A new family of locally 5–arc transitive graphs ⋆

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Abstract

We present a new family of locally 5–arc transitive graphs. The graphs constructed are bipartite with valency $\{2^m+1, 2^m\}$. The actions of the automorphism group on the two bipartite halves are distinctly different and the corresponding amalgams are new.

Key words: locally s–arc transitive graphs

1 Introduction

Let Γ be a graph with vertex set $V\Gamma$. An s*–arc* is an $(s+1)$–tuple $(v_0, v_1, \ldots, v_s)$ of vertices in Γ such that each v_i is adjacent to v_{i+1} while $v_i \neq v_{i+2}$. We say that Γ is *locally* s*–arc transitive* if for each $v \in V\Gamma$, the stabiliser in $\mathrm{Aut}(\Gamma)$ of v acts transitively on the set of s–arcs whose initial vertex is v. Locally s–arc transitive graphs have been the subject of much investigation, for example the results in [8–11], and examples of locally s–arc transitive graphs with large values of s are of particular interest. Stellmacher [7] has proved that the

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largest possible value of s when all vertices have valency at least three is 9. This bound is attained by the incidence graphs of the classical generalised octagons.

The purpose of this paper is to construct an infinite family of locally 5–arc transitive graphs $\mathcal{G}(p,m)$ where p is a prime and m is a positive integer. The graph $\mathcal{G}(p,m)$ is constructed as a coset graph of a group G, where if $p=2$ then

$$G \cong (\mathrm{PSL}(2,p^m)^{p^m} \rtimes \mathrm{AGL}(1,p^m)) \rtimes C_m$$

while if p is odd then

$$G \cong (\mathrm{PSL}(2,p^m)^{p^m}.2 \rtimes \mathrm{AGL}(1,p^m)) \rtimes C_m.$$

For an edge $\{v,w\}$, the vertex stabilisers satisfy

$$G_v \cong (\mathrm{PGL}(2,p^m) \times \mathrm{AGL}(1,p^m)) \rtimes C_m,$$
$$G_w \cong ((C_p^{2m} \rtimes C_{p^m-1}) \rtimes \mathrm{AGL}(1,p^m)) \rtimes C_m,$$

and

$$G_{vw} \cong (\mathrm{AGL}(1,p^m) \times \mathrm{AGL}(1,p^m)) \rtimes C_m.$$

The detailed structural information is stated in Theorem 1.1. The discovery of the graphs $\mathcal{G}(p,m)$ arose from a global analysis of locally s–arc transitive graphs initiated by the authors in [1]. The quasiprimitive types mentioned in Theorem 1.1(3) are defined below.

Theorem 1.1 *Let p be a prime and m a positive integer such that $(p,m)\neq(2,1),(3,1)$. Let $\Gamma=\mathcal{G}(p,m)$ be the graph constructed in Construction 2.2. Then Γ is bipartite, connected, and of valency $\{p^m+1,p^m\}$, $\mathrm{Aut}(\Gamma)$ is the group G above, and further, the following statements hold.*

(1) If $p=2$ then Γ is locally 5–arc transitive but not locally 6–arc transitive.
(2) If p is odd then Γ is locally 3–arc transitive but not locally 4–arc transitive.
(3) Let Δ_1 be the set of all vertices of valency p^m+1 and Δ_2 be the set of all vertices of valency p^m. Then $G^{\Delta_1}\cong G$ is primitive of Simple Diagonal type, and $G^{\Delta_2}\cong G$ is quasiprimitive of Product Action type.
(4) For an edge $\{v,w\}$ with $v\in\Delta_1$, $w\in\Delta_2$, the local actions are $G_v^{\Gamma(v)}\cong \mathrm{P\Gamma L}(2,p^m)$ and $G_w^{\Gamma(w)}\cong \mathrm{A\Gamma L}(1,p^m)$.

The graphs $\mathcal{G}(p,m)$ are constructed in Section 2 and we prove Theorem 1.1 in Section 3.

We have the following remarks.

(1) The amalgams (G_v,G_w,G_{vw}) for the locally 5–arc transitive graphs $\mathcal{G}(2,m)$ are different from those occurring in any other infinite family of locally

5–arc transitive graphs known to the authors. It would be interesting to determine if G is the smallest completion of this amalgam.

