

Homogeneous Factorisations of Graphs and Digraphs [★]

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Abstract

A *homogeneous factorisation* $(M, G, \Gamma, \mathcal{P})$ is a partition \mathcal{P} of the arc set of a digraph Γ such that there exist vertex transitive groups $M < G \leq \text{Aut}(\Gamma)$ such that M fixes each part of \mathcal{P} setwise while G acts transitively on \mathcal{P} . Homogeneous factorisations of complete graphs have previously been studied by the second and fourth authors, and are a generalisation of vertex-transitive self-complementary digraphs. In this paper we initiate the study of homogeneous factorisations of arbitrary graphs and digraphs. We give a generic group theoretic construction and show that all homogeneous factorisations can be constructed in this way. We also show that the important homogeneous factorisations to study are those where G acts transitively on the set of arcs of Γ , M is a normal subgroup of G and G/M is a cyclic group of prime order.

Key words: Homogeneous factorisations, vertex-transitive graphs

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1 Introduction

The study of homogeneous factorisations of complete graphs was initiated in [18] by the second and fourth authors, motivated in part by the important special subclass corresponding to vertex-transitive self-complementary digraphs. The aim of this paper is to set the foundations for the study of homogeneous factorisations of arbitrary graphs and digraphs.

Let Γ be a graph or digraph with vertex set $V\Gamma$ and arc set $A\Gamma$. An *isomorphic factorisation* of Γ is a partition $\{P_1, P_2, \dots, P_k\}$ of the arc set $A\Gamma$ such that the induced subdigraphs $(V\Gamma, P_1), (V\Gamma, P_2), \dots, (V\Gamma, P_k)$ are pairwise isomorphic. Such factorisations have been extensively studied, see for example [12,13]. Homogeneous factorisations are a special class of isomorphic factorisations. Let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a partition of the arc set $A\Gamma$ of Γ with $k \geq 2$, and let $M < G$ be subgroups of $\text{Aut}(\Gamma)$, which are transitive on the vertex set $V\Gamma$ of Γ , such that

- (i) M fixes each P_i setwise, and
- (ii) the partition \mathcal{P} is G -invariant and the induced action $G^{\mathcal{P}}$ of G on \mathcal{P} is transitive.

We say that $(M, G, \Gamma, \mathcal{P})$ is a *homogeneous factorisation* (of the digraph Γ) of index k . We call the elements of \mathcal{P} *factors* and sometimes also refer to the corresponding subdigraphs as factors. A subset O of $V\Gamma \times V\Gamma$ is called *symmetric* if $(u, v) \in O$ if and only if $(v, u) \in O$. If each part of the partition \mathcal{P} is symmetric, then the factorisation is called *symmetric*. If the arc set of Γ is symmetric then Γ may be regarded as an undirected graph whose edges are the unordered pairs $\{u, v\}$ such that (u, v) is an arc. In this setting, symmetric homogeneous factorisations may be regarded as partitions of the edge set of Γ as opposed to the arc set.

1.1 Relationship to self-complementary digraphs

Let Γ be a digraph on n vertices and let $\bar{\Gamma}$ be the complement of Γ , that is, (u, v) is an arc of $\bar{\Gamma}$ if and only if it is not an arc of Γ . We say that Γ is *self-complementary* if Γ is isomorphic to $\bar{\Gamma}$. Vertex-transitive self-complementary digraphs have been extensively studied, see for example [22,28–30] and have been shown to exist if and only if $n \equiv 1 \pmod{2}$ while vertex-transitive self-complementary graphs exist if and only if $p^k \equiv 1 \pmod{4}$ for every prime power p^k dividing n such that p^{k+1} does not [23]. For a homogeneous factorisation (M, G, K_n, \mathcal{P}) of index 2 on the vertex set V , the factors $\Gamma_1 = (V, P_1)$

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and $\Gamma_2 = (V, P_2)$ are a pair of vertex-transitive self-complementary digraphs (or graphs if \mathcal{P} is symmetric). Conversely, if $\Gamma = (V, P)$ is a vertex-transitive self-complementary (di)graph of order n and $\sigma : \Gamma \rightarrow \bar{\Gamma}$ is an isomorphism then $(\text{Aut}(\Gamma), \langle \text{Aut}(\Gamma), \sigma \rangle, K_n, \{P, \bar{P}\})$ is a homogeneous factorisation of index 2, where $\bar{\Gamma} = (V, \bar{P})$. Thus the study of self-complementary vertex-transitive digraphs is equivalent to the study of a special class of homogeneous factorisations. It was this connection which motivated the initial work of the second and fourth authors on homogeneous factorisations of complete graphs and gave rise to the first infinite family of vertex-transitive self-complementary graphs which are not Cayley graphs [16]. Particular attention has been paid to the cases where Γ is a certain type of graph, for example vertex-transitive graphs [15,24,33,34], circulant graphs [1,8,14,20], and vertex-transitive non-Cayley graphs [16].

Homogeneous factorisations of index two of digraphs other than the complete graph give rise to what have previously been referred to as *relative self-complementary graphs*. In [5], cyclically almost self-complementary circulant graphs were studied. In our terminology, these are precisely the symmetric homogeneous factorisations of the complete multipartite graph $K_{2,\dots,2}$ of index 2 where M is a cyclic group acting regularly on vertices. Also, the case where $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index 2 which is not symmetric and for which each part in \mathcal{P} is an orbit of M , is equivalent to the existence of a half-arc-transitive group M of automorphisms of Γ such that $\text{Aut}(\Gamma)$ has an element σ which switches the orientation of the edges of Γ induced by the action of M . (A group of automorphisms of Γ is *half-arc-transitive* if it is vertex and edge-transitive but not arc-transitive.) Half-arc-transitive graphs have received much attention and such triples (M, σ, Γ) were used in [21] to define new families of *semisymmetric graphs*, that is, regular edge-transitive graphs which are not vertex-transitive.

1.2 Existence of a factorisation

In Section 2 we investigate restrictions on M , G , Γ and \mathcal{P} needed for the existence of a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$. This investigation leads us to posing the following problem.

- Problem 1.1** (1) *Determine which digraphs Γ admit a homogeneous factorisation.*
- (2) *Determine the family \mathcal{G} of vertex-transitive digraphs that occur as factors of homogeneous factorisations. Further, for $\Sigma \in \mathcal{G}$ describe the digraphs Γ for which Σ occurs as a factor of a homogeneous factorisation of Γ .*

For example, we see in Lemma 2.11 that the Petersen graph does not belong to \mathcal{G} . However, we also construct an infinite family of homogeneous factorisations where the connected components of each factor are Petersen graphs.

We give a generic construction of a homogeneous factorisation in Construction 3.1. In fact we prove the following necessary and sufficient conditions on the existence of a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ in Section 3 and show that all homogeneous factorisations arise from this construction. Given a set Ω and a permutation group X , we denote the set of orbits of X on Ω by $\text{Orb}(X, \Omega)$.

Theorem 1.2 *Let Γ be a digraph, and let $M \triangleleft G$ be vertex-transitive subgroups of $\text{Aut}(\Gamma)$. Let \mathcal{O} denote the set of orbits of G on $\text{Orb}(M, A\Gamma)$ and k be an integer, $k \geq 2$. Then:*

- (i) *There exists a partition \mathcal{P} of $A\Gamma$ such that $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index k if and only if, for each $O \in \mathcal{O}$, there exists a G -invariant partition \mathcal{B}_O of O with k parts such that the actions of G on \mathcal{B}_O and \mathcal{P} are equivalent.*
- (ii) *There exists a partition \mathcal{P} of $A\Gamma$, such that $(M, G, \Gamma, \mathcal{P})$ is a symmetric homogeneous factorisation of index k if and only if the conditions in part (i) hold and also, for each symmetric $O \in \mathcal{O}$ (if any exist), there exists a partition \mathcal{B}_O as in part (i), each part of which is symmetric.*

Given a vertex $v \in V\Gamma$, we let $\Gamma^+(v) = \{w \in V\Gamma \mid (v, w) \in A\Gamma\}$. Instead of analysing the global action of a group G to determine if a digraph Γ has a homogeneous factorisation, it is often easier to study the local action, by which we mean the action of the vertex stabiliser G_v on $\Gamma^+(v)$. It transpires that all the information about the action of G on the factors of a homogeneous factorisation is encoded in the action of G_v on an induced partition of $\Gamma^+(v)$. This enables us to give a local version (Theorem 3.5) of the characterisation in Theorem 1.2.

1.3 Basic factorisations

A *cyclic homogeneous factorisation* $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation such that $G^{\mathcal{P}}$ is a cyclic group. In Theorem 4.5 we show that, given a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ of index k , there exist a prime p dividing k , a subgroup G' of G containing M and a partition \mathcal{Q} of $A\Gamma$ refined by \mathcal{P} , such that $(M, G', \Gamma, \mathcal{Q})$ is a cyclic homogeneous factorisation of index p . Thus understanding cyclic homogeneous factorisations is crucial for understanding homogeneous factorisations in general.

It was shown in [18] that when studying cyclic homogeneous factorisations of

complete graphs, one can reduce to the case where G acts primitively on vertices. This was done via a quotienting process which provided a related homogeneous factorisation of a quotient graph. Cyclic homogeneous factorisations of complete graphs where G is vertex-primitive were studied in [10] and are related to the exceptionality of permutation groups, a concept studied in [11]. However, in the case of a general digraph Γ , it is not possible to take a quotient of an arbitrary cyclic homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$. We overcome this obstacle by studying arc-transitive, cyclic homogeneous factorisations. A homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ for which G acts transitively on $A\Gamma$ is called *arc-transitive*. If $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index k and O is an orbit of G on $A\Gamma$ then Construction 4.2 yields a homogeneous factorisation $(M, G, \Gamma_O, \mathcal{P}_O)$, where Γ_O is the subdigraph of Γ whose arc set is O . Furthermore, $G^{\mathcal{P}}$ is permutationally isomorphic to $G^{\mathcal{P}_O}$. Thus in light of the discussion in the previous paragraph, given a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ we can derive a cyclic homogeneous factorisation $(M, G', \Gamma, \mathcal{Q}_O)$ of prime index p for which G' acts transitively on the set of arcs of Γ_O . Hence arc-transitive, cyclic homogeneous factorisations $(M, G, \Gamma, \mathcal{P})$ are an important family to study. Lim [19] has recently investigated arc-transitive homogeneous factorisations of complete graphs. In his study he also required that M should be arc-transitive on each factor and came close to a classification of this type of factorisation.

Let $(M, G, \Gamma, \mathcal{P})$ be an arc-transitive, cyclic homogeneous factorisation such that $G/M \cong C_p$ for some prime p . We say that $(M, G, \Gamma, \mathcal{P})$ is *basic* if Γ is connected and every nontrivial normal subgroup of G contained in M has at most two orbits on vertices. If M contains a normal subgroup N of G which has at least three orbits on $V\Gamma$, we can either induce a homogeneous factorisation on the quotient digraph of Γ modulo the N -orbits or we can study the bipartite subdigraph induced on a pair of adjacent N -orbits. In the latter case we induce either a homogeneous factorisation or a closely related factorisation called a bihomogeneous factorisation. Let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a partition of the arc set of some bipartite digraph Γ and suppose that there exist, $N < H \leq \text{Aut}(\Gamma)$ such that N fixes each P_i setwise while H acts transitively on \mathcal{P} . If both N and H are transitive on each part of the bipartition of Γ , but are not transitive on $V\Gamma$, then we call $(N, H, \Gamma, \mathcal{P})$ a *bihomogeneous factorisation* of index k . Bi-homogeneous factorisations share many of the properties of homogeneous factorisations and often arise when studying factorisations induced on bipartite subdigraphs.

A transitive permutation group G is said to be *quasiprimitive* if every normal subgroup of G is transitive and is called *biquasiprimitive* if every nontrivial normal subgroup has at most two orbits and there exists at least one nontrivial intransitive normal subgroup. We prove the following characterisation of basic factorisations in Section 5.

Theorem 1.3 *Let $\mathcal{F} = (M, G, \Gamma, \mathcal{P})$ be an arc-transitive, cyclic homogeneous factorisation of prime index p such that $G/M \cong C_p$. If \mathcal{F} is basic then one of the following holds.*

- (1) G is quasiprimitive or biquasiprimitive on $V\Gamma$,
- (2) $G = C_p \times M$ where M is quasiprimitive or biquasiprimitive on $V\Gamma$.

Furthermore, each of these four possibilities arises. Conversely, if \mathcal{F} is not basic then there is a smaller graph and a corresponding smaller arc-transitive, cyclic homogeneous or bihomogeneous factorisation.

The possible structures of finite quasiprimitive permutation groups were analysed by the fourth author and a summary is given in [26] identifying eight types of such groups. Similarly the structure of biquasiprimitive groups was studied in [27]. This suggests the following problem.

