations. Finally, if  $|B| = 7^3 \cdot 2^2 \cdot 3$  then  $B$  must meet  $A$  in a subgroup of order 2. The subgroup  $\langle a \rangle$  is the unique cyclic 3-subgroup of  $D$  which is normalised by  $\langle y, b \rangle$  and not contained in  $T$ . Hence the only factorisation when  $|B| = 7^3 \cdot 2^2 \cdot 3$  is for  $B = M \rtimes \langle y, a, b \rangle$ . Thus  $G = AB$  for seven conjugacy classes of subgroups  $B \leq 7_+^{1+2} \rtimes (S_3 \times C_6)$ , three of these give exact factorisations.

Next suppose that  $B \leq 7^2 \cdot \text{SL}(2, 7) \cdot 2 = D$ . Since  $|G : D| = 244800$  it follows that 244800 divides  $|A|$ . Then as  $A$  is not contained in  $T$  it follows that  $A = \text{PSp}(4, 4) \cdot 4$ . Thus  $|G : A| = 2058$  divides  $|B|$  and so  $7^2 \rtimes C_7 \leq B$ . Hence if  $B \neq D$ , then either  $B = 7^2 \cdot \text{SL}(2, 7)$  or  $B$  normalises a subgroup of order 7 from the conjugacy class  $7C$  [3, p 104]. We do not have the first case as there  $B \leq T$  and so we are in the second case. Hence  $B \leq 7_+^{1+2} \rtimes (S_3 \times C_6)$ , the normaliser of a  $7C$  subgroup [3, p 104], and we have already determined all such factorisations.

$G = Ru$ . By [9], we have  $C = \text{PSL}(2, 29)$  and  $D = {}^2F_4(2)$ . Since  $|G : D| = 4060$  it follows that 4060 divides  $|A|$ . Thus  $A = C$  and  $|G : A| = 11980800$  divides  $|B|$ . Hence  $B = D$ .

$G = Suz$ . By [9] we have  $C = G_2(4)$  and  $D$  is either  $\text{PSU}(5, 2)$  or  $3^5 \rtimes M_{11}$ . Suppose first that  $B \leq D = \text{PSU}(5, 2)$ . Then  $|G : D| = 32760$  divides  $|A|$ . Looking at the subgroup structure of  $C$  in [3, p 97], it follows that  $A = C$ . Hence  $|G : A| = 1782$  divides  $|B|$ . However, by [3, p 73],  $D$  has no such proper subgroups. Hence  $B = D = \text{PSU}(5, 2)$ .

Suppose next that  $B \leq D = 3^5 \rtimes M_{11}$ . Then  $|G : D| = 232960$  divides  $|A|$  and so  $A = C = G_2(4)$ . Now by [9, p 121],  $C \cap D = 3.A_6$ . By Lemma 2.1(4), if  $B < D$  then  $D = B(C \cap D)$ . Letting  $Z = 3^5 \triangleleft D$  we have  $|Z \cap C \cap D| = 3$  and by Lemma 2.1(3), it follows that  $M_{11} = (BZ/Z)((C \cap D)Z/Z)$ . Hence  $M_{11}$  has a factorisation with one factor isomorphic to  $A_6$ . This contradicts our list of factorisations of  $M_{11}$  and so  $B = D$ .

$G = \text{Aut}(Suz)$ . By [9] we have  $C = G_2(4) \cdot 2$  and  $D$  is either  $\text{PSU}(5, 2) \cdot 2$  or  $3^5 \rtimes (M_{11} \times C_2)$ . Moreover, both factorisations give rise to factorisations of  $T$ .

Suppose first that  $B \leq D = \text{PSU}(5, 2) \cdot 2$ . Then  $|G : D| = 32760$  divides  $|A|$ . Since  $A$  is not contained in  $T$ , looking at the subgroup structure of  $C$  it follows that  $A = C$ . Hence  $|G : A| = 1782$  divides  $|B|$ . Again we have that  $B$  is not contained in  $T$  and so  $B = D$ . Hence we only obtain the maximal factorisation, which gives rise to a factorisation of  $T$ .