(2) It will be shown in [4, Corollary 1.2] that the graphs $\mathcal{G}(2,m)$ are the only examples of locally s–arc transitive graphs for $s \geq 4$ such that the automorphism group has different quasiprimitive types on the two parts of the bipartition.

(3) The two smallest members $\mathcal{G}(2,1)$ and $\mathcal{G}(3,1)$ are in some sense 'exceptional' to the Theorem 1.1: the automorphism groups are soluble and $\mathcal{G}(3,1)$ is disconnected, see Section 4. For $\Gamma = \mathcal{G}(2,1)$ we still have $\mathrm{Aut}(\Gamma) = G$, and (1) and (4) of Theorem 1.1 hold. For $\Gamma = \mathcal{G}(3,1)$, $G \neq \mathrm{Aut}(\Gamma)$, and (2) and (4) of Theorem 1.1 hold. The graph $\mathcal{G}(2,1)$ can be constructed by placing a vertex at the midpoint of every edge of the complete bipartite graph $K_{3,3}$ (see Lemma 4.1).

(4) When p is odd, G_w is transitive on the set of 4–arcs starting at w, while G_v has two orbits of equal size on the set of 4–arcs starting at v (see Propsition 3.2).

(5) The subgroup that fixes pointwise v, w, and all vertices adjacent to either, is a p-group acting trivially on the set of vertices at distance two from v, and having $p^m - 1$ orbits of length p^m on the set of vertices at distance two from w (see Lemma 3.1).

(6) For $(p,m) \neq (2,1),(3,1)$, if $p = 2$, let $N = \mathrm{PSL}(2,p^m)^{p^m} = \mathrm{soc}(G)$ and if p is odd, let $N = \mathrm{PSL}(2,p^m)^{p^m}.2 \lhd G$. Then Γ is locally $(N,3)$–arc transitive. Now let $M = \mathrm{PSL}(2,p^m)^{p^m-2} \lhd N$. Then M acts intransitively on both Δ_1 and Δ_2. By [1, Lemma 5.1], the quotient graph Γ_M is locally $(N/M,3)$–arc transitive and Γ is a cover of Γ_M. Furthermore, Γ_M is one of the graphs constructed in [3, Example 2.2].

Let Γ be a finite locally (G,s)–arc transitive graph with $s \geq 2$ and suppose that G acts intransitively on $V\Gamma$. Then Γ is bipartite and the two parts Δ_1 and Δ_2 of the bipartition are orbits of G. In [1], a global analysis of such graphs was initiated which showed that the important graphs to study are those where G acts faithfully on both Δ_1 and Δ_2 and G acts quasiprimitively on at least one of these two sets. (A permutation group G acts *quasiprimitively* on a set Ω if every nontrivial normal subgroup of G is transitive.) Let us call such graphs *basic* locally (G,s)–arc transitive graphs. All locally s–arc transitive graphs were proved to be covers of one of these basic graphs or of a complete bipartite graph. Quasiprimitive groups have been classified in an O'Nan–Scott like theorem into eight types (see [6]). The two types of particular interest for this discussion are called SD (Simple Diagonal) and PA (Product Action). Let G be a quasiprimitive permutation group on a set Ω, let $\omega \in \Omega$, and suppose that G has a unique minimal normal subgroup $N = T^k$, for some finite nonabelian simple group T and $k \geq 2$. Then G has type SD if $N_\omega \cong T$. The group G has type PA if $N_\omega \neq 1$ and N_ω projects onto a proper subgroup of each simple direct factor of N. In the latter case we may assume that $N_\omega \leqslant H^k$ for some proper subgroup H of T.

Suppose now that Γ is a basic locally (G, s)–arc transitive graph and that G is quasiprimitive on both bipartite halves Δ_1 and Δ_2. The possible quasiprimitive types for the actions of G on Δ_1 and Δ_2 were determined in [1]. It was proved that, if the two types of actions are different, then one type is SD and the other is PA. Graphs with this property are referred to as being of type $\{\mathrm{SD}, \mathrm{PA}\}$. Furthermore, it is not possible for both the quasiprimitive actions to be of type SD.