Question 1.4 *Determine the types of quasiprimitive and biquasiprimitive groups which arise for basic, arc-transitive, cyclic homogeneous factorisations.*

2 Restrictions on M, G, Γ and \mathcal{P}

In this section we investigate restrictions, if any, that hold on the parameters M, G, Γ and \mathcal{P} of a homogeneous factorisation of index k .

2.1 On the normality of M

Let $(M, G, \Gamma, \mathcal{P})$ be a homogeneous factorisation. Then M is not necessarily normal in G . However, the kernel K of the action of G on \mathcal{P} is such that $M \leq K \triangleleft G$, and $(K, G, \Gamma, \mathcal{P})$ is also a homogeneous factorisation. Thus assuming that $M \triangleleft G$ does not reduce the collection of digraphs Γ and partitions of $A\Gamma$ which give rise to homogeneous factorisations.

Let $(M, G, \Gamma, \mathcal{P})$ be a homogeneous factorisation of index k . Each $P_i \in \mathcal{P}$ is a union of orbits of M on $A\Gamma$ and so since G acts transitively on \mathcal{P} it follows that G fixes no orbit of M on $A\Gamma$. The 2-closure of M , denoted by $M^{(2)}$, is the group of all permutations of $V\Gamma$ which fixes setwise each orbit of M on $(V\Gamma)^{(2)}$, the set of ordered pairs of distinct elements from $V\Gamma$. If $M = M^{(2)}$ we say that M is 2-closed. The concept of 2-closure was introduced and initially studied by Wielandt in [32]. Here we relate the existence of a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ involving a given transitive group M with a property of the 2-closure $M^{(2)}$.

Lemma 2.1 *Let M be a transitive permutation group on a finite set V . Then the following are equivalent.*

- (1) *There exist a digraph Γ with vertex set V , a subgroup G of $\text{Aut}(\Gamma)$ with $M \triangleleft G$, and a partition \mathcal{P} of $A\Gamma$ such that $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation.*
- (2) $N_{\text{Sym}(V)}(M) \not\leq M^{(2)}$.

PROOF. If $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation with $M \triangleleft G$, then G fixes no orbit of M on $A\Gamma$ and hence G is not contained in $M^{(2)}$. Since $G \leq N_{\text{Sym}(V)}(M)$ it follows that $N_{\text{Sym}(V)}(M)$ is not contained in $M^{(2)}$.

Conversely, suppose that $G = N_{\text{Sym}(V)}(M)$ is not contained in $M^{(2)}$. Then since $M \triangleleft G$ it follows that G preserves the set of orbitals of M . Furthermore, as G is not contained in $M^{(2)}$, G has a nontrivial orbit A on the set of orbitals of M . Let Γ be the digraph with vertex set V and arc set the union of the M -orbitals in A , and let \mathcal{P} be the set of orbitals of M contained in A . Then $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation with $M \triangleleft G$. ■

If we are given a graph Γ which is not a complete graph and an $M < \text{Aut}(\Gamma)$ which is vertex transitive then $A\Gamma \neq (V\Gamma)^{(2)}$ and $M^{(2)}$ is not always the appropriate subgroup to examine to decide whether there is a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ for some G and \mathcal{P} . We define $M^{(\Gamma)}$, called the Γ -closure of M , as the group of all permutations of $V\Gamma$ which fix setwise each orbit of M on $A\Gamma$. If $\Gamma = K_n$, then $M^{(\Gamma)} = M^{(2)}$, and in general $M^{(2)} \leq M^{(\Gamma)}$.

We have the following lemma. We say that $\text{Orb}(M, A\Gamma)$ is G -invariant if for each $P \in \text{Orb}(M, A\Gamma)$ and $g \in G$, the image P^g lies in $\text{Orb}(M, A\Gamma)$.

Lemma 2.2 *Let Γ be a digraph, and let $M, G \leq \text{Aut}(\Gamma)$ with M vertex-transitive. Then $\text{Orb}(M, A\Gamma)$ is G -invariant if and only if G normalises $M^{(\Gamma)}$.*

PROOF. Suppose that G normalises $M^{(\Gamma)}$, and let $g \in G$ and $P \in \text{Orb}(M, A\Gamma)$. Then for all $m \in M$, there exists $m' \in M^{(\Gamma)}$ such that $gm = m'g$. Hence $(P^g)^m = P^{gm} = P^{m'g} = P^g$ and so P^g is M -invariant. A similar argument shows that P^g is an orbit of M . Thus $\text{Orb}(M, A\Gamma)$ is G -invariant.

Conversely, suppose that G preserves $\text{Orb}(M, A\Gamma)$ and let $m \in M^{(\Gamma)}$. Let P be an orbit of M on $A\Gamma$. Then for all $g \in G$, $P^{g^{-1}}$ is an orbit of M on $A\Gamma$ and so $P^{g^{-1}m} = P^{g^{-1}}$. Hence $P^{g^{-1}mg} = P$. As this holds for all $P \in \text{Orb}(M, A\Gamma)$, it follows that $g^{-1}mg \in M^{(\Gamma)}$ and so G normalises $M^{(\Gamma)}$. ■

It was mentioned in [18] that for all homogeneous factorisations $(M, G, \Gamma, \mathcal{P})$ of complete graphs then known to the authors, the partition $\text{Orb}(M, A\Gamma)$ was G -invariant. This however is not necessarily the case. Suppose that $\mathcal{F} =$

$(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation, where $\Gamma = K_n$, and suppose that M contains a regular subgroup N which is not normal in G . Then (N, G, K_n, \mathcal{P}) is also a homogeneous factorisation and since N is regular, $N^{(2)} = N$ (see [32, Theorem 5.12]). Now G does not normalise N and hence does not normalise $N^{(2)}$ and so by Lemma 2.2, the partition $\text{Orb}(N, A\Gamma)$ is not G -invariant. If \mathcal{F} is one of the homogeneous factorisations given in [10, Theorem 1.2 (iii)], then the socle of G has a regular subgroup N which is not normal in G . Hence these factorisations provide explicit examples of this behaviour.

Lemma 2.2 has the following corollary, the statement and proof of which are analogous to those for Lemma 2.1.

Corollary 2.3 *Let $\Gamma = (V, A\Gamma)$ be a digraph, and M a vertex-transitive subgroup of $\text{Aut}(\Gamma)$. Then the following are equivalent.*

- (1) *There exist G satisfying $M \leq G \leq \text{Aut}(\Gamma)$, a G -invariant subdigraph $\Gamma' = (V, A')$ and a partition \mathcal{P} of A' such that $(M, G, \Gamma', \mathcal{P})$ is a homogeneous factorisation.*
- (2) $N_{\text{Aut}(\Gamma)}(M) \not\leq M^{(\Gamma)}$.

We conclude this section with the following striking consequence of Lemma 2.1.

Proposition 2.4 *For a given group M , there is a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ for some G, Γ and \mathcal{P} if and only if $|M| > 2$.*

PROOF. Suppose $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation. Then $|M| \geq 2$. Suppose $|M| = 2$. Then $\Gamma = K_2$, but then M is transitive on $A\Gamma$; contradiction. Hence $|M| > 2$.

Conversely, let M be a group of order greater than two. We claim that M has a nontrivial automorphism σ . If M is nonabelian then some element of M induces a nontrivial automorphism by conjugation. On the other hand, if M is abelian then it is the direct product of cyclic groups of prime power order. Cyclic groups of order at least three have nontrivial automorphisms and these induce nontrivial automorphisms of M , while if M is the direct product of more than one cyclic group of order two then there is an automorphism of M which permutes them.

Let $N = M \rtimes \langle \sigma \rangle$ be a subgroup of $\text{Sym}(M)$, such that M acts by right multiplication and σ acts by conjugation. Since M is regular, [32, Theorem 5.12] implies that $M = M^{(2)}$. Further, $M \triangleleft N$ and so Lemma 2.1 implies that there exists a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$. ■

2.2 Restrictions on \mathcal{P}

Recall that given a digraph Γ and vertex v then $\Gamma^+(v) = \{w \in V\Gamma \mid (v, w) \in A\Gamma\}$. We also define $\Gamma^-(v) = \{w \in V\Gamma \mid (w, v) \in A\Gamma\}$. The *in-degree* of a vertex $v \in V\Gamma$ is the size of $\Gamma^-(v)$ while the *out-degree* of $v \in V\Gamma$ is the size of $\Gamma^+(v)$. We say that Γ is *regular* if there is a constant r such that for each vertex v , both the in-degree and out-degree of v is r . We call r the *valency* of Γ . Note that if Γ is vertex-transitive, then Γ is regular.

For homogeneous factorisations of the complete graphs K_n , the index k divides $n - 1$ (see [18, Lemma 2.5]). We have an analogous restriction in the general case.

Lemma 2.5 *Let $(M, G, \Gamma, \mathcal{P})$ be a homogeneous factorisation of index k . Then k divides the valency of Γ .*

PROOF. Let Γ_i be a factor of \mathcal{P} . Then since M is a vertex-transitive subgroup of $\text{Aut}(\Gamma_i)$, Γ_i is regular of valency r_i . Furthermore, as G permutes the Γ_i transitively, r_i is independent of i . Hence, given a vertex v of Γ , its valency is equal to kr_i as required. ■

Given an integer λ , a λ -*factor of a digraph* Γ is a subdigraph of Γ with the same vertex set as Γ which has valency λ . The case where $\lambda = 1$ is of particular interest. For a digraph Σ of valency 1, each vertex has in-degree and out-degree 1, and hence each connected component of Σ is either a directed cycle of length at least 3 or an undirected graph on two vertices. Moreover, if Σ is vertex-transitive then its components are pairwise isomorphic. If a digraph of valency 1 can be regarded as an undirected graph, that is to say, if each of its connected components is an undirected graph K_2 , then we call it a *perfect matching*.

Let $\mathcal{F} = (M, G, \Gamma, \mathcal{P})$ be a homogeneous factorisation, such that all the factors are λ -factors of Γ . Then \mathcal{F} is called a *homogeneous λ -factorisation* of Γ . The following is a simple corollary of Lemma 2.5 and its proof is omitted.

Corollary 2.6 *If $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation and Γ has prime valency p , then $(M, G, \Gamma, \mathcal{P})$ is a homogeneous 1-factorisation.*

A *walk* in a digraph Γ between $u, v \in V\Gamma$ is a sequence of vertices $u = v_0, v_1, \dots, v_n = v$, such that for each $i \in \{1, \dots, n\}$ at least one of the pairs (v_{i-1}, v_i) or (v_i, v_{i-1}) is an arc of Γ . A digraph Γ is *connected* if for any two vertices $u, v \in V\Gamma$ there exists a walk between u and v .

Lemma 2.7 *If $(M, G, \Gamma, \mathcal{P})$ is a homogeneous 1-factorisation of a connected digraph Γ , then M acts regularly on V and coincides with the kernel of the*

action of G on \mathcal{P} . In particular, Γ is a Cayley graph of M , and M is normal in G .

PROOF. Let $v \in V$, $g \in M_v$, and $u \in V \setminus \{v\}$. Since Γ is connected, there exists a walk $v = v_0, v_1, \dots, v_{r-1}, v_r = u$ in Γ from v to u . We will show that $v_i^g = v_i$ for each $i \in \{0, 1, \dots, r\}$. Suppose this is not true, and let $i \in \{0, 1, \dots, r\}$ be minimal, such that $v_i^g \neq v_i$. Since $v_0^g = v_0$, $i > 0$. Suppose first that $(v_{i-1}, v_i) \in A\Gamma$. Since M acts trivially on \mathcal{P} , the arcs (v_{i-1}, v_i) and $(v_{i-1}^g, v_i^g) = (v_{i-1}, v_i^g)$ belong to the same member A_i of \mathcal{P} , so v_{i-1} has out-degree at least two in the factor $\Gamma_i \in \mathcal{P}$, contradicting the assumption that (M, G, V, \mathcal{P}) is a 1-factorisation of Γ . A similar argument shows that $(v_i, v_{i-1}) \in A\Gamma$ leads to a contradiction. Thus g fixes every vertex $u \in V$ and so $g = 1$. Hence M acts regularly on V . Let K be the kernel of the action of G on \mathcal{P} . Then $M \leq K$ and (K, G, V, \mathcal{P}) is also a homogeneous 1-factorisation of Γ . Our argument above shows that K acts regularly on V and so $M = K$. ■

Corollary 2.8 *Suppose Γ is a connected digraph of prime valency. Then Γ has a nontrivial homogeneous factorisation if and only if there exists $M, G \leq \text{Aut}(\Gamma)$ such that $M \triangleleft G$, Γ is a Cayley digraph for M and G is arc-transitive.*

PROOF. Let p be the valency of Γ . If Γ has a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$, then Corollary 2.6 implies that $(M, G, \Gamma, \mathcal{P})$ is a homogeneous 1-factorisation. Then by Lemma 2.7, M acts regularly on vertices, $M \triangleleft G$ and so Γ is a Cayley digraph of M . Since each factor $\Gamma_i = (V, P_i)$ is a 1-factor it follows that M is transitive on each part $P_i \in \mathcal{P}$. Then since G acts transitively on \mathcal{P} , it follows that G is transitive on $\cup P_i = A\Gamma$.

