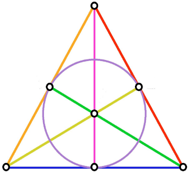
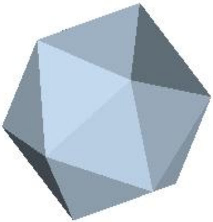
Moonshine and the monster

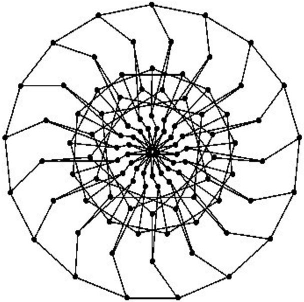
Michael Giudici

Groups

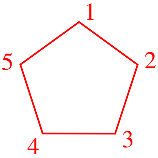
Group theory is the abstract study of symmetry.

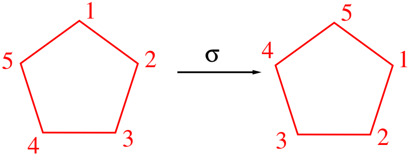






An example



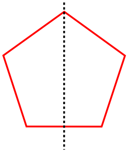


An example II

This gives 5 symmetries of the pentagon.

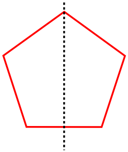
We call the set of these 5 symmetries the cyclic group C_5 of order 5.

More symmetries



A reflection τ about the y-axis is also a symmetry.

More symmetries



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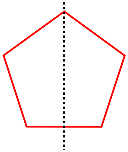
The product of a rotation by θ with τ is a reflection about the y axis rotated by $-\theta/2$.

More symmetries



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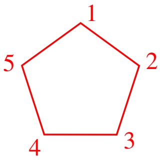

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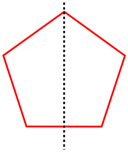

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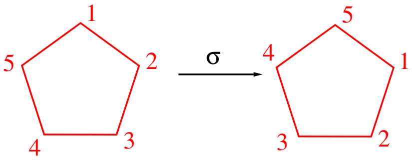

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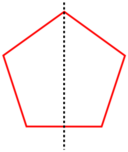

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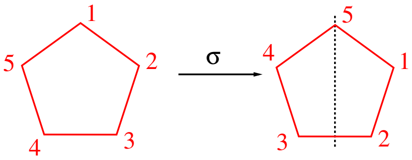

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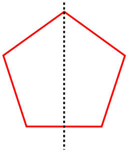

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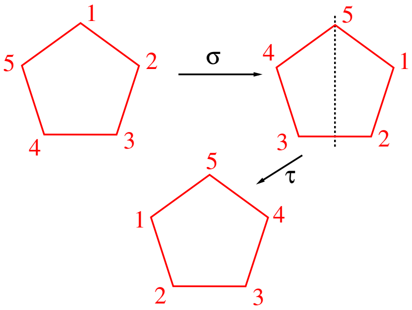

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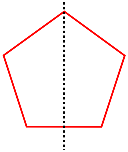

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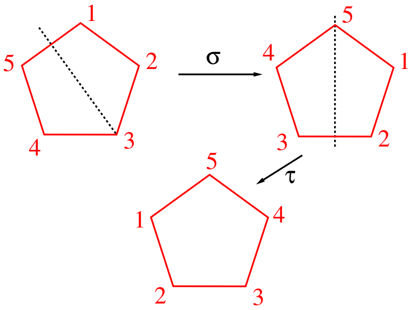

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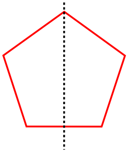

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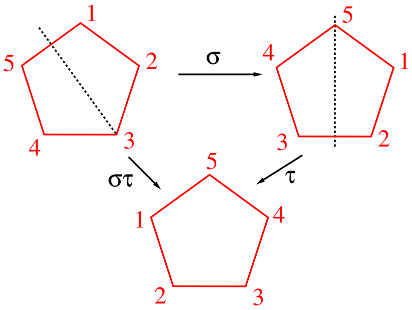

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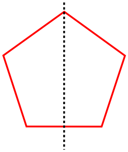

A reflection τ about the y-axis is also a symmetry.

The product of a rotation by θ with τ is a reflection about the y axis rotated by $-\theta/2$.

This gives a total of 10 symmetries

$$\{1, \sigma, \sigma^2, \sigma^3, \sigma^4, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau, \sigma^4\tau\}$$

which we call the dihedral group D_{10}.

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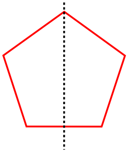

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Note that combining any two elements gives us another element of D_{10}.