By Theorem 1.1, $\Gamma = \mathcal{G}(p, m)$ is of type $\{\mathrm{SD}, \mathrm{PA}\}$ whenever $(p, m) \neq (2, 1), (3, 1)$ and the socle of $\mathrm{Aut}(\Gamma)$ is isomorphic to $\mathrm{PSL}(2, p^m)^{p^m}$. Our construction arose from a detailed investigation of locally $(G, 2)$–arc transitive graphs of type $\{\mathrm{SD}, \mathrm{PA}\}$ in [4]. In that paper a general construction is given which yields locally $(G, 3)$–arc transitive graphs such that G has socle $\mathrm{Sz}(2^m)^{2^r}$, $\mathrm{Ree}(3^m)^{3^r}$, $\mathrm{PSU}(3, p^m)^{p^r}$ or $\mathrm{PSL}(2, p^m)^{p^r}$, where m divides r, and also provides locally $(G, 2)$–arc transitive graphs for which G has socle $\mathrm{PSL}(d, p^m)^{p^m}$ for $d \geq 3$. Small members of these families, where the socle has two simple direct factors had previously been given in [1, Example 4.1] and [2, Example 2.5].

2 Construction

In this section we present our construction. First we recall some elementary properties of coset graphs.

Given a group G with subgroups L and R, the coset graph $Cos(G, L, R)$ is defined to be the bipartite graph with vertex set $\Delta_1 \dot\cup \Delta_2$ where Δ_1 is the set $[G : L]$ of right cosets of L in G and $\Delta_2 = [G : R]$. Two vertices Lx and Ry are adjacent if and only if $xy^{-1} \in LR$. The group G acts as a group of automorphisms of $Cos(G, L, R)$ by right multiplication. We note the following lemma from [1]. (A subgroup L of a group G is *core-free* if it contains no nontrivial normal subgroup of G.)

Lemma 2.1 *For a group G with proper subgroups L, R such that $L \cap R$ is core-free in G, the graph $\Gamma = Cos(G, L, R)$ satisfies the following properties:*

(1) Γ is connected if and only if $\langle L, R\rangle = G$;
(2) $G \leqslant \mathrm{Aut}(\Gamma)$, and Γ is G–edge transitive and G–vertex intransitive;
(3) G acts faithfully on both Δ_1 and Δ_2 if and only if both L and R are core-free.
(4) Γ is G–locally primitive if and only if $L \cap R$ is a maximal subgroup of both L and R.
(5) Γ is locally $(G, 2)$–arc transitive if and only if L acts 2–transitively on $[L : L \cap R]$ and R acts 2–transitively on $[R : L \cap R]$.

Conversely, if Γ is a G–edge transitive but not G–vertex transitive graph, and v and w are adjacent vertices then $\Gamma \cong Cos(G, G_v, G_w)$.

We will construct the graphs $\mathcal{G}(p, m)$ as coset graphs for certain groups G and use Lemma 2.1 to verify connectivity and local 2–arc transitivity.

2.1 The group G

In this subsection we define the following group G, where p is a prime and m is a positive integer.

$$G \cong \begin{cases} (\mathrm{PSL}(2,p^m)^{p^m} \rtimes \mathrm{AGL}(1,p^m)) \rtimes C_m & \text{for } p = 2, \\ (\mathrm{PSL}(2,p^m)^{p^m}.2 \rtimes \mathrm{AGL}(1,p^m)) \rtimes C_m & \text{for } p \text{ odd.} \end{cases} \qquad (1)$$

The group $T = \mathrm{PSL}(2, p^m)$ acts 2–transitively on the set Ω of $p^m + 1$ points of the projective line over $\mathrm{GF}(p^m)$. Let $A = \mathrm{P\Gamma L}(2, p^m)$, $\omega_1 \in \Omega$ and let $H = A_{\omega_1}$. Then $H = M \rtimes (P \rtimes B) \cong \mathrm{A\Gamma L}(1, p^m)$, that is, $M \cong C_p^m$, $P = \langle t_1 \rangle \cong C_{p^m-1}$ and $B = \langle t_2 \rangle \cong C_m$. Also, there exists $\omega_2 \in \Omega \backslash \{\omega_1\}$ such that $A_{\omega_1\omega_2} = P \rtimes B$. Furthermore, M is isomorphic to the additive group of $\mathrm{GF}(p^m)$ and in this setting P is the group of automorphisms of $\mathrm{GF}(p^m)$ induced by field multiplications while B is the group of automorphisms induced by field automorphisms. Note that M is the unique minimal normal subgroup of H and H acts transitively by conjugation on the nontrivial elements of M.