Conversely, suppose that Γ is a Cayley digraph for M and G is arc-transitive. Then M is regular on $V\Gamma$ and so M has p orbits on the arcs of Γ . As G is arc-transitive the M -orbits in $A\Gamma$ form the partition \mathcal{P} of a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$. ■

Thus requiring Γ to have prime valency and the factors to be 1-factors forces Γ to be an arc-transitive Cayley digraph. This suggested to us that there may be restrictions on the digraphs that occur as factors of homogeneous factorisations, and for a given factor that does arise, there may be restrictions on the graphs Γ that it factorises. This motivated Problem 1.1 posed in the Introduction. We note without proof the following simple restrictions.

Lemma 2.9 *Let Σ be a digraph with n vertices and valency r .*

- (1) *If $r \geq n/2$ then Σ cannot be a factor of a homogeneous factorisation.*
- (2) *If $r = \frac{n-1}{2}$, then Σ is a factor of a homogeneous factorisation if and only if Σ is a vertex-transitive self-complementary digraph.*
- (3) *If $r = \frac{n-2}{2}$, then Σ is a factor of a homogeneous factorisation if and only*

if it is a factor of a homogeneous factorisation of $K_{2,2,\dots,2}$ of index 2.

Case 3 of Lemma 2.9 is investigated in [25]. One of the smallest cases with $r < \frac{n-2}{2}$ and $r > 2$ is the Petersen graph P with $r = 3$ and $n = 10$. It transpires that P cannot occur as a factor. However, there exist homogeneous factorisations where the connected components of the factors are Petersen graphs. Before demonstrating this we need first to introduce the concept of a coset graph.

Let G be a group with a core-free subgroup H . Given $g \in G \setminus N_G(H)$ such that $g^2 \in H$, we define the coset graph $\Gamma = \text{Cos}(G, H, HgH)$ to have vertex set $[G : H]$, the set of right cosets of H in G , with two cosets Hx and Hy adjacent if and only if $xy^{-1} \in HgH$. The group G acts by right multiplication as an arc-transitive group of automorphisms of Γ . The valency of Γ is equal to $|H : H \cap H^g|$ and Γ is connected if and only if $\langle H, g \rangle = G$. For more details and proofs of these assertions see for example [6].

Construction 2.10 Let $p \equiv 1 \pmod{10}$ be a prime, $M = \text{PSL}(2, p)$ and $G = \text{PGL}(2, p)$. Then M has two conjugacy classes of maximal subgroups isomorphic to A_5 which are fused in G . Let $L_1 \leq M$ be such that $L_1 \cong A_5$, and let $H \leq L_1$ be such that $H \cong S_3$. All subgroups of M isomorphic to S_3 are conjugate. Hence there exists an involution $\tau \in G \setminus M$ that normalises H , and $K = \langle H, \tau \rangle = S_3 \times C_2$.

Let $g_1 \in L_1$ be an involution such that $\langle H, g_1 \rangle = L_1$ and $H \cap H^{g_1} = C_2$. Let $L_2 = L_1^{\tau}$ and $g_2 = g_1^{\tau}$. Then $L_2 < M$ is not conjugate in M to L_1 , $L_1 \cap L_2 = H$, and $\langle L_1, L_2 \rangle = M$. Further, $H \cap H^{g_2} = C_2$.

For each $i = 1, 2$, let $\Gamma_i = \text{Cos}(M, H, Hg_iH)$ and let $\mathcal{P} = \{A\Gamma_1, A\Gamma_2\}$. Finally, let Γ be the graph with vertex set $[M : H]$ and arc set $A\Gamma_1 \cup A\Gamma_2$.

We now have the following lemma.

Lemma 2.11 (1) *The Petersen graph does not occur as a factor in a homogeneous factorisation.*

(2) *The 4-tuple $(M, G, \Gamma, \mathcal{P})$ in Construction 2.10 is a homogeneous factorisation of index 2 such that each factor is a disjoint union of Petersen graphs.*

PROOF.

(1) Suppose that $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index $k \geq 2$ with a factor Γ_1 isomorphic to the Petersen graph. Then Γ has 10 vertices and by Lemma 2.5, $k \leq 3$. The only vertex transitive subgroups M of $\text{Aut}(\Gamma_1)$ are $M = A_5$, S_5 or $\text{AGL}(1, 5)$. In the first two cases $A_5 = \text{soc}(M) \triangleleft G$. As A_5 is primitive on $V\Gamma$ it follows that $G \leq S_5 = \text{Aut}(\Gamma_1)$,

which fixes $A\Gamma_1$ setwise and this is a contradiction as G permutes the factors transitively. Thus $M = \text{AGL}(1, 5)$. Let $v \in V\Gamma$. Then $|M_v| = 2$ and M_v has four orbits of length 2 on $V\Gamma \setminus \{v\}$ and one orbit of length 1. As each factor is isomorphic to the Petersen graph, the neighbourhood of v in each factor has size three. However, each neighbourhood of v in a factor is fixed setwise by M_v and so is a union of M_v -orbits. Thus there can only be one factor of valency three, contradicting the existence of a factorisation with $k \geq 2$.

- (2) Recall the notation in Construction 2.10, and in particular note that $\Gamma_i = \text{Cos}(M, H, Hg_iH)$. Now as $\langle H, g_i \rangle = L_i \neq M$ for each i , then each Γ_i is disconnected with connected components isomorphic to $\Sigma_i = \text{Cos}(L_i, H, Hg_iH)$. Each Σ_i has $|L_i : H| = 10$ vertices and has valency $|H : H \cap H^{g_i}| = 3$. Furthermore, each Σ_i has a group of automorphisms isomorphic to A_5 and so is isomorphic to the Petersen graph. Thus all that remains to be proven is that $\mathcal{P} = \{A\Gamma_1, A\Gamma_2\}$ is a partition of the arc set of Γ and that the two parts are interchanged by τ . Now both Γ_1 and Γ_2 have the same vertex set $[M : H]$ and two vertices Hx and Hy are adjacent in Γ_1 if and only if $xy^{-1} \in Hg_1H$ while they are adjacent in Γ_2 if and only if $xy^{-1} \in Hg_2H$. As $L_1 \cap L_2 = H$ and $g_1 \in L_1$ while $g_2 \in L_2$, it follows that the two double cosets Hg_1H and Hg_2H are disjoint. Thus the arc sets of Γ_1 and Γ_2 are disjoint and so \mathcal{P} is a partition of $A\Gamma$. The element τ acts on $V\Gamma$ by mapping each coset Hx to Hx^τ . Furthermore, if (Hx, Hy) is an arc in Γ_1 then $xy^{-1} \in Hg_1H$. Hence as τ normalises H it follows that $x^\tau(y^{-1})^\tau \in (Hg_1H)^\tau = Hg_1^\tau H = Hg_2H$. Thus τ interchanges $A\Gamma_1$ and $A\Gamma_2$ and so $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index 2. ■

2.3 Restrictions on Γ

We have the following necessary condition on Γ for the existence of a homogeneous factorisation.

Lemma 2.12 *If a graph Γ has a homogeneous factorisation then there exists a pair of vertex-transitive subgroups $M, G \leq \text{Aut}(\Gamma)$ such that $M \triangleleft G$ and $\text{Orb}(M, A\Gamma) \neq \text{Orb}(G, A\Gamma)$.*

PROOF. If $(N, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation and M is the kernel of the action of G on \mathcal{P} , then $M \triangleleft G$, and M and G are vertex-transitive. Furthermore, if $P \in \mathcal{P}$, then P is a union of orbits of M on $A\Gamma$ and G fixes none of these M -orbits setwise. Hence $\text{Orb}(M, A\Gamma) \neq \text{Orb}(G, A\Gamma)$. ■

This simple condition allows us to show easily that certain graphs do not have homogeneous factorisations. For example, we could use it to exclude the

Petersen graph.

Lemma 2.13 *The Petersen graph admits no homogeneous factorisation.*

PROOF. The only vertex-transitive automorphism groups are S_5 , A_5 and $\text{AGL}(1, 5)$. Since $\text{AGL}(1, 5)$ is not normal in either A_5 or S_5 , and since S_5 and A_5 are both arc-transitive, there are no subgroups M, G satisfying the conditions of Lemma 2.12. Thus the Petersen graph has no homogeneous factorisation. ■

It is interesting to study the homogeneous factorisations for certain special families of graphs. For example, in [9], a study of the case where Γ is a complete multipartite graph with s parts of size k , was begun. Sufficient and necessary conditions on k, s and t for the existence of a homogeneous factorisation of index k were determined. The existence of homogeneous factorisations in the Johnson graphs has been investigated by Cuaresma [3] who, as well as constructing some interesting factorisations of $J(n, 2)$ and $J(n, 3)$ for various n , has proved the following.

Theorem 2.14 *The Johnson graphs $J(n, r)$ with $4 \leq r \leq n/2$ do not admit a homogeneous factorisation.*

It would be interesting to know which families of distance transitive graphs admit homogeneous factorisations. Also the question of determining precisely the values of n for which the complete graph K_n admits a homogeneous factorisation (M, G, K_n, \mathcal{P}) is still open. In the case where $M \triangleleft G$ and G/M is cyclic of order k , the possible values of n for a given k were determined in [18, Theorem 1.1]. The following lemma provides an additional partial answer.

Lemma 2.15 *Let $\Gamma = K_n$ such that $n - 1$ is a prime. Then there exists a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ if and only if $n = 2^r$ or 3.*

PROOF. Suppose first that $n = 2^r$ or 3. Let M be an elementary abelian group of order n and $G = M \rtimes C_{n-1} \cong \text{AGL}(1, 2^r)$. Then G acts 2-transitively on the n vertices of K_n and so is arc transitive. As $M \triangleleft G$, it follows that the set of orbits of M on arcs forms a homogeneous factorisation.

Conversely, suppose that $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index k . Without loss of generality, we may assume that M is the kernel of the action of G on \mathcal{P} . Since Γ has prime valency $n - 1$, Lemma 2.5 implies that $k = n - 1$. Also, there exists an element $g \in G$ which induces a k -cycle on \mathcal{P} . Then $(M, \langle M, g \rangle, \Gamma, \mathcal{P})$ is a homogeneous factorisation such that $\langle M, g \rangle / M \cong C_k$. Suppose that n has prime power factorisation $n = p_1^{d_1} p_2^{d_2} \dots p_l^{d_l}$. Then by [18, Theorem 1.1], $p_i^{d_i} \equiv 1 \pmod{k}$ and in particular $p_i^{d_i} \geq k + 1$ for each $i \in \{1, \dots, l\}$. However, $k = n - 1$ and so $l = 1$ and $n = p_1^{d_1} = k + 1$, with p_1

and k both prime. Thus either $n = 3$ or n is a power of 2. ■

2.4 Primitivity of G and connectivity of Γ

First we consider the case where G is imprimitive on vertices, that is to say, G is transitive on $V\Gamma$ and has a nontrivial block of imprimitivity B in $V\Gamma$; ($B \subseteq V\Gamma$, $1 < |B| < |V\Gamma|$ and for all $g \in G$, $B^g = B$ or $B^g \cap B = \emptyset$). The set $\{B^g \mid g \in G\}$ is a *system of imprimitivity* for G and each B^g is a block of imprimitivity.

For a subset U of $V\Gamma$, the *subdigraph of Γ induced on U* is denoted by $\Gamma[U]$ and is the digraph with vertex set U which contains all arcs of Γ between vertices in U .

Lemma 2.16 *Let $(M, G, \Gamma, \mathcal{P})$ be a homogeneous factorisation of index k and let B be a block of imprimitivity of G on $V\Gamma$ such that there is an arc between some pair of vertices in B . Then $(M_B^B, G_B^B, \Gamma[B], \mathcal{P}^B)$ is a homogeneous factorisation of index k .*

PROOF. Since B is a block of imprimitivity for G and M on $V\Gamma$, it follows that M_B^B and G_B^B are transitive on B . Moreover, since M is transitive on $V\Gamma$ we have $G = MG_B$ and it follows that $G_B^B = G^{\mathcal{P}}$ and in particular G_B^B is transitive. For each $P_i \in \mathcal{P}$, let $P_i^B = P_i \cap (B \times B)$. By our assumption, at least one of the P_i^B is nonempty. Moreover, since $\mathcal{P} = \{P_1, \dots, P_k\}$ is a G -invariant partition of $A\Gamma$ it follows that $\mathcal{P}^B = \{P_1^B, \dots, P_k^B\}$ is a G_B^B -invariant partition of the arc set of $\Gamma[B]$, and since G_B^B is transitive, also G_B^B is transitive on \mathcal{P}^B . Thus each P_i^B is nonempty. Also M_B^B fixes each P_i^B setwise. Hence $(M_B^B, G_B^B, \Gamma[B], \mathcal{P}^B)$ is a homogeneous factorisation as required. ■

One important application of Lemma 2.16 is in the case where Γ is disconnected and B is the vertex set of a connected component of Γ . Such a set B is a block of imprimitivity for any vertex-transitive subgroup of $\text{Aut}(\Gamma)$. In this case $\Gamma[B]$ is a connected component of Γ and we have a converse to Lemma 2.16. Its proof is straightforward and is omitted.