Suppose next that  $B \leq D = 3^5 \rtimes (M_{11} \times C_2)$ . Then  $|G : D| = 232960$  divides  $|A|$ , so  $A = C$ . Thus  $B < D$  and  $|G : A| = 1782$  divides  $|B|$ . Let  $Z = 3^5 \triangleleft D$ . By [9, p 120] and the fact that  $G = AD$  we have  $A \cap D = 3.A_6.2$ . Moreover,  $A \cap D \cap Z = C_3$ . The normaliser of  $A_6$  in  $M_{11} \times C_2$  is  $M_{10} \times C_2$ , and  $D = G_2(4).2$  does not contain  $A_6 \times C_2$  as a section (see [3, p 97]). Thus  $(A \cap D)Z/Z \cong M_{10}$ . Letting  $Y$  be the largest normal  $\{2, 3\}$ -subgroup of  $D$ , we have  $Z \triangleleft Y = 3^5 \rtimes C_2$  and  $A \cap D \cap Y = C_3$ . Hence  $(A \cap D)Y/Y \cong M_{10}$ . By Lemma 2.1(4),  $D = B(A \cap D)$  and by Lemma 2.1(3),  $M_{11} = (BY/Y)((A \cap D)Y/Y)$ . Looking at our list of factorisations of  $M_{11}$  we see that there are three choices for  $BY/Y$ . Hence  $BY$  is one of  $3^5 \rtimes (C_{11} \times C_2)$ ,  $3^5 \rtimes ((C_{11} \times C_5) \times C_2)$  or  $3^5 \rtimes (\text{PSL}(2, 11) \times C_2)$ . Now  $1782 = 11 \cdot 3^4 \cdot 2$  divides  $|B|$ . Moreover, as 11 is a primitive prime divisor of  $3^5 - 1$  and  $B$  contains an element of order 11, it follows that  $Z \leq B$ . Thus if  $BY = 3^5 \rtimes (C_{11} \times C_2)$  or  $3^5 \rtimes ((C_{11} \times C_5) \times C_2)$  it follows that  $B = BY$ . Since  $(A \cap D)Z/Z \cong M_{10}$ , in both cases the involutions in  $B$  do not lie in  $A \cap D$  and so  $|B \cap A| = 3$  or 15 respectively. Hence  $G = AB$  for the first two choices for  $B$ . This leaves us to consider the case  $BY = 3^5 \rtimes (\text{PSL}(2, 11) \times C_2)$ . Here either  $B = 3^5 \rtimes \text{PSL}(2, 11)$  or  $B = BY$ . In the first case  $B \leq T$  and so if  $G = AB$  then by Lemma 2.1(5), we have  $T = (A \cap T)B$ . This contradicts our determination of all factorisations of  $T = \text{Suz}$ . Hence we must have  $B = BY = 3^5 \rtimes (\text{PSL}(2, 11) \times C_2)$ . Moreover, since  $G = A(3^5 \rtimes (C_{11} \times C_2))$  is a factorisation it follows that  $G = AB$  is a factorisation.

$G = \text{Fi}_{22}$ . By [9] we have  $C = {}^2F_4(2)'$  and  $D = 2 \cdot \text{PSU}(6, 2) = D$ . Now  $|G : D| = 3510$  and so 3510 divides  $|A|$ . Looking at the subgroup structure of  $C$  in [3, p 74] we see that  $A = C$ . Now  $|G : A| = 3592512$  divides  $|B|$ . Letting  $Z$  be the centre of  $D$  it follows that 1796256 divides  $|BZ/Z|$ . Looking at the subgroup structure of  $\text{PSU}(6, 2)$  in [3, p 115] it follows that  $BZ/Z = \text{PSU}(6, 2)$  and so  $B = D$ .

