S_3-involution graphs

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on joint work with Alice Devillers

An M_{11}-graph

Witt design $S(4, 5, 11)$: collection of 5-subsets (pentads) of an 11-set such that any 4-subset is contained in a unique pentad.

Automorphism group is the Mathieu group M_{11} which acts 4-transitively on the 11-set.

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Define a graph Σ

- vertices: 3-subsets of an 11-set
- adjacency: complement of union is a pentad.

Another definition

M_{11} also has a 3-transitive action on a set of size 12.

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Theorem (Devillers,MG,Li,Praeger)

Σ is isomorphic to the graph defined as follows:

- *vertices: elements of $\mathcal{O}$*
- *adjacency: intersection is a 3-subset*

$J(12,4)$ can be decomposed into 12 copies of Σ with the 12 copies transitively permuted by M_{12}.

A $\mathrm{PSL}(2, 11)$-graph

$\mathrm{PSL}(2, 11)$ has a 2-transitive action on 11 points.

Has two orbits on 3-subsets and these have lengths 55 and 110.

The orbit of length 55 forms a $2 - (11, 3, 3)$ design (Petersen design).

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Define a graph

- vertices: blocks of Petersen design
- adjacency: intersection is a 2-subset

An alternative definition

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Equivalent definition of Σ is:

- vertices: involutions of M_{11}
- adjacency: generate an S_3 with normaliser $S_3 \times S_3$.

The $\mathrm{PSL}(2, 11)$ graph has a similar definition

Tower of graphs

Devillers, MG, Li, Praeger

Using this definition get a tower of graphs for the groups

$$A_5 < \mathrm{PSL}(2,11) < M_{11} < M_{12}$$

The graph for A_5 is the line graph of the Petersen graph.

The graph for M_{12} is the Johnson graph $J(12,4)$.

In general

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NB: The product of adjacent involutions has order three.

Reminiscent of:

- Fischer's 3-transposition groups
- Coxeter graphs
- commuting involution graphs

Automorphisms

G acts by conjugation as a group of automorphisms

G-vertex-transitive if and only if X is single conjugacy class

G-arc-transitive if and only if $\mathcal{S}$ is a single conjugacy class

If $g \in \mathrm{Aut}(G)$ fixes X and $\mathcal{S}$ then g induces automorphism of $\Gamma(G, X, \mathcal{S})$.

Full automorphism group can be much larger than G,
eg $G = M_{12}$ and $\Gamma(G, X, \mathcal{S}) = J(12, 4)$

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Thus $\Gamma(G, X, \mathcal{S}) \cong J(n, 2)$.

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The other class yields the Johnson graph $J(11, 3)$.

Complete graphs

$$G = \mathrm{AGL}(1, 3^n) = \{t_{a,b} : x \mapsto ax + b \mid a, b \in \mathrm{GF}(3^n), a \neq 0\}$$

Unique class of involutions $X = \{t_{-1,b} \mid b \in \mathrm{GF}(3^n)\}$.

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$\Gamma(G, X, \mathcal{S}) \cong K_{3^n}$

Theorem

If $\Gamma(G, X, \mathcal{S})$ *is the complete graph on* X *for some group* G *then* $|X| = 3^n$ *for some positive integer* n*.*

Paley graphs

n even

$$\begin{aligned} G &= \{t_{a,b} : x \mapsto ax + b \mid a, b \in \mathrm{GF}(3^n), a = \Box \neq 0\} \\ &\cong C_3^n \rtimes C_{(3^n-1)/2} \end{aligned}$$

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$\Gamma(G, X, \mathcal{S}_1)$ is the Paley graph for $\mathrm{GF}(3^n)$.

Triangles

Each $S \in \mathcal{S}$ contains three involutions and these form a triangle in $\Gamma(G, X, \mathcal{S})$.

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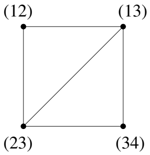

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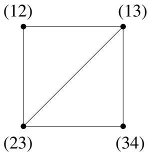

Theorem

If no $S \in \mathcal{S}$ is contained in a subgroup of G of the form $C_3^2 \rtimes C_2$ or $C_n^2 \rtimes S_3$, then the only triangles of $\Gamma(G, X, \mathcal{S})$ are those given by the subgroups of $\mathcal{S}$.

$G = \mathrm{PSL}(2, q)$

Unique conjugacy class of involutions

One or two conjugacy classes of S_3-subgroups but if two then fused in $\mathrm{PGL}(2, q)$.

$q \pmod{12}$	$\lvert X \rvert$	$\lvert \mathcal{S} \rvert$	valency
$4, 8$	$q^2 - 1$	$\lvert G \rvert/6$	q
1	$q(q+1)/2$	$\lvert G \rvert/12$	$(q-1)/2$
3	$q(q-1)/2$	0	
5	$q(q+1)/2$	$\lvert G \rvert/6$	$q-1$
7	$q(q-1)/2$	$\lvert G \rvert/6$	$q+1$
9	$q(q+1)/2$	$\lvert G \rvert/6$	$q-1$
11	$q(q-1)/2$	$\lvert G \rvert/12$	$(q+1)/2$

Theorem

$G = \mathrm{PSL}(2, q)$ for $q \geq 4$,
X the unique conjugacy class of involutions,
$\mathcal{S}$ a conjugacy class of S_3-subgroups.

The size of the largest clique is

- 3^e *if* $q = 9^e$,
- 4 *if* $q = 25^e$,
- 3 *otherwise.*

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For $q = 9^e$ the subgraphs induced on the parabolics $C_3^{2e} \rtimes C_{(9^e-1)/2}$ are Paley graphs.

Duality

The dual graph of $\Gamma(G, X, \mathcal{S})$ is the graph with

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Theorem

$\Gamma(\mathrm{PSL}(2, q), X, \mathcal{S})$ is isomorphic to its dual graph if and only if $q = 11$ or 13.