Lemma 2.17 *If $M < G \leq \text{Aut}(\Gamma)$ with M vertex-transitive, if Γ is disconnected with connected component $\Gamma[B]$, and if $(M_B^B, G_B^B, \Gamma[B], \mathcal{P}^B)$ is a homogeneous factorisation of index k , then $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index k where each $P \in \mathcal{P}$ is the set $\{(u^g, v^g) \mid g \in G, (u, v) \in P^B\}$ for some part $P^B \in \mathcal{P}^B$.*

In view of Lemma 2.16 and its converse Lemma 2.17 in the special case where Γ is disconnected, there is no loss of generality in restricting the study of

homogeneous factorisations to the case of connected graphs Γ .

Let $(M, G, \Gamma, \mathcal{P})$ be a homogeneous factorisation and let \mathcal{B} be a G -invariant partition of $V\Gamma$. The *quotient digraph* $\Gamma_{\mathcal{B}}$ is the digraph with vertex set \mathcal{B} and arcs (B_1, B_2) for all B_1, B_2 where there exist $v \in B_1, w \in B_2$ such that (v, w) is an arc in Γ . Note that with this definition, $\Gamma_{\mathcal{B}}$ may contain loops, which correspond to arcs inside blocks. As \mathcal{B} is G -invariant, the set of arcs of Γ which are contained inside a block is fixed setwise by G . Hence if G is arc-transitive and Γ is connected then $\Gamma_{\mathcal{B}}$ contains no loops.

As discussed in [18, Section 4.2], for the complete graph case, there is no natural way of defining a homogeneous factorisation of $\Gamma_{\mathcal{B}}$ since arcs between distinct blocks B_1 and B_2 may lie in different parts of \mathcal{P} . We will see in Section 4.2 that if all the arcs between B_1 and B_2 lie in the same part then it is possible to define a natural ‘‘quotient’’ homogeneous factorisation.

We have the following lemma for the case where G acts imprimitively on \mathcal{P} and we leave the proof to the reader.

Lemma 2.18 *Suppose $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation such that $G^{\mathcal{P}}$ is imprimitive with system of imprimitivity $\{B_1, \dots, B_t\}$ where $t \geq 2$. For each $i = 1, 2, \dots, t$, let $P'_i = \cup_{A \in B_i} A$, and let $\mathcal{P}' = \{P'_1, \dots, P'_t\}$. Then $(M, G, \Gamma, \mathcal{P}')$ is a homogeneous factorisation of index t .*

3 Group theoretic constructions and characterisations

3.1 A generic construction

Actions of a group G on a set V and of a group H on a set U are *permutationally isomorphic* if there exist a group isomorphism $\phi: G \rightarrow H$ and a bijection $f: V \rightarrow U$, such that $f(v)^{\phi(g)} = f(v^g)$, for each $v \in V$ and $g \in G$. If $G = H$ and there is a bijection f so that the identity map ϕ has this property, then we say that G acts on V and U *equivalently*. Note that if G acts on V and U transitively, the latter condition is equivalent to requiring that the vertex stabiliser G_v of a vertex $v \in V$ be equal to the vertex stabiliser G_u of some vertex $u \in U$. For a subset $A \subseteq V^{(2)}$, A^* denotes the set $\{(u, v) \mid (v, u) \in A\}$. Similarly, for a set \mathcal{A} of subsets of $V^{(2)}$, let $\mathcal{A}^* = \{A^* \mid A \in \mathcal{A}\}$. The set \mathcal{A} is said to be *symmetric* if $\mathcal{A}^* = \mathcal{A}$.

We have the following general construction for a homogeneous factorisation. This is a generalisation of [18, Construction 3.2].

Construction 3.1 (i) *Let Γ be a digraph with vertex-transitive automor-*

phism groups M and G such that $M \triangleleft G$ and $\text{Orb}(M, A\Gamma) \neq \text{Orb}(G, A\Gamma)$. Let \mathcal{O} be the set of G -orbits on $\text{Orb}(M, A\Gamma)$ and label the elements of \mathcal{O} by O_1, O_2, \dots, O_t . Suppose that for each O_i , there exists a G -invariant partition \mathcal{B}_i of O_i with k parts, such that the actions of G on the \mathcal{B}_i are pairwise equivalent. We construct a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ of index k .

Each orbit of G on $A\Gamma$ is a union of M -orbits. For each $i \in \{1, 2, \dots, t\}$ there exists a bijection $f_i: \mathcal{B}_1 \rightarrow \mathcal{B}_i$, such that $f_i(B^g) = f_i(B)^g$ for any $B \in \mathcal{B}_1$ and any $g \in G$. (We may assume that f_1 is the identity map.) Choose $B_{1,1} \in \mathcal{B}_1$ and let $J = \{g_1, g_2, \dots, g_k\}$ be a set of coset representatives for the setwise stabiliser $G_{B_{1,1}}$ in G , where $g_1 = 1$. Further, for each $i \in \{1, \dots, t\}$ and $j \in \{1, \dots, k\}$, let $B_{i,j} = f_i((B_{1,1})^{g_j}) = f_i(B_{1,1})^{g_j}$. Observe that $\mathcal{B}_i = \{B_{i,1}, B_{i,2}, \dots, B_{i,k}\}$, and that $G_{B_{1,j}} = G_{B_{i,j}}$ for all i, j . We now form a partition of $A\Gamma$ with k parts such that the j^{th} part A_j is the union of all the arcs in each of the $B_{i,j}$ for $i = 1, 2, \dots, t$. That is, for $j \in \{1, \dots, k\}$,

$$A_j = \bigcup_{i=1}^t \left(\bigcup_{C \in \mathcal{B}_{i,j}} C \right).$$

Define $\mathcal{P} = \{A_1, A_2, \dots, A_k\}$. We claim that $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation. The fact that \mathcal{P} is a partition of $A\Gamma$ follows directly from the construction of the sets A_j . Since $\text{Orb}(M, A\Gamma)$ is a refinement of \mathcal{P} , each set A_j is preserved by M . The next step is to show that \mathcal{P} is a G -invariant partition. Let $g \in G$. Then for each $l \in \{1, \dots, k\}$, there exists a unique $j \in \{1, \dots, k\}$ such that $g_l g \in G_{B_{1,1}} g_j$ and so

$$\begin{aligned} A_l^g &= \bigcup_{i=1}^t \left(\bigcup_{C \in \mathcal{B}_{i,l}} C \right)^g = \bigcup_{i=1}^t \left(\bigcup_{C \in f_i(B_{1,1})^{g_l g}} C \right) = \bigcup_{i=1}^t \left(\bigcup_{C \in f_i((B_{1,1})^{g_l g})} C \right) \\ &= \bigcup_{i=1}^t \left(\bigcup_{C \in f_i((B_{1,1})^{g_j})} C \right) = \bigcup_{i=1}^t \left(\bigcup_{C \in \mathcal{B}_{i,j}} C \right) = A_j. \end{aligned}$$

Hence \mathcal{P} is G -invariant. The fact that $G^{\mathcal{P}}$ is transitive follows from the transitivity of G on each \mathcal{B}_i . Thus $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index k .

(ii) Suppose in addition, that for each symmetric $O_j \in \mathcal{O}$ each part of \mathcal{B}_j is symmetric. Then the construction in part (i) may be adjusted to produce a symmetric homogeneous factorisation of Γ .

Let $O_i \in \mathcal{O}$ be non-symmetric, that is $O_i^* \neq O_i$. Now $O_i^* \in \mathcal{O}$, and so $O_i^* = O_{i'}$ for some $i' \in \{1, \dots, t\}$. Observe that $\mathcal{B}_i^* = \{B^* \mid B \in \mathcal{B}_i\}$ is a G -invariant partition of $O_{i'}$ (in general different than $\mathcal{B}_{i'}$), and that the bijection $f_i^*: \mathcal{B}_1 \rightarrow \mathcal{B}_i^*$, defined by $f_i^*(B) = f_i(B)^*$, gives rise to an equivalence of the actions of G on \mathcal{B}_1 and \mathcal{B}_i^* . If we replace $\mathcal{B}_{i'}$ by \mathcal{B}_i^* and $f_{i'}$ by f_i^* , for one orbit in each non-symmetric pair $\{O_i, O_i^*\}$ of G -orbits in \mathcal{O} , then the homogeneous

factorisation $(M, G, \Gamma, \mathcal{P})$, defined as in part (i), is symmetric. Namely, by the above adjustments we have achieved that for each $i \in \{1, \dots, t\}$ there exists $i' \in \{1, \dots, t\}$, such that for each $j \in \{1, \dots, k\}$ we have $(B_{i,j})^* = B_{i',j}$, and so $A_j^* = A_j$.

Construction 3.1 shows the sufficiency of the conditions in Theorem 1.2. Thus it remains to show that they are necessary.

Proof of Theorem 1.2 Suppose that $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation. Since M acts on \mathcal{P} trivially, $\text{Orb}(M, A\Gamma)$ is a refinement of \mathcal{P} . Let \mathcal{O} be the set of G -orbits in $\text{Orb}(M, A\Gamma)$. For each $O \in \mathcal{O}$ and $A \in \mathcal{P}$, let $B_{O,A} = \{C \mid C \in O, C \subseteq A\}$ and let $\mathcal{B}_O = \{B_{O,A} \mid A \in \mathcal{P}\}$. Clearly, \mathcal{B}_O is a partition of O . For $g \in G$, since $O^g = O$, we have $(B_{O,A})^g = \{C \mid C \in O^g, C \subseteq A^g\} = B_{O,A^g}$, which shows that \mathcal{B}_O is G -invariant and that the action of G on each \mathcal{B}_O is equivalent to the action of G on \mathcal{P} . If, in addition, $(M, G, \Gamma, \mathcal{P})$ is a symmetric factorisation and O is a symmetric G -orbit on $\text{Orb}(M, A\Gamma)$, then for each $A \in \mathcal{P}$ and each $C \in B_{O,A}$, we have $C^* \in B_{O,A}$. Hence $B_{O,A}$ is symmetric. ■

Let G be a transitive permutation group acting on a set Ω . Let $\alpha \in \Omega$. Then G has a system of imprimitivity \mathcal{B} on Ω if and only if G_α is not a maximal subgroup of G , that is, if and only if there exists R , such that $G_\alpha < R < G$. Furthermore, the block of imprimitivity for G containing α is the R -orbit α^R containing α and the action of G on \mathcal{B} is equivalent to the action of G on the set of right cosets of R in G . See for example [4, Theorem 1.5A]. Hence we have the following corollary of Theorem 1.2 which provides us with a useful group theoretic method for recognising the existence of homogeneous factorisations.

Corollary 3.2 *Let Γ be a digraph, $M \triangleleft G \leq \text{Aut}(\Gamma)$, such that M is vertex-transitive and let \mathcal{O} be the set of G -orbits on $\text{Orb}(M, A\Gamma)$. Then*

- (i) *There exists a partition \mathcal{P} of $A\Gamma$ such that $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index k if and only if there exists a subgroup $R < G$ of index k which for each $C \in \text{Orb}(M, A\Gamma)$, contains a G -conjugate of the setwise stabiliser G_C (or equivalently, for each $O \in \mathcal{O}$ there exists $C \in O$ such that G_C is contained in R).*
- (ii) *There exists a partition \mathcal{P} of $A\Gamma$, such that $(M, G, \Gamma, \mathcal{P})$ is a symmetric homogeneous factorisation of index k if and only if, in addition to the requirement in part (i), for each symmetric $O \in \mathcal{O}$ there exists $C \in O$ such that G_C is contained in R and $C^* = C^h$ for some $h \in R$.*

PROOF. Part (i) follows naturally from Construction 3.1 and Theorem 1.2 and the discussion preceding the corollary. All that remains to be proven is part (ii).

Suppose first that for each symmetric $O \in \mathcal{O}$ there exists $C \in O$ and $h \in R$, such that $G_C \leq R$ and $C^* = C^h$. Let \mathcal{B}_O be the system of imprimitivity for G on O corresponding to R , and let $B = C^R$ be the block containing C . Then $B^* = (C^R)^* = (C^*)^R = C^{hR} = B$, and for each part $B^g \in \mathcal{B}_O$, $(B^g)^* = (B^*)^g = B^g$. By part (ii) of Construction 3.1, $(M, G, \Gamma, \mathcal{P})$ is a symmetric homogeneous factorisation.