Abstract definition of a group

A group is a set G together with an operation $*$ such that

(i)	$g * h \in G$ for all $g, h \in G$.	(Closure)
(ii)	$(g * h) * k = g * (h * k)$	(Associativity)
(iii)	there exists an element $e \in G$ such that $g * e = e * g = g$ for all $g \in G$.	(Identity)
(iv)	for all $g \in G$ there exists $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.	(Inverse)

A subgroup of G is a subset of G which is itself a group.

Some other examples

- $\mathbb{Z}$ with addition
- $\mathrm{GL}(n, \mathbb{C})$ with matrix multiplication
- $\mathrm{Sym}(\Omega)$: all permutations of a set Ω with composition

Group representations

We would like an easy way of representing the elements in a group so that multiplication is easy.

One way is as a group of matrices with elements from $\mathbb{C}$, that is, as a subgroup of $\mathrm{GL}(n, \mathbb{C})$.

For example D_{10} can be represented as 2×2 matrices as follows:

$$\sigma = \begin{pmatrix} \cos(\frac{2\pi}{5}) & \sin(\frac{2\pi}{5}) \\ \sin(\frac{2\pi}{5}) & -\cos(\frac{2\pi}{5}) \end{pmatrix}, \ \tau = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Which n?

Any finite group G can be represented as a subgroup of $\mathrm{GL}(n, \mathbb{C})$ for some n.

We would like n to be minimal in some sense.

Irreducible representations

Let $G \leqslant \mathrm{GL}(n, \mathbb{C})$ be finite.

Suppose that there is subspace U of $\mathbb{C}^n$ which is fixed by every element of G.

Then there is also a subspace W fixed by G such that

$$\mathbb{C}^n = U \oplus W.$$

With respect to a suitable basis we can write each $g \in G$ in the form

$$\begin{pmatrix} A_g & 0 \\ 0 & B_g \end{pmatrix}$$

where $A_g \in \mathrm{GL}(m, \mathbb{C})$ and $B_g \in \mathrm{GL}(n - m, \mathbb{C})$.

Irreducible representations II

We say that a representation of G is irreducible if there is no nontrivial proper subspace fixed by G, that is, G cannot be broken down as above.

We are interested in the irreducible representations of groups.

For a given group they may not all have the same dimension.

Any representation can be built from gluing together irreducible representations.

Simple groups

Given a subgroup H of G and element $g \in G$, let

$$g^{-1}Hg = \{g^{-1} * h * g \mid h \in H\}$$

We say that H is a normal subgroup if $g^{-1}Hg = H$ for all $g \in G$.

Any group G has $\{e\}$ and G as normal subgroups.

We call a group G simple if the only normal subgroups are $\{e\}$ and G.

Back to pentagons

A reflection followed by a rotation followed by the same reflection is another rotation.

Thus the group C_5 of rotations is a normal subgroup of D_{10}.

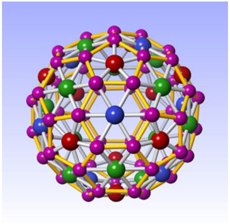
C_5 is simple as it has no nontrivial proper subgroups.

Simple groups II

Jordan-Hölder Theorem

Every finite group has a unique set of simple groups from which it is built.

The simple groups are the atoms of group theory.



Classification of Finite Simple Groups

All finite simple groups have been classified and are one of the following:

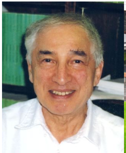
1. cyclic of prime order
2. a member of one of 17 infinite families
3. one of 26 sporadics.

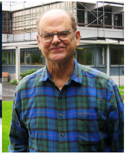
Completed in the early 80s.

Proof spread over hundreds of papers totalling more than 10, 000 pages.

nyt

The Odd Order Theorem





Theorem (Feit-Thompson, 1963)

Every noncyclic simple group has even order.

Paper filled a whole issue of the Pacific Journal of Mathematics

For this and other work, Thompson received the Fields medal in 1970 and the Abel prize in 2008.

The sporadics

First ones were discovered by Mathieu in 1860 and 1873.

$$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$$

with orders

$$7\,920, \quad 95\,040, \quad 443\,520, \quad 10\,200\,960, \quad 244\,823\,040$$

Next one discovered in 1966 by Janko: J_1 having order $175\,560$.

The monster

The largest of the sporadics is known as the Monster, denoted $\mathbb{M}$.

Existence was conjectured in 1973 by Fischer and Griess.

Constructed in 1982 by Griess.