$G = \text{Aut}(\text{Fi}_{22})$ . By [9],  $C = {}^2F_4(2)$  and  $D = 2 \cdot \text{PSU}(6, 2).2$ . Now  $|G : D| = 3510$  divides  $|A|$ . Since  $A$  is not contained in  $T$  it follows that  $A = C$ . Similarly, we see that  $B = D$ . Moreover,  $G = CD$  gives rise to a factorisation of  $\text{Fi}_{22}$ .

$G = C_{01}$ . From [9] there are four maximal factorisations  $G = CD$  where  $C$  is one of  $C_{02}$  or  $C_{03}$ , and  $D$  is one of  $(3 \cdot \text{Suz}).2$  or  $(A_4 \times G_2(4)).2$ . Since neither  $C_{02}$  nor  $C_{03}$  has a factorisation (see [9]) it follows from Lemma 2.1(4), that  $A = C_{02}$  or  $C_{03}$ .

Suppose first that  $A = C_{02}$  and  $B \leq (3 \cdot \text{Suz}).2 = D$ . Then  $A \cap D = \text{PSU}(5, 2).2$ , the stabiliser in  $D$  of a type 2 vector. Let  $Z$  be the minimal normal subgroup of  $D$ . Then if  $B \neq D$  we have

$$D/Z = (BZ/Z)((A \cap D)Z/Z)$$

and hence

$$Suz.2 = (BZ/Z)(PSU(5, 2).2).$$

Looking at the factorisations of  $Suz.2$  (and remembering the ones which are not corefree) we have that  $BZ/Z$  is one of  $Suz$ ,  $G_2(4)$  or  $G_2(4).2$ . Hence  $BZ = 3 \cdot Suz$ ,  $3 \times G_2(4)$  or  $(3 \times G_2(4)).2$ . Now  $G_2(4)$  [11, §§3,4] and  $3 \cdot Suz$  [12, p 155] are both transitive on type 2 vectors and so we have factorisations  $G = Co_2B$  where  $B$  is one of  $3 \cdot Suz$ ,  $G_2(4)$ ,  $3 \times G_2(4)$ ,  $G_2(4).2$  and  $(3 \times G_2(4)).2$

Next suppose that  $B \leq (A_4 \times G_2(4)).2 = D$ . Now  $D$  contains a unique subgroup  $S$  isomorphic to  $G_2(4)$ . Moreover  $|G : A| = 2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$  divides  $|B|$ . Hence  $3^2 \cdot 5 \cdot 7 \cdot 13$  divides  $|B \cap S|$ . Looking at the subgroup structure of  $G_2(4)$  it follows that  $S \leq B$ . Moreover, since  $G_2(4)$  is transitive on type 2 vectors it follows that we have factorisations  $G = Co_2B$  for any  $B$  with  $G_2(4) \leq B \leq (A_4 \times G_2(4)).2$ . Since there is a unique conjugacy class of  $G_2(4)$ 's in  $G$  [13, p 163], we have already seen four of these factorisations.

Next suppose that  $A = Co_3$  and  $B \leq (A_4 \times G_2(4)).2 = D$ . Let  $S_1$  be a subgroup of  $D$  isomorphic to  $G_2(4).2$ . Then by [9, p 122]  $A \cap S_1 \cong A_5$  and  $G = Co_3S_1$  is a factorisation. Moreover,  $|G : A| = 8386560$  must divide  $|B|$  and so  $698880$  must divide  $|B \cap S_1|$ . Looking at the subgroups of  $G_2(4).2$  it follows that  $S_1 \leq B$  and so we get factorisations  $G = Co_3B$  for any  $B$  such that  $G_2(4).2 \leq B \leq (A_4 \times G_2(4)).2$ .