Conversely suppose that $(M, G, \Gamma, \mathcal{P})$ is symmetric. By Theorem 1.2, for each symmetric O in \mathcal{O} , there exists a system of imprimitivity \mathcal{B}_O for G on O such that the actions of G on the \mathcal{B}_O are pairwise equivalent and each part of each \mathcal{B}_O is symmetric. Hence there exists an $R < G$ such that, for each O , R is the stabiliser of a block B_O in \mathcal{B}_O . Furthermore, let $C \in B_O$. Then $G_C \leq R$. Since B_O is symmetric, $C^* \in B_O$, and then since $B_O = C^R$ there exists $h \in R$, such that $C^* = C^h$, as required. \blacksquare

3.2 Local actions

In this subsection we show that, in some sense, the complete information about a homogeneous factorisation is already contained in the local action of a vertex stabiliser and in the partition of neighbours induced by the homogeneous factorisation. This provides us with a useful tool for finding and analysing homogeneous factorisations.

Let Γ be a digraph, M a vertex-transitive subgroup of $\text{Aut}(\Gamma)$ and \mathcal{P} a partition of $A\Gamma$ refined by $\text{Orb}(M, A\Gamma)$. Fix a vertex $v \in V\Gamma$ and for each $P \in \mathcal{P}$, define $P(v) = \{u \mid (v, u) \in P\}$. Then $\mathcal{P}(v) = \{P(v) \mid P \in \mathcal{P}\}$ is a partition of $\Gamma^+(v)$ which is refined by $\text{Orb}(M_v, \Gamma^+(v))$. Conversely, if \mathcal{S} is a partition of $\Gamma^+(v)$ refined by $\text{Orb}(M_v, \Gamma^+(v))$ then

$$\mathcal{P}_{\mathcal{S}} = \{ \{(v^g, w^g) \mid w \in S, g \in M\} \mid S \in \mathcal{S} \}$$

is a partition of $A\Gamma$ refined by $\text{Orb}(M, A\Gamma)$. Moreover $\mathcal{P}_{\mathcal{S}}(v) = \mathcal{S}$.

Proposition 3.3 *Let Γ be a digraph, $M \triangleleft G \leq \text{Aut}(\Gamma)$, such that M is vertex-transitive and let $v \in V\Gamma$.*

(i) *If $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index k , then $\mathcal{P}(v)$ is a G_v -invariant partition of $\Gamma^+(v)$ of size k which is refined by $\text{Orb}(M_v, \Gamma^+(v))$ and $G^{\mathcal{P}}$ is permutationally isomorphic to $G_v^{\mathcal{P}(v)}$.*

(ii) *If \mathcal{S} is a G_v -invariant partition of $\Gamma^+(v)$ of size k refined by $\text{Orb}(M_v, \Gamma^+(v))$ upon which G_v acts transitively, then $(M, G, \Gamma, \mathcal{P}_{\mathcal{S}})$ is a homogeneous factorisation of index k and $G^{\mathcal{P}_{\mathcal{S}}}$ is permutationally isomorphic to $G_v^{\mathcal{S}}$. Moreover, $\mathcal{S} = \mathcal{P}_{\mathcal{S}}(v)$.*

PROOF. The first half of part (i) follows directly from the definition of $\mathcal{P}(v)$. Let us now prove that the permutation groups $G^{\mathcal{P}}$ and $G_v^{\mathcal{P}(v)}$ are permutationally isomorphic. Observe first that for each $g \in G_v$ and $P \in \mathcal{P}$, we have $P(v)^g = P^g(v)$. Thus the actions of G_v on \mathcal{P} and $\mathcal{P}(v)$ are equivalent and so the corresponding permutation groups $G_v^{\mathcal{P}(v)}$ and $G_v^{\mathcal{P}}$ are permutationally isomorphic. Since M is transitive on $V\Gamma$, $G = MG_v$. Moreover, since M lies in the kernel of the action of G on \mathcal{P} it follows that $G^{\mathcal{P}} = (MG_v)^{\mathcal{P}} = G_v^{\mathcal{P}}$. Thus $G_v^{\mathcal{P}(v)}$ and $G^{\mathcal{P}}$ are permutationally isomorphic. This completes the proof of part (i).

For the proof of part (ii), first note that $\mathcal{P}_{\mathcal{S}}$ has k parts and that by definition, each part of $\mathcal{P}_{\mathcal{S}}$ is invariant under M . Furthermore, $\mathcal{S} = \mathcal{P}_{\mathcal{S}}(v)$. If $(u, w) \in A\Gamma$, then there exists $g \in M$ such that $(u, w)^g = (v, w^g)$. Then as \mathcal{S} is a partition of $\Gamma^+(v)$, there exists a unique $S \in \mathcal{S}$ such that $w^g \in S$. Furthermore, if there exists $h \in M \setminus \{g\}$ such that $u^h = v$ and $w^h = w^g$, then $h \in gM_v$ and so $w^h \in S$. Hence (u, w) belongs to a unique part of $\mathcal{P}_{\mathcal{S}}$. Thus $\mathcal{P}_{\mathcal{S}}$ is a partition of $A\Gamma$. Now as M is vertex-transitive, $G = MG_v$ and so $G^{\mathcal{P}_{\mathcal{S}}}$ is permutationally isomorphic to $G_v^{\mathcal{P}_{\mathcal{S}}}$. Furthermore, give $S_1, S_2 \in \mathcal{S}$ and $g \in G_v$ such that $S_1^g = S_2$ then

$$\{(v, w) \mid w \in S_1\}^g = \{(v, u) \mid u \in S_2\}.$$

The normality of M in G then implies that

$$\begin{aligned} \{(v, w) \mid w \in S_1\}^{Mg} &= \{(v, w) \mid w \in S_1\}^{gM} \\ &= \{(v, u) \mid u \in S_2\}^M \end{aligned}$$

Thus G_v and hence $G = MG_v$ preserve the partition $\mathcal{P}_{\mathcal{S}}$ and $G^{\mathcal{P}_{\mathcal{S}}}$ is permutationally isomorphic to $G_v^{\mathcal{P}_{\mathcal{S}}}$. Furthermore, $(M, G, \Gamma, \mathcal{P}_{\mathcal{S}})$ is a homogeneous factorisation of index k . ■

Note that if in part (i) of the above lemma we let $\mathcal{P} = \text{Orb}(M, A\Gamma)$, then the bijection $f: \mathcal{P} \rightarrow \mathcal{P}(v)$ mapping $P \in \mathcal{P}$ to $P(v)$, which gives rise to an isomorphism of the permutation groups $G^{\mathcal{P}}$ and $G_v^{\mathcal{P}(v)}$, is just the restriction of the usual one-to-one correspondence between the *orbitals of M on $V\Gamma$* (that is, orbits of the induced action of M on $V\Gamma^{(2)}$) and the *suborbits of M on $V\Gamma$ relative to v* , (that is, the orbits of the vertex-stabiliser M_v on $V\Gamma \setminus \{v\}$). If S is a suborbit corresponding to an orbital A , then the suborbit corresponding to the orbital A^* is denoted by S^* , and is called the suborbit *paired to S* . If $S = S^*$, then S is said to be *symmetric* (or *self-paired*). For a set \mathcal{S} of suborbits of M let $\mathcal{S}^* = \{S^* \mid S \in \mathcal{S}\}$. If $\mathcal{S}^* = \mathcal{S}$, then we say that \mathcal{S} is *symmetric*. We state the following lemma which is essentially [31, Lemma 16.1].

Lemma 3.4 *Let M be a transitive permutation group on a set V , and let $S = u^{M_v}$ be a suborbit of M relative to $v \in V$. Then $S^* = \{v^{g^{-1}} \mid g \in M, v^g \in S\}$.*

In particular, S is symmetric if and only if, for each $g \in M$ such that $v^g \in S$, also $v^{g^{-1}} \in S$.

Proposition 3.3 and Lemma 3.4 can now be used to translate Construction 3.1 to a “local version”. This gives rise to the following “local” counterparts of Theorem 1.2 and Corollary 3.2. The proofs are omitted.

Theorem 3.5 *Let Γ be a digraph, and let $M \triangleleft G$, be vertex-transitive subgroups of $\text{Aut}\Gamma$. For a vertex $v \in V\Gamma$ let \mathcal{O}_v denote the set of G_v -orbits of the induced action of G_v on $\text{Orb}(M_v, \Gamma^+(v))$. Then:*

- (i) *There exists a partition \mathcal{P} of $A\Gamma$ such that $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index k if and only if for each $\mathcal{S} \in \mathcal{O}_v$ there exists a G_v -invariant partition $\mathcal{B}_{\mathcal{S}}$ of \mathcal{S} with k parts, such that the actions of G_v on the $\mathcal{B}_{\mathcal{S}}$, $\mathcal{S} \in \mathcal{O}_v$, are pairwise equivalent.*
- (ii) *There exists a partition \mathcal{P} of $A\Gamma$, such that $(M, G, \Gamma, \mathcal{P})$ is a symmetric homogeneous factorisation of index k if and only if, in addition to the requirements in part (i), for each symmetric $\mathcal{S} \in \mathcal{O}_v$ (if any such exist), there exists a partition $\mathcal{B}_{\mathcal{S}}$ as in part (i), each part of which is symmetric.*

Corollary 3.6 *Let Γ, M, G and \mathcal{O}_v be as in Theorem 3.5. Then:*

- (i) *There exists a partition \mathcal{P} of $A\Gamma$ such that $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index k if and only if there exists a subgroup $R < G_v$ of index k such that, for each $S \in \text{Orb}(M_v, \Gamma^+(v))$, R contains a G_v -conjugate of the setwise stabiliser $(G_v)_S$ (or equivalently, for each $\mathcal{S} \in \mathcal{O}_v$ there exists $S \in \mathcal{S}$ such that $(G_v)_S$ is contained in R).*
- (ii) *There exists a partition \mathcal{P} of $A\Gamma$, such that $(M, G, \Gamma, \mathcal{P})$ is a symmetric homogeneous factorisation of index k if and only if, in addition to requirements in part (i), for each symmetric $\mathcal{S} \in \mathcal{O}_v$ there exists $S \in \mathcal{S}$ such that $(G_v)_S$ is contained in R and $S^* = S^h$ for some $h \in R$.*

3.3 Cayley homogeneous factorisations

Cayley digraphs form a large family of vertex-transitive digraphs. Given a group M and a subset S not containing the identity, the *Cayley digraph* $\text{Cay}(M, S)$ of M with respect to S is the digraph with vertex set M such that (x, y) is an arc if and only if $xy^{-1} \in S$. The group M induces a regular group of automorphisms of Γ by acting on the set of vertices by right multiplication. It is well known that a digraph Γ is a Cayley digraph if and only if $\text{Aut}(\Gamma)$ contains a regular subgroup. Let $S^{-1} = \{s^{-1} \mid s \in S\}$. If $S = S^{-1}$ we say that S is *self-inverse*. In this case, (x, y) is an arc of $\text{Cay}(M, S)$ if and only if (y, x) is an arc, and so Γ may be regarded as an undirected graph.

Let Γ be a digraph with automorphism group G such that G has a normal

subgroup M acting regularly on $V\Gamma$. Then by standard permutation group theory (see for example [4, Section 1.7]), we can identify the vertices of Γ with the elements of M such that M acts by right multiplication and, for $v = 1_M$, G_v acts by conjugation. Furthermore, under this identification, $\Gamma = \text{Cay}(M, S)$ where $S = \Gamma^+(v)$. Note then, that for any $w \in \Gamma^+(v)$, $(G_v)_w$ is the centraliser of w in G_v . Also, for a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$, each factor is a Cayley digraph of M . We call any homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ for which $M \triangleleft G$ and M acts regularly on $V\Gamma$, an *M -Cayley homogeneous factorisation* of Γ .

For M -Cayley homogeneous factorisations, the local version of Construction 3.1 is particularly nice and we give it here.

Construction 3.7 (i) Let M be a group and let S be a subset of $M \setminus \{1\}$ preserved by some $H \leq \text{Aut}(M)$. Let $\mathcal{O} = \{O_1, \dots, O_r\}$ be the set of H -orbits in S . For each $i \in \{1, \dots, r\}$, choose $x_i \in O_i$. Let R be a proper subgroup of H containing $C_H(x_i)$ for each i . For each $i \in \{1, \dots, r\}$ let $B_i \subseteq O_i$ denote the R -orbit of the element x_i , and let $S_1 = B_1 \cup \dots \cup B_r$. Choose a set $\{h_1, h_2, \dots, h_k\}$ of coset representatives of R in H such that $h_1 = 1$. For each $i \in \{2, \dots, k\}$, define $S_i = S_1^{h_i}$, and let P_i be the arc-set of the Cayley digraph $\Gamma_i = \text{Cay}(M, S_i)$. Note that $S = \cup_{i=1}^k S_i$. Finally, let $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$, and let $G = M \rtimes H$ be a permutation group acting on M in such a way that M acts by right multiplication, and H acts as a subgroup $\text{Aut}(M)$. Then M and G are vertex-transitive subgroups of $\text{Aut}(\Gamma)$, where $\Gamma = \text{Cay}(M, S)$, and \mathcal{P} is a partition of $A\Gamma$. Furthermore, M fixes each P_i setwise while \mathcal{P} is G -invariant and G permutes the parts of \mathcal{P} transitively. Hence $(M, G, \Gamma, \mathcal{P})$ is an M -Cayley homogeneous factorisation of index $k = |H : R|$.