It has order

$$2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{2}.17.19.23.29.31.41.47.59.71$$

$$= 808017424794512875886459904961710757005754368000000000$$

$$\approx 8 \times 10^{53}$$

The monster II

The dimensions of the smallest irreducible representations over $\mathbb{C}$ of the Monster are

$$1, \quad 196\,883, \quad 21\,296\,876, \quad 842\,609\,326$$

Smallest permutation representation is on

$$97\,239\,461\,142\,009\,186\,000 \text{ points.}$$

Contains 20 of the sporadics.

McKay's observation

John McKay noticed in 1978 that

$$196\,884 = 1 + 196\,883$$

To see the significance of this we need to understand modular functions.

Modular functions

The modular group is the group $\mathrm{PSL}(2,\mathbb{Z})$ of all transformations

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}; \quad a, b, c, d \in \mathbb{Z},\ ad - bc = 1$$

of the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} |\, \mathrm{Im}(\tau) > 0\}$.

A modular function is a function f on $\mathbb{H}$ (with some continuity assumptions) such that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau)$$

for all transformations in $\mathrm{PSL}(2,\mathbb{Z})$.

Modular functions II

Invariant under $\tau \mapsto \tau + 1$ and so are periodic with period 1.

Thus have a Fourier expansion

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n q^n, \quad \text{where } q = e^{2\pi i \tau}.$$

The j-function

There is a unique modular function with Fourier expansion

$$f(q) = q^{-1} + \sum_{n=1}^{\infty} a_n q^n$$

and this is

$$j(\tau) = q^{-1} + 196\,884q + 21\,493\,760q^2 + 864\,299\,970q^3 + \cdots$$

All other modular functions can be written in terms of j.

Study goes back to Klein.

Some numerology

$$196\,884 = 1 + 196\,883$$

Is this a coincidence?

Thompson noticed that the next couple of coefficients of j can also be written in terms of the dimensions of irreducible representations of $\mathbb{M}$.

$$21\,493\,760 = 1 + 196\,883 + 21\,296\,876$$

$$864\,299\,970 = 2 \cdot 1 + 2 \cdot 196\,883 + 21\,296\,876 + 842\,609\,326$$

Hauptmodul

We can also look at functions on $\mathbb{H}$ which are invariant under some subgroup G of $\mathrm{PSL}(2, \mathbb{R})$.

In particular, genus 0 subgroups G containing

$$\Gamma_0(N) = \left\{ \tau \mapsto \frac{a\tau + b}{c\tau + d} \in \mathrm{PSL}(2, \mathbb{Z}) \mid N \text{ divides } c \right\}$$

for some integer N.

Hauptmodul II

In such cases there is a unique function J_G with q-expansion

$$J_G(\tau) = q^{-1} + \sum_{n=1}^{\infty} a_n q^n$$

J_G is called the Hauptmodul for G.

Another numerical coincidence

Ogg (1975) remarked that the primes for which the group

$$\left\langle \Gamma_0(p), \frac{1}{\sqrt{p}} \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \right\rangle$$

has genus 0 are precisely

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71$$

These are the prime divisors of $|\mathbb{M}|$.

Ogg offered a bottle of Jack Daniels to anyone who could explain the coincidence.

Moonshine Conjectures

Conway and Norton in a paper in 1979 entitled ‘Monstrous Moonshine’ conjectured:

Conjecture 1

There is a graded $\mathbb{M}$-module

$$V = \oplus_{n \geq -1} V_n$$

where the dimension of V_n is the coefficient of q^n in the j-function.

V is known as the Moonshine module.

Moonshine Conjectures II

If Conjecture 1 holds, then each $g \in \mathbb{M}$ acts on V_n and has character value $\chi_n(g) = Trace(g_{|V_n})$.

Can then form the Thompson-McKay series

$$T_g(q) = \sum_{n \geq -1} \chi_n(g) q^n$$

Conjecture 2

Given the moonshine module, for each $g \in \mathbb{M}$, there is a genus 0 subgroup G of $\mathrm{PSL}(2, \mathbb{R})$ such that $T_g(q)$ is the Hauptmodul for G.

The proof

Frenkel, Lepowsky and Meurman (1984) constructed a graded infinite dimensional representation $V^\natural$ for $\mathbb{M}$.

Borcherds (1992) proved the Moonshine conjectures in work involving vertex operator algebras, generalised Kac-Moody algebras and the Monster Lie algebra.

Vertex operator algebras are used in physics and there is a string theory involving the moonshine module.

Received Fields medal in 1998 for his work.

Why Moonshine?

‘The stuff we were getting was not supported by logical argument. It had the feeling of mysterious moonbeams lighting up dancing Irish leprechauns. Moonshine can also refer to illicitly distilled spirits, and it seemed almost illicit to be working on this stuff.’

- Conway in Mark Ronan’s ‘Symmetry and the Monster’.