Next suppose that  $A = Co_3$  and  $B \leq (3 \cdot Suz).2 = D$ . Then  $A \cap D = 3^5 \rtimes (\text{PSL}(2, 11) \times C_2)$ , the stabiliser of a type 3 vector, and if  $B \neq D$  we have  $D = B(A \cap D)$ . Letting  $Z$  be the minimal normal subgroup of  $D$  we get

$$D/Z = (BZ/Z)((A \cap D)Z/Z)$$

and hence since  $A \cap D \cap Z = 1$  we have

$$Suz.2 = (BZ/Z)(3^5 \rtimes (\text{PSL}(2, 11) \times C_2)).$$

Looking at the factorisations of  $Suz.2$  it follows that  $BZ/Z = G_2(4).2$  or  $Suz$ . Hence  $BZ = (3 \times G_2(4)).2$  or  $3 \cdot Suz$  and so  $B = G_2(4).2$ ,  $(3 \times G_2(4)).2$  or  $3 \cdot Suz$ . There is only one conjugacy class of  $G_2(4)$ 's in  $G$  so we have already seen that the first two possibilities give us factorisations. Hence we also have  $G = Co_3(3 \cdot Suz)$ .

This completes the proof of Theorems 1.1 and 1.2. Theorem 1.3 then follows.

## 4 The tables

Table 1  
Factorisations of sporadics

$G$	$A$	$B$	Comments
$M_{11}$	$C_{11}$	$M_{10}$	exact $B$ is stabiliser of a point in Witt design
	$C_{11} \rtimes C_5$	$M_{10}$	
	$C_{11} \rtimes C_5$	$M_{9.2}$	exact $B$ is stabiliser of a 2-set in action on 11 points
	$\text{PSL}(2, 11)$	$M_{10}$	$A$ is stabiliser of a point in action on 12 points
	$\text{PSL}(2, 11)$	$M_{9.2}$	
	$\text{PSL}(2, 11)$	$M_9$	
	$\text{PSL}(2, 11)$	$\text{AGL}(1, 9)$	
$M_{12}$	$M_{11}$	transitive on $[G : A]$	2 conjugacy classes for $A$ ; given $A$ , 28 classes for $B$ , 3 of which are regular  maximal choices for $B$ are $M_{11}$ , $M_{10.2}$ , $\text{PSL}(2, 11)$ , $M_9.S_3$ , $2 \times S_5$ , $C_4^2.D_{12}$ and $A_4 \times S_3$ .
	$\text{PSL}(2, 11)$	$M_{9.2}$	exact, 2 classes for $B$ $A$ maximal in $T$
	$\text{PSL}(2, 11)$	$M_9.S_3$	$A$ maximal in $T$ , 2 classes for $B$ $B$ is stabiliser of a 3-set in action on 12 points
	$\text{PSL}(2, 11)$	$S_6$	$A$ maximal in $T$ , 2 classes for $B$
	$\text{PSL}(2, 11)$	$M_{10}$	$A$ maximal in $T$ , 2 classes for $B$ $B$ is stabiliser of 2 points in action on 12 points
	$\text{PSL}(2, 11)$	$M_{10.2}$	$A$ maximal in $T$ , 2 classes for $B$ $B$ is stabiliser of a 2-set in action on 12 points
$J_2$	$\text{PSU}(3, 3)$	$A_5 \times C_5$	$A$ is vertex stabiliser for rank 3 graph
	$\text{PSU}(3, 3)$	$A_5 \times D_{10}$	