(ii) If, in addition, for each self-inverse $O_j \in \mathcal{O}$ there exists $h \in R$, such that $x_j^{-1} = x_j^h$, then the construction in part (i) can be adjusted to produce a symmetric M -Cayley homogeneous factorisation of the same index in the following way: The assumptions imply that \mathcal{O} is self-inverse. Hence, for each $O_i \in \mathcal{O}$ which is not self-inverse, the set O_i^{-1} also belongs to \mathcal{O} , so $O_i^{-1} = O_{i'}$ for some $i' \in \{1, \dots, k\}$. We can now replace $x_{i'}$ by x_i^{-1} (or even with any $(x_i^{-1})^h$, $h \in R$) for one orbit $O_{i'}$ of each pair $\{O_i, O_{i'}\}$ of non-self-inverse H -orbits in \mathcal{O} . Since $C_H(x) = C_H(x^{-1})$ for each $x \in M$, the group R will still contain all vertex stabilisers $C_H(x_i)$. The construction described in part (i) will then give rise to a symmetric M -Cayley homogeneous factorisation.

4 Reduction Methods

When studying homogeneous factorisations, it often proves useful to study “smaller” related homogeneous factorisations. In this section we discuss several ways of obtaining such related factorisations.

4.1 Subgraphs

Let Γ be a digraph and let U be a subset of $V\Gamma$. Recall that $\Gamma[U]$ is the subdigraph induced on U . Our first reduction method is to look at the homogeneous factorisation induced on $\Gamma[U]$ for a certain type of subset U . In its most general setting we have the following construction.

Construction 4.1 *Let $(M, G, \Gamma, \mathcal{P})$ be a homogeneous factorisation of index k . Let U be a proper subset of $V\Gamma$ such that $\Gamma_0 = \Gamma[U]$ is not the empty graph. Let $M_0 \leq M_U$ and G_0 be a subgroup of G_U which contains M_0 , and suppose that M_0 acts transitively on U and G_0 acts transitively on \mathcal{P} .*

For each $P_i \in \mathcal{P}$, let $Q_i = P_i \cap (U \times U)$, that is, Q_i is the set of all arcs of Γ_0 which lie in P_i . Since G_0 acts transitively on \mathcal{P} and fixes $U \times U$ setwise, it follows that each Q_i is nonempty, and that $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ is a G_0 -invariant partition of $A\Gamma_0$ consisting of k parts. Furthermore, M_0 fixes each Q_i setwise and G_0 permutes the set \mathcal{Q} transitively. Thus $(M_0^U, G_0^U, \Gamma_0, \mathcal{Q})$ is a homogeneous factorisation of index k .

Some of the ways in which Construction 4.1 can be applied are the following.

- (1) If U is taken to be a block of imprimitivity for G on $V\Gamma$, $M_0 = M_U$ and $G_0 = G_U$, then the homogeneous factorisation produced by Construction 4.1 is the same as the one in the proof of Lemma 2.16.
- (2) Let $v \in V\Gamma$ and suppose that $M_0 \leq M$ does not fix v . Assume that $E \leq G_v$ is such that $E^{\mathcal{P}}$ is transitive and $E \leq N_G(M_0)$. Let $G_0 = M_0E$ and $U = v^{M_0}$. Then G_0 fixes U setwise and acts transitively on \mathcal{P} . Thus if $\Gamma[U]$ is nonempty we can use Construction 4.1 to construct a homogeneous factorisation. This method was given in [18, Lemma 3.5] for the case where Γ is a complete graph and was exploited in the situation where M_0 is a Sylow p -subgroup of M for some prime p dividing $|V\Gamma|$. This in turn was a generalisation of Muzychuk's method for analysing vertex-transitive self-complementary graphs in [23].
- (3) Let $(M, G, \Gamma, \mathcal{P})$ be a homogeneous factorisation of index k such that $M \triangleleft G$. Let p be a prime dividing $|M|$ and let S be a Sylow p -subgroup of M . Suppose that S fixes some arc of Γ pointwise and let U be the set of all vertices fixed by S . Then $U \neq \emptyset$ and $\Gamma[U]$ is not the empty graph. Let $M_0 = N_M(S)$ and $G_0 = N_G(S)$. Both M_0 and G_0 fix U setwise. By [2, Ex 1.28a], M_0 acts transitively on U . Furthermore, by the Frattini argument $G = G_0M$ and so G_0 acts transitively on \mathcal{P} . Thus we can use Construction 4.1 to find a partition \mathcal{Q} of the arc set of $\Gamma[U]$ such that $(M_0^U, G_0^U, \Gamma[U], \mathcal{Q})$ is a homogeneous factorisation of index k .

Instead of studying the subdigraph of Γ induced on a subset of vertices we can also look at the subdigraph of Γ induced by a subset of arcs.

Construction 4.2 Let $(M, G, \Gamma, \mathcal{P})$ be a homogeneous factorisation of index k such that G is not arc-transitive. Let O be an orbit of G on arcs and for each $P_i \in \mathcal{P}$ let $O_i = P_i \cap O$. As M fixes each P_i setwise, M fixes each O_i setwise. Furthermore, for each $g \in G$, $O_i^g = P_i^g \cap O$. Thus $\mathcal{P}_O = \{O_1, O_2, \dots, O_k\}$ is a partition of O of size k which is preserved by G and $G^{\mathcal{P}}$ is permutationally isomorphic to $G^{\mathcal{P}_O}$. If we let Γ_O be the subdigraph of Γ whose arc set is given by O then $(M, G, \Gamma_O, \mathcal{P}_O)$ is an arc-transitive homogeneous factorisation of index k .

We can also reconstruct $(M, G, \Gamma, \mathcal{P})$ from $(M, G, \Gamma_O, \mathcal{P}_O)$ in the following way: Suppose we are given the arc-transitive homogeneous factorisation $\mathcal{F} = (M, G, \Gamma_O, \mathcal{P}_O)$, where Γ_O is a spanning subdigraph of Γ , and we know that \mathcal{F} has been obtained from a homogeneous factorisation of Γ via Construction 4.2. Let $O_1 = O, O_2, \dots, O_l$ be the orbits of G on $A\Gamma$. Then for each O_i there exists a partition $\mathcal{P}_{O_i} = \{O_{i1}, \dots, O_{ik}\}$ which is invariant under G such that M fixes each O_{ij} setwise. Furthermore, each $G^{\mathcal{P}_{O_i}}$ is permutationally isomorphic to $G^{\mathcal{P}}$. Thus without loss of generality, we may assume that if $g \in G$ and $(O_{1j})^g = O_{1m}$, then $(O_{ij})^g = O_{im}$ for all $i = 1, \dots, l$. If we let $P_i = O_{1i} \cup O_{2i} \cup \dots \cup O_{li}$ for each $i = 1, \dots, k$ and let $\mathcal{P} = \{P_1, \dots, P_k\}$, then $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index k .

Note that Construction 4.2 shows that a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ can be viewed as a union of arc-transitive homogeneous factorisations. This construction will be illustrated in Examples 5.1, 5.2 and 5.4.

The subdigraphs given in Constructions 4.1 and 4.2 may or may not be connected. However, if the subdigraph is disconnected then Construction 4.1 can always be applied to provide us with a homogeneous factorisation of one of the connected components.

4.2 Monochromatic quotients

Let $(M, G, \Gamma, \mathcal{P})$ be a homogeneous factorisation of index k and suppose that \mathcal{B} is a system of imprimitivity for G on $V\Gamma$. It would be beneficial if we could construct a homogeneous factorisation of the quotient graph $\Gamma_{\mathcal{B}}$. However, as discussed in Section 2.4, \mathcal{P} may not naturally induce a partition of $A\Gamma_{\mathcal{B}}$. In the case where all arcs from a block B_1 to a block B_2 lie in the same part of \mathcal{P} , it is possible to define a homogeneous factorisation of $\Gamma_{\mathcal{B}}$.

Proposition 4.3 *Let $(M, G, \Gamma, \mathcal{P})$ be a homogeneous factorisation of index k and let \mathcal{B} be a system of imprimitivity for G on $V\Gamma$. Suppose that for any two blocks $B_1, B_2 \in \mathcal{B}$ all arcs from B_1 to B_2 lie in the same part of \mathcal{P} . Then there exists a partition \mathcal{Q} of the arc set of $\Gamma_{\mathcal{B}}$ such that $(M^{\mathcal{B}}, G^{\mathcal{B}}, \Gamma_{\mathcal{B}}, \mathcal{Q})$ is a homogeneous factorisation of index k . Furthermore, if G is transitive on $A\Gamma$*

then $G^{\mathcal{B}}$ is transitive on $A\Gamma_{\mathcal{B}}$.

PROOF. For each $P_i \in \mathcal{P}$, let

$$Q_i = \{(B_l, B_j) \mid \text{there exists } (v, w) \in P_i \text{ such that } v \in B_l \text{ and } w \in B_j\}.$$

Note that possibly $B_l = B_j$ in this definition. Also each $Q_i \neq \emptyset$ since $P_i \neq \emptyset$. Let $\mathcal{Q} = \{Q_1, \dots, Q_k\}$. Then since, for all l, j , all arcs from B_l to B_j lie in the same part of \mathcal{P} , it follows that \mathcal{Q} is a partition of $A\Gamma_{\mathcal{B}}$ with k parts. Since M and G are vertex-transitive, they both induce transitive groups $M^{\mathcal{B}}$ and $G^{\mathcal{B}}$ respectively, on the vertex set of $V\Gamma_{\mathcal{B}}$. Furthermore, since M fixes each P_i setwise it follows that $M^{\mathcal{B}}$ fixes each Q_i setwise. Similarly, if $g \in G$ maps P_i to P_j then $g^{\mathcal{B}}$ maps Q_i to Q_j . Thus \mathcal{Q} is $G^{\mathcal{B}}$ -invariant and $G^{\mathcal{B}}$ acts transitively on \mathcal{Q} , so $(M^{\mathcal{B}}, G^{\mathcal{B}}, \Gamma_{\mathcal{B}}, \mathcal{Q})$ is a homogeneous factorisation of index k . If (B_1, B_2) and (B_3, B_4) are arcs in $\Gamma_{\mathcal{B}}$ then there exist arcs $(v, w), (u, x) \in A\Gamma$ such that $v \in B_1, w \in B_2, u \in B_3$ and $x \in B_4$. If G is arc transitive on Γ then there exists $g \in G$ such that $(v, w)^g = (u, x)$. Then as each B_i is a block of imprimitivity for G , it follows that $(B_1, B_2)^g = (B_3, B_4)$ and so $G^{\mathcal{B}}$ is transitive on the set of arcs of $\Gamma_{\mathcal{B}}$. \blacksquare

We demonstrate Proposition 4.3 in the following example.

Example 4.4 Let p be a prime such that $p \equiv 1 \pmod{4}$ and let σ be an automorphism of C_p of order 4. Let $a \in C_p \setminus \{1\}$ and define $\Sigma = \text{Cay}(C_p, S)$ where $S = \{a, a^{-1}, a^\sigma, (a^{-1})^\sigma\}$. Let $n \geq 2$ and let Γ be the graph with vertex set $\{1, 2, \dots, n\} \times V\Sigma$ such that $((i, v), (j, w))$ is an arc if and only if (v, w) is an arc in Σ . Note that Γ is the lexicographic product of Σ and the empty graph on n vertices. We let

$$P_1 = \{((i, v), (j, w)) \mid vw^{-1} \in \{a, a^{-1}\}\}$$

and

$$P_2 = \{((i, v), (j, w)) \mid vw^{-1} \in \{a^\sigma, (a^{-1})^\sigma\}\}.$$

Then $\mathcal{P} = \{P_1, P_2\}$ is a partition of $A\Gamma$. Let $M = S_n \text{ wr } C_p$ in its imprimitive action on $V\Gamma$. Then M fixes P_1 and P_2 setwise. Let $G = S_n \text{ wr } \langle C_p, \sigma \rangle$. Then $M \triangleleft G \leq \text{Aut}(\Gamma)$ and G acts transitively on \mathcal{P} . Thus $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index 2.

Note that G acts imprimitively on $V\Gamma$ with system of imprimitivity

$$\mathcal{B} = \{\{(i, v) \mid i = 1, 2, \dots, n\} \mid v \in V\Sigma\}.$$

Moreover, $\Gamma_{\mathcal{B}} \cong \Sigma$, $M^{\mathcal{B}} = C_p$ and $G^{\mathcal{B}} = \langle C_p, \sigma \rangle$. Let $B_1 = \{(i, v) \mid i = 1, 2, \dots, n\}$ and $B_2 = \{(i, w) \mid i = 1, 2, \dots, n\}$ be elements of \mathcal{B} and suppose that B_1 and B_2 are adjacent in $\Gamma_{\mathcal{B}}$. Then v and w are adjacent in Σ and so $vw^{-1} \in S$. Hence each arc in Γ from B_1 to B_2 lies in the same P_i .

Thus by Proposition 4.3, the partition \mathcal{Q} of $A\Sigma$ induced by \mathcal{P} is such that $(M^{\mathcal{B}}, G^{\mathcal{B}}, \Gamma_{\mathcal{B}}, \mathcal{Q})$ is a homogeneous factorisation of index 2. Note that this is the homogeneous factorisation of index 2 of Σ whose two parts of the partition are the arc sets of $\text{Cay}(C_p, \{a, a^{-1}\})$ and $\text{Cay}(C_p, \{a^\sigma, (a^{-1})^\sigma\})$.

4.3 Cyclic homogeneous factorisations

We say that a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ is *cyclic* if $G^{\mathcal{P}}$ is cyclic. The following result shows that given any homogeneous factorisation, we can construct a cyclic homogeneous factorisation of Γ of prime index which is refined by the original partition \mathcal{P} . The construction is not elementary as it relies on a result of Fein, Kantor and Schacher [7] which in turn relies heavily on the Classification of Finite Simple Groups. It is a generalisation of [18, Theorem 3.6].

Theorem 4.5 *Let $(M, G, \Gamma, \mathcal{P})$ be a homogeneous factorisation of index k such that $M \triangleleft G$ and let $v \in V\Gamma$. Then there exists a cyclic homogeneous factorisation $(M, H, \Gamma, \mathcal{Q})$ of index p , where p is a certain prime divisor of k , \mathcal{Q} is a partition of $A\Gamma$ refined by \mathcal{P} and $H = \langle M, \sigma \rangle$ for some $\sigma \in G_v \setminus M_v$ such that σ^p acts trivially on \mathcal{Q} .*

PROOF. Since M is transitive on $V\Gamma$ it follows that $G = MG_v$. This implies that $G^{\mathcal{P}} = G_v^{\mathcal{P}}$ and so $G_v^{\mathcal{P}}$ is a transitive group. By a result of Fein, Kantor and Schacher [7], there exists a fixed point free permutation $\bar{\sigma} \in G_v^{\mathcal{P}}$ of prime power order p^e say. If $\sigma' \in G_v$ is such that $(\sigma')^{\mathcal{P}} = \bar{\sigma}$, then the order of σ' is $p^d q$, for some $d \geq e$, and q relatively prime to p . Now $(\sigma')^q$ induces $(\bar{\sigma})^q$ on \mathcal{P} . As q is coprime to p , $(\bar{\sigma})^q$ has no fixed points on \mathcal{P} . Furthermore, the order of $\sigma := (\sigma')^q$ is p^d and $\sigma \in G_v \setminus M_v$.

Let O_1, O_2, \dots, O_r be the orbits of $\langle \sigma \rangle$ on \mathcal{P} and let P_1, P_2, \dots, P_r be representatives of these orbits. For each $j \in \{1, \dots, r\}$, let

$$S_j = P_j^{\langle \sigma^p \rangle} = P_j \cup P_j^{\sigma^p} \cup P_j^{\sigma^{2p}} \cup \dots \cup P_j^{\sigma^{p^d - p}}.$$

Let $Q_1 = S_1 \cup S_2 \cup \dots \cup S_r$ and for $i \in \{2, \dots, p\}$, let $Q_i = Q_1^{\sigma^{i-1}}$. Then each Q_i is M -invariant and σ transitively permutes the Q_i . Hence letting $\mathcal{Q} = \{Q_1, \dots, Q_p\}$, we have that $(M, \langle M, \sigma \rangle, \Gamma, \mathcal{Q})$ is a homogeneous factorisation of index p such that \mathcal{P} refines \mathcal{Q} and σ^p acts trivially on \mathcal{Q} . ■

Thus if we are given a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ of index k there is a prime p dividing k , a possibly coarser partition \mathcal{Q} and a possibly smaller group G' such that $(M, G', \Gamma, \mathcal{Q})$ is a cyclic homogeneous factorisation of index p .

5 Arc-transitive, cyclic homogeneous factorisations

Recall that a homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ is called arc-transitive if G acts transitively on $A\Gamma$. Let $(M, G, \Gamma, \mathcal{P})$ be a homogeneous factorisation of index k . We have the following method of reducing $(M, G, \Gamma, \mathcal{P})$ to an arc-transitive, cyclic homogeneous factorisation of prime index of a connected digraph.

- (1) Using Theorem 4.5, we construct a cyclic homogeneous factorisation $(M, G', \Gamma, \mathcal{Q})$ of index p , for some prime p dividing k and G' such that $M \leq G' \leq G$. Furthermore, by the discussion in Section 2.1, if we let M' be the kernel of the action of G' on \mathcal{Q} then $(M', G', \Gamma, \mathcal{Q})$ is a homogeneous factorisation of index p and $G'/M' \cong C_p$.
- (2) Next, we use Construction 4.2 to construct a cyclic homogeneous factorisation $(M', G', \Sigma, \mathcal{R})$ of index p where Σ is a subdigraph of Γ with the same vertex set but for which G' is arc-transitive. Furthermore, $(G')^{\mathcal{Q}}$ is permutationally isomorphic to $(G')^{\mathcal{R}}$.
- (3) Finally, if Σ is not connected we use Construction 4.1 to construct a cyclic homogeneous factorisation of index p of a connected component of Σ and this factorisation will be arc-transitive.

Hence given any homogeneous factorisation we can construct a related arc-transitive, cyclic homogeneous factorisation of a connected digraph that is a subdigraph of the original digraph. Furthermore, by Lemma 2.17, any cyclic homogeneous factorisation in case 3 can be extended to a cyclic homogeneous factorisation of Σ . In turn, as discussed in Construction 4.2, the cyclic homogeneous factorisation obtained in step 1 can be viewed as being a union of each of the homogeneous factorisations obtained in step 2 by varying the choice of orbit of G on $A\Gamma$. *Thus it is the class of arc-transitive, cyclic homogeneous factorisations of index p which we concentrate on.*

We demonstrate the reduction process to an arc-transitive, cyclic homogeneous factorisation of prime index of a connected digraph in the following four examples. These show that each of the four basic cases in Theorem 1.3 occurs.

Example 5.1 Let $M = C_2^d$ for $d \neq 6$, $S = M \setminus \{1\}$, $\Gamma = \text{Cay}(M, S) \cong K_{2^d}$, $H = \text{GL}(1, 2^d) \cong C_{2^d-1}$ and $G = M \rtimes H \cong \text{AGL}(1, 2^d)$. Then H acts regularly on S and so letting $R = 1$ we can use Construction 3.7 to obtain a partition \mathcal{P} of the arc set of Γ such that $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index $2^d - 1$.

- (1) Let p be a primitive prime divisor of $2^d - 1$, that is, p does not divide $2^i - 1$ for all $i < d$, and let σ be an element of H of order p . Such a prime exists (see [35]). Then σ has no fixed points on S and normalises no nontrivial,

- proper subgroup of M . Let $G' = M \rtimes \langle \sigma \rangle$. Then using the construction in the proof of Theorem 4.5, there is a partition \mathcal{Q} of $A\Gamma$ refined by \mathcal{P} such that $(M, G', \Gamma, \mathcal{Q})$ is a cyclic homogeneous factorisation of index p .
- (2) Now $|G'| = 2^d p$ while Γ has $2^d(2^d - 1)$ arcs. Thus if $2^d - 1$ is not prime, G' then is not arc-transitive. Let O be an orbit of G' on $A\Gamma$ and let Γ_O be the subdigraph of Γ with arc set O . The digraph Γ_O has valency p . Then using Construction 4.2, \mathcal{Q} induces a partition \mathcal{Q}_O of $A\Gamma_O$ such that $(M, G', \Gamma_O, \mathcal{Q}_O)$ is an arc transitive, cyclic homogeneous factorisation of index p .
 - (3) Finally, G' acts primitively on $V\Gamma_O$, and so Γ_O is connected and we do not need to perform a further reduction.

As G' acts primitively on $V\Gamma_O$, the homogeneous factorisation $(M, G', \Gamma_O, \mathcal{Q}_O)$ is basic and is of type (1) of Theorem 1.3. Note that we can also use Construction 4.2 to reconstruct $(M, G', \Gamma, \mathcal{Q})$ from $(M, G', \Gamma_O, \mathcal{Q}_O)$.

Example 5.2 We start by constructing the homogeneous factorisation given in [9, Example 4.4]. Let $M = C_2^4$, L a subgroup of M of order 2^3 , $S = M \setminus L$ and $\Gamma = \text{Cay}(M, S) \cong K_{8,8}$. Let $H = \text{GL}(3, 2)$ such that H acts on L as $\text{GL}(3, 2)$ fixing the vertex $v = 1_M$ while it acts transitively on the eight points of S as $\text{PSL}(2, 7)$ and let $G = M \rtimes H$. Thus we can use Construction 3.7 to find a partition \mathcal{P} of $A\Gamma$ such that $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index 8.

- (1) Since 2^3 is the largest power of 2 which divides $|H|$ and $|S| = 2^3$ it follows that the stabiliser in H of a point in S has order coprime to 2. Thus we can find $\sigma \in H$ of order 2 which fixes no points of S . Hence letting $G' = M \rtimes \langle \sigma \rangle$ and using the construction in the proof of Theorem 4.5, we have a partition \mathcal{Q} of $A\Gamma$ refined by \mathcal{P} such that $(M, G', \Gamma, \mathcal{Q})$ is a cyclic homogeneous factorisation of index 2.
- (2) Now G' does not act transitively on the set of arcs of Γ so we choose an orbit O of G' on $A\Gamma$ and form the new graph Γ_O which has vertex set $V\Gamma$ and arc set O . We can then use Construction 4.2 to construct an arc-transitive, cyclic homogeneous factorisation of Γ' .
- (3) The graph Γ_O is disconnected and has connected component $\Sigma \cong K_{2,2}$ which contains the vertices $U = \{1, x, x^\sigma, xx^\sigma\}$ for some $x \in S$. Then using Construction 4.1, we obtain the arc-transitive, cyclic homogeneous factorisation $(M_U^U, G_U^U, \Sigma, \mathcal{R})$ of index 2, where $M_U^U = \langle x, x^\sigma \rangle \cong C_2^2$, $G_U^U = M_U^U \rtimes \langle \sigma \rangle$ and

$$\mathcal{R} = \{ \{(1, x), (xx^\sigma, x^\sigma)\}, \{(1, x^\sigma), (xx^\sigma, x)\} \}.$$

The only nontrivial, proper normal subgroups of G_U^U are M_U^U , $\langle xx^\sigma \rangle$, $\langle xx^\sigma, \sigma \rangle$ and $\langle xx^\sigma, x\sigma \rangle$. The first is vertex-transitive, while the last three each have two orbits. Hence G_U^U is biquasiprimitive on $V\Sigma$ and so $(M_U^U, G_U^U, \Sigma, \mathcal{R})$ is basic of type (1) in Theorem 1.3.

Example 5.3 Let V be the set of $2^{2^e} - 1$ nonzero vectors of a 2-dimensional vector space over $\text{GF}(2^e)$. Let $M = \text{SL}(2, 2^e)$ and $G = \text{GL}(2, 2^e) = \text{SL}(2, 2^e) \times C_{2^e-1}$, where the cyclic group C_{2^e-1} is the group of all scalar matrices. Both M and G act transitively on V . Let $v \in V$. Then G_v fixes all scalar multiples of v while acting transitively on the remaining vectors. Let w be a vector in V which is not a scalar multiple of v and define Γ to be the graph with vertex set V and arc set $(v, w)^G$. Then Γ has valency $2^e(2^e - 1)$ and G acts transitively on the set of arcs in Γ . Also the set of neighbours of v is the set of all vectors which are not scalar multiples of v . The connected component of Γ containing v contains at least $2^e(2^e - 1) + 1$ vertices. As this is more than half the vertices of Γ and G is vertex-transitive, it follows that Γ is connected. Now $M_v \cong C_2^e$ and has $2^e - 1$ orbits of length 2^e on the set of neighbours of v . As $M_v \triangleleft G_v$, the partition of $\Gamma(v)$ given by the set of M_v -orbits is G_v -invariant. Hence by Theorem 3.5, we can construct a partition \mathcal{P} of $A\Gamma$ such that $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index $2^e - 1$. Since M is simple it acts quasiprimively on $V\Gamma$. Then in the cases where $2^e - 1$ is a prime, that is, a Mersenne prime, $(M, G, \Gamma, \mathcal{P})$ is an arc-transitive, cyclic homogeneous factorisation of prime index $2^e - 1$ which is basic of type (2) of Theorem 1.3.