Table 2  
More factorisations of sporadics

$G$	$A$	$B$	Comments
$M_{23}$	$M_{22}$	$C_{23}$	exact $B$ is stabiliser of a point of Witt design
	$M_{22}$	$C_{23} \times C_{11}$	
	$\text{PSL}(3, 4)$	$C_{23} \times C_{11}$	exact $A$ is stabiliser of a 2-set in action on 23 points
	$2^4 \times A_7$	$C_{23} \times C_{11}$	exact $A$ is stabiliser of a heptad in Witt design
$M_{24}$	$M_{23}$	transitive on 24 points	173 classes for $B$ , 8 regular, maximal transitive subgroups are $M_{12}.2$ , $2^6.(3 \cdot S_6)$ , $\text{PSL}(2, 23)$ , $\text{PSL}(2, 7)$ , and $2^6.(\text{PSL}(3, 2) \times S_3)$
	$\text{PSL}(2, 23)$	$\text{PSL}(3, 4)$	exact
	$\text{PSL}(2, 23)$	$\text{PSL}(3, 4).S_3$	$B$ is stabiliser of a 3-set
	$\text{PSL}(2, 23)$	$2^4 \times A_7$	exact $B$ is stabiliser of flag
	$\text{PSL}(2, 23)$	$2^4 \times A_8$	$B$ is stabiliser of octad in Witt design
	$\text{PSL}(2, 23)$	$M_{22}.2$	$B$ is stabiliser of a 2-set
	$\text{PSL}(2, 23)$	$M_{22}$	$B$ is stabiliser of 2 points
$HS$	$M_{22}$	$\text{PSU}(3, 5).2$	$A$ is stabiliser of a vertex in Higman-Sims graph 2 classes for $B$
	$M_{22}$	$5_+^{1+2} \times C_8 \times C_2$	
	$M_{22}$	$(C_5 \times C_4) \times A_5$	
	$M_{22}$	$(C_5 \times C_2) \times A_5$	
	$M_{22}$	$C_5 \times A_5$	
$He$	$\text{Sp}(4, 4)$	$7^2 \times \text{SL}(2, 7)$	
	$\text{Sp}(4, 4).2$	$7^2 \times \text{SL}(2, 7)$	
$Ru$	$\text{PSL}(2, 29)$	${}^2F_4(2)$	$A$ is vertex stabiliser for rank 3 graph
$Suz$	$G_2(4)$	$\text{PSU}(5, 2)$	$A$ is vertex stabiliser for rank 3 graph
	$G_2(4)$	$3^5 \times M_{11}$	

Table 3  
Even more factorisations of sporadics

$G$	$A$	$B$	Comments
$Fi_{22}$	${}^2F_4(2)'$	$2 \cdot \text{PSU}(6, 2)$	$B$ is centraliser of an involution
$Co_1$	$Co_2$	$(3 \cdot \text{Suz}).2$	$A$ is stabiliser of type 2 vector $B$ is centraliser of element of order 3
	$Co_2$	$3 \cdot \text{Suz}$	
	$Co_2$	$G_2(4) \leq B \leq (A_4 \times G_2(4)).2$	any such subgroup
	$Co_3$	$(3 \cdot \text{Suz}).2$	$A$ is stabiliser of type 3 vector
	$Co_3$	$3 \cdot \text{Suz}$	
	$Co_3$	$G_2(4).2 \leq B \leq (A_4 \times G_2(4)).2$	any such subgroup

Table 4  
Factorisations of automorphism groups of sporadics

$G$	$A$	$B$	Comments
$\text{Aut}(M_{22})$	$\text{PSL}(3, 4)$	$C_{11} \times C_2$	exact $A$ stabiliser of point in Witt design
	$\text{PSL}(3, 4)$	$C_{11} \times C_{10}$	
	$\text{PSL}(3, 4)$	$\text{PGL}(2, 11)$	
$\text{Aut}(J_2)$	$G_2(2)$	$B \leq 5^2 \times (C_4 \times S_3)$	6 conjugacy classes for $B$ , 2 exact
$\text{Aut}(HS)$	$M_{22}.2$	$B \leq 5_+^{1+2}.[2^5]$	10 conjugacy classes for $B$ , 4 exact $A$ is vertex stabiliser
$\text{Aut}(He)$	$\text{Sp}(4, 4).4$	$B \leq 7_+^{1+2} \times (S_3 \times C_6)$	7 conjugacy classes for $B$ , 3 exact $A$ is vertex stabiliser for Held graph
$\text{Aut}(Suz)$	$G_2(4).2$	$3^5 \times (C_{11} \times C_2)$	
	$G_2(4).2$	$3^5 \times ((C_{11} \times C_5) \times C_2)$	
	$G_2(4).2$	$3^5 \times (\text{PSL}(2, 11) \times C_2)$	