Example 5.4 We start with the homogeneous factorisation from [9, Example 5.2] which we reconstruct here. Let $M = S_5$, $L = A_5$, H be the group of automorphisms of M induced by conjugation by L and $G = M \rtimes H$. Then H has three orbits on the set of elements of $M \setminus L$ with orbit representatives $x_1 = (1, 2)$, $x_2 = (1, 2)(3, 4, 5)$ and $x_3 = (1, 4, 2, 3)$. Now $C_H(x_2), C_H(x_3) \leq C_H(x_1) \cong S_3$. Thus letting $\Gamma = \text{Cay}(M, M \setminus L) \cong K_{60,60}$ and $R = C_H(x_1)$ we can use Construction 3.7 to find a partition \mathcal{P} of $A\Gamma$ such that $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index 10.

- (1) Let σ be the automorphism of M which is induced by conjugation by $(1, 2, 3, 4, 5)$. Then $\sigma \in H$ and has order 5. Also, since $R \cong S_3$ is the stabiliser of a factor of \mathcal{P} it follows that σ acts on \mathcal{P} without fixed points. Let $G' = M \rtimes \langle \sigma \rangle$. Then we can use the construction in the proof of Theorem 4.5 to find a partition \mathcal{Q} of $A\Gamma$ which is refined by \mathcal{P} such that $(M, G', \Gamma, \mathcal{Q})$ is a cyclic homogeneous factorisation of index 5.
- (2) Let v be the vertex of Γ given by the element 1_M . Then $(G')_v = \langle \sigma \rangle$ and since $|\Gamma(v)| = 60$ it follows that G' does not transitively on the set of arcs of Γ . Let w be the vertex of Γ given by the vertex $(1, 2)$ and $O = (v, w)^{G'}$. Note that $(w, v) \in O$ and so O is symmetric. Then letting Γ_O be the subgraph of Γ with arc set O and \mathcal{P}_O be the partition of $A\Gamma_O$ induced by \mathcal{Q} , we have from Construction 4.2 that $(M, G', \Gamma_O, \mathcal{P}_O)$ is an arc-transitive, cyclic homogeneous factorisation.
- (3) The element $g = (1, 2) \in M$ interchanges v and w and $G' = \langle (G')_v, g \rangle$. Thus by [6, Theorem 2.1], Γ_O is connected.

Now $G' \cong M \times C_5$ and M is biquasiprimitive on $V\Gamma_O$. Hence $(M, G', \Gamma_O, \mathcal{P}_O)$ is basic of type (2) in Theorem 1.3.

We begin an analysis of the structure of arc-transitive, cyclic homogeneous factorisations of prime index with the following lemma.

Lemma 5.5 *Let $(M, G, \Gamma, \mathcal{P})$ be an arc-transitive homogeneous factorisation of the connected digraph Γ . Let \mathcal{B} be a nontrivial system of imprimitivity for G on $V\Gamma$ and let K be the kernel of the action of G on \mathcal{P} . Then no block of \mathcal{B} contains an arc and either*

- (1) *given any two distinct blocks $B_1, B_2 \in \mathcal{B}$, all arcs from B_1 to B_2 lie in the same part P_i of \mathcal{P} , or*
- (2) *$G \setminus K$ contains an element which fixes two adjacent blocks of the quotient graph $\Gamma_{\mathcal{B}}$.*

PROOF. Suppose that (u, v) is an arc contained in the block $B \in \mathcal{B}$. Since B is a block of imprimitivity for G , it follows that for each $g \in G$, $(u, v)^g \in B^g \in \mathcal{B}$. Hence each $B \in \mathcal{B}$ is a union of connected components of Γ , contradicting Γ being connected. Thus no block of \mathcal{B} contains an arc.

Suppose now that there are two distinct blocks $B_1, B_2 \in \mathcal{B}$ and distinct $P_i, P_j \in \mathcal{P}$ such that there exist arcs $(u, v) \in P_i \cap (B_1 \times B_2)$ and $(w, x) \in P_j \cap (B_1 \times B_2)$. Then as G is arc-transitive, there exists $g \in G$ such that $(u, v)^g = (w, x)$ and so $P_i^g = P_j$ which implies that $g \notin K$. Furthermore, B_1 and B_2 are blocks of imprimitivity for G and so g fixes the two adjacent blocks B_1 and B_2 . ■

If we have a homogeneous factorisation for which case 1 of Lemma 5.5 holds then we can use Proposition 4.3 to find a homogeneous factorisation of $\Gamma_{\mathcal{B}}$. If case 1 does not hold, then there exists an element $g \in G \setminus K$ such that g fixes two adjacent blocks B_1 and B_2 . In this case we consider the subdigraph $\Gamma[B_1 \cup B_2]$. To obtain an appropriate homogeneous factorisation of $\Gamma[B_1 \cup B_2]$ in this latter case, we look at systems of imprimitivity which arise from the intransitive normal subgroups of M .

Proposition 5.6 *Let $(M, G, \Gamma, \mathcal{P})$ be an arc-transitive, cyclic homogeneous factorisation of the connected digraph Γ of prime index p where $G/M \cong C_p$. Suppose that G has an intransitive normal subgroup N such that $N \leq M$ and that there exists a pair of N -orbits B_1 and B_2 such that there are arcs in $B_1 \times B_2$ lying in different parts of \mathcal{P} . Furthermore, suppose that Γ has arcs in both $B_1 \times B_2$ and $B_2 \times B_1$. Then there is a homogeneous factorisation $(\overline{M}, \overline{G}, \Gamma[B_1 \cup B_2], \mathcal{P}')$ of index p and \overline{G} is arc-transitive on $\Gamma[B_1 \cup B_2]$.*

PROOF. By Lemma 5.5, there exists $\sigma \in G \setminus M$ such that σ fixes both B_1 and B_2 setwise. Now $G = \langle M, \sigma \rangle$ and $G_{B_1 \cup B_2} = \langle M_{B_1 \cup B_2}, \sigma \rangle$. Then as $\langle \sigma \rangle$ is

transitive on \mathcal{P} and Γ has arcs in both $B_1 \times B_2$ and $B_2 \times B_1$ it follows that both $B_1 \times B_2$ and $B_2 \times B_1$ contain an arc from each part of \mathcal{P} . Thus $P \in \mathcal{P}$ contains an arc $(u, v) \in B_1 \times B_2$ and an arc $(x, w) \in B_2 \times B_1$. Since G is arc-transitive, there exists $g \in G$ such that $(u, v)^g = (x, w)$ and hence g interchanges B_1 and B_2 . Furthermore, g fixes P setwise and so, since $G^P = C_p$ it follows that g fixes each part of \mathcal{P} and hence belongs to M . Thus $M_{B_1 \cup B_2}$ interchanges B_1 and B_2 . Also $N \leq M_{B_1 \cup B_2}$ and N is transitive on both B_1 and B_2 . Hence $M_{B_1 \cup B_2}$ is transitive on $B_1 \cup B_2$. Hence letting $\overline{M} = M_{B_1 \cup B_2}^{B_1 \cup B_2}$ and $\overline{G} = G_{B_1 \cup B_2}^{B_1 \cup B_2}$, Construction 4.1 yields a homogeneous factorisation $(\overline{M}, \overline{G}, \Gamma[B_1 \cup B_2], \mathcal{P}')$ of index p , where \mathcal{P}' is the partition of the arc set of $\Gamma[B_1 \cup B_2]$ induced by \mathcal{P} . Furthermore, since G is arc-transitive and B_1 and B_2 are blocks of imprimitivity for G it follows that \overline{G} acts transitively on the set of arcs of $\Gamma[B_1 \cup B_2]$. ■

In light of Lemma 5.5 and Proposition 5.6, it appears that the only case of an arc-transitive, cyclic homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ of index p such that G has an intransitive normal subgroup $N \leq G$, where we cannot reduce to a smaller homogeneous factorisation (by applying either Proposition 4.3 or 5.6), is when for each pair of N -orbits B_1, B_2 such that there exists an arc $(u, v) \in B_1 \times B_2$, there are arcs in $B_1 \times B_2$ from at least two different parts of \mathcal{P} and all the arcs of $\Gamma[B_1 \cup B_2]$ lie in $B_1 \times B_2$, that is, there are no arcs in $B_2 \times B_1$. In this case we have that $G_{B_1 \cup B_2}$ is not transitive on $B_1 \cup B_2$. The homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ does give rise to a factorisation of $\Gamma[B_1 \cup B_2]$ with p parts but it is not a homogeneous one. The factorisation obtained is a bihomogeneous factorisation (as defined at the end of Section 1).

Proposition 5.7 *Let $(M, G, \Gamma, \mathcal{P})$ be an arc-transitive cyclic homogeneous factorisation of the connected digraph Γ of prime index p where $G/M \cong C_p$. Suppose that G has an intransitive normal subgroup N such that $N \leq M$ and that there exists a pair of N -orbits B_1 and B_2 such that there are arcs from B_1 to B_2 lying in different parts of \mathcal{P} . Furthermore, suppose that Γ has arcs going from B_1 to B_2 but no arcs from B_2 to B_1 . Then there is a bihomogeneous factorisation $(\overline{M}, \overline{G}, \Gamma[B_1 \cup B_2], \mathcal{P}')$ of index p and \overline{G} is arc-transitive on $\Gamma[B_1 \cup B_2]$.*

PROOF. The proof is analogous to that of Proposition 5.6. ■

We recap our progress thus far. Let $(M, G, \Gamma, \mathcal{P})$ be a cyclic homogeneous factorisation of prime index p such that $G/M \cong C_p$ and G is arc-transitive. Then we have three reduction methods. First of all, if Γ is disconnected then we can use Construction 4.1 to reduce to a connected graph with a homogeneous factorisation having the same properties. Once Γ is connected and if G has a normal subgroup N contained in M which is intransitive on vertices we can then apply one of the reductions given in Propositions 4.3, 5.6 and 5.7. We

can now prove Theorem 1.3.

Proof of Theorem 1.3 First suppose that \mathcal{F} is basic, that is to say, Γ is connected and every normal subgroup of G contained in N has at most two orbits on vertices. Hence G is quasiprimitive, or biquasiprimitive on $V\Gamma$ so that part (1) holds, or G has a normal subgroup N not contained in M such that N has more than two orbits on vertices. It remains to consider the last case. Since $N \cap M$ has more than two orbits and is normal in G , we have $N \cap M = 1$ and so N centralises M . Furthermore, as $G/M \cong C_p$ we have $G = N \times M$ and $N \cong C_p$, that is $G = C_p \times M$. As all normal subgroups of M are then normal subgroups of G we have that M is either quasiprimitive or biquasiprimitive on $V\Gamma$ and so part (2) holds. Furthermore, note that Examples 5.1, 5.2, 5.3 and 5.4 provide examples for each of the four cases given in the statement of the theorem.

Conversely, suppose that \mathcal{F} is not basic. If Γ is not connected then we can apply Construction 4.1. Suppose now that Γ is connected. Then G has a normal subgroup N contained in M with more than two orbits on vertices. Then by Lemma 5.5, either

- (1) given any two N -orbits B_1 and B_2 , any arcs from B_1 to B_2 lie in the same part P_i of \mathcal{P} , or
- (2) some $\sigma \in G \setminus M$ fixes two adjacent blocks B_1 and B_2 setwise.

In the first case we can use Proposition 4.3 to get a homogeneous factorisation of $\Gamma_{\mathcal{B}}$, where \mathcal{B} is the set of orbits of N . Note that, since G is arc-transitive and Γ is connected, $\Gamma_{\mathcal{B}}$ does not contain loops. In the second case, if there are arcs from B_1 to B_2 and from B_2 to B_1 then we can apply Proposition 5.6 to find an arc-transitive, cyclic homogeneous factorisation $(\overline{M}, \overline{G}, \Gamma[B_1 \cup B_2], \mathcal{P}')$ of index p such that $\Gamma[B_1 \cup B_2] \neq \Gamma$. If there are only arcs going from B_1 to B_2 then we apply Proposition 5.7 to find an arc-transitive, cyclic bihomogeneous factorisation $(\overline{M}, \overline{G}, \Gamma[B_1 \cup B_2], \mathcal{P}')$ of index p such that $\Gamma[B_1 \cup B_2] \neq \Gamma$. ■

Theorem 1.3 states that if we have an arc-transitive, cyclic homogeneous factorisation of prime index which is not basic then we can obtain a smaller graph with a corresponding arc transitive, cyclic homogeneous or bihomogeneous factorisation. This is done by continually applying the reductions in Propositions 4.3, 5.6 and 5.7 and also using Construction 4.1 to reduce to a connected graph. This leads to the following question.

Question 5.8 *Is it possible to use the reductions in Propositions 4.3, 5.6 and 5.7 and Construction 4.1 to produce arbitrarily long sequences of homogeneous factorisations?*

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