Table 5  
Exact factorisations

$G$	$A$	$B$	Comments
$M_{11}$	$C_{11}$	$M_{10}$	
	$C_{11} \rtimes C_5$	$M_{9,2}$	
$M_{12}$	$M_{11}$	$A_4, D_{12}, C_6 \times C_2$	
	$\text{PSL}(2, 11)$	$M_{9,2}$	$A$ maximal, 2 classes for $B$
$\text{Aut}(M_{12})$	$M_{11}$	$S_4, D_{24}, D_8 \rtimes C_3$	
		$B_{24}^1$	2 classes for $B$ , one meets $T$ in $D_{12}$ , other meets $T$ in $C_6 \times C_2$
	$\text{PGL}(2, 11)$	$M_{9,2}$	$A \cap T$ maximal in $T$
$\text{Aut}(M_{22})$	$\text{P}\Sigma\text{L}(3, 4)$	$C_{11} \rtimes C_2$	
$M_{23}$	$M_{22}$	$C_{23}$	
	$C_{23} \rtimes C_{11}$	$\text{P}\Sigma\text{L}(3, 4), 2^4 \rtimes A_7$	
$M_{24}$	$M_{23}$	$A_4 \times C_2, D_{24}, D_8 \times C_3$	
		$S_4$	3 classes for $B$
		$B_{24}^1$	2 classes $B$
	$\text{PSL}(2, 23)$	$\text{P}\Sigma\text{L}(3, 4), 2^4 \rtimes A_7$	
$\text{Aut}(J_2)$	$G_2(2)$	$B_{100,1}, B_{100,2}^2$	
$\text{Aut}(HS)$	$M_{22}.2$	$B_{100,1}, B_{100,2}, B_{100,3}, B_{100,4}^2$	
$\text{Aut}(He)$	$\text{Sp}(4, 4).4$	$7_+^{1+2} \rtimes C_{6,1}, 7_+^{1+2} \rtimes C_{6,2}^3$	
		$7_+^{1+2} \rtimes C_{6,3}^3$	

<sup>1</sup>  $B_{24} \cong \langle x, y, z \mid z^6 = x^2 = y^2 = 1, y^x = yz^3, z^x = z^{-1}, z^y = z \rangle$

<sup>2</sup>  $B_{100,1} \cong \langle x, y, z \mid x^4 = y^5 = z^5 = 1, y^x = y^2, z^x = z^2, z^y = z \rangle$

$B_{100,2} \cong \langle x, y, z \mid x^4 = y^5 = z^5 = 1, y^x = y^3, z^x = z^2, z^y = z \rangle$

$B_{100,3} \cong \langle x, y, z \mid x^4 = y^5 = z^5 = 1, y^x = y^3, z^x = z^4, z^y = z \rangle$

$B_{100,4} \cong \langle x, y, z \mid x^4 = y^5 = z^5 = 1, y^x = y^2, z^x = z, z^y = z \rangle$

<sup>3</sup>  $\text{Aut}(He)$  has a maximal subgroup  $7_+^{1+2} \rtimes (S_3 \times C_6)$ . Letting  $S_3 = \langle x, y \rangle$  with  $x^3 = y^2 = 1$  and  $C_6 = \langle a, b \rangle$  with  $a^3 = b^2 = 1$  then  $S_3 \times C_6$  contains  $C_{6,1} \langle a, b \rangle$ ,  $C_{6,2} = \langle xa, b \rangle$  and  $C_{6,3} = \langle a, yb \rangle$ , all cyclic of order 6.

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