Let F be the group of all functions $f : \mathrm{GF}(p^m) \to A$ with multiplication defined pointwise, that is, $(fg)(\beta) = f(\beta)g(\beta)$ for all $\beta \in \mathrm{GF}(p^m)$. Then $F \cong A^{p^m}$. Let N denote the subgroup of F consisting of all functions $f : \mathrm{GF}(p^m) \to T$. Then $N \lhd F$ and $N \cong T^{p^m}$. For each $\alpha \in \mathrm{GF}(p^m)$ we define the automorphism σ_α of F by

$$f^{\sigma_\alpha}(\beta) = f(\beta - \alpha), \text{ for } f \in F.$$

Note that $\sigma_\alpha^{-1} = \sigma_{-\alpha}$ and $U = \{\sigma_\alpha \mid \alpha \in \mathrm{GF}(p^m)\}$ is isomorphic to the additive group of $\mathrm{GF}(p^m)$. For each $\alpha \in \mathrm{GF}(p^m)^* = \mathrm{GF}(p^m)\backslash\{0\}$ we also define the automorphism τ_α of F by

$$f^{\tau_\alpha}(\beta) = f(\beta\alpha^{-1}), \text{ for } f \in F.$$

Then $\tau_\alpha^{-1} = \tau_{\alpha^{-1}}$ and $\{\tau_\alpha \mid \alpha \in \mathrm{GF}(p^m)^*\}$ is isomorphic to the multiplicative group of $\mathrm{GF}(p^m)$. Furthermore, for all $\alpha \in \mathrm{GF}(p^m)$, $\gamma \in \mathrm{GF}(p^m)^*$ and $f \in F$

we have

$$\begin{aligned} f^{\tau_\gamma^{-1}\sigma_\alpha\tau_\gamma}(\beta) &= f^{\tau_{\gamma^{-1}}\sigma_\alpha}(\beta\gamma^{-1}) \\ &= f^{\tau_{\gamma^{-1}}}(\beta\gamma^{-1}-\alpha) \\ &= f(\beta-\alpha\gamma) \\ &= f^{\sigma_{\alpha\gamma}}(\beta) \end{aligned}$$

and hence $\tau_\gamma^{-1}\sigma_\alpha\tau_\gamma = \sigma_{\alpha\gamma}$. Thus

$$K = \langle \sigma_\alpha, \tau_\gamma \mid \alpha \in \mathrm{GF}(p^m), \gamma \in \mathrm{GF}(p^m)^* \rangle \cong \mathrm{AGL}(1, p^m). \tag{2}$$

Note that each element of K can be expressed as $\sigma_\alpha\tau_\gamma$ for some (unique) $\alpha, \gamma \in \mathrm{GF}(p^m)$ with $\gamma \neq 0$. We also define the automorphism ρ of F by

$$f^\rho(\beta) = f(\beta^{p^{m-1}}), \text{ for } f \in F.$$

Then $\rho^{-1}\tau_\gamma\rho = \tau_{\gamma^p}$ and $\rho^{-1}\sigma_\alpha\rho = \sigma_{\alpha^p}$. Hence $\langle K, \rho\rangle \cong \mathrm{A\Gamma L}(1, p^m)$.

Now M (as defined in the first paragraph of this subsection) is isomorphic to the additive group of $\mathrm{GF}(p^m)$ but we write the operation in M as multiplication. Each $\alpha \in \mathrm{GF}(p^m)^*$ induces an automorphism of M, which for the additive group $\mathrm{GF}(p^m)$ would be multiplication by α. We denote the image of $l \in M$ under this automorphism by l^α. We also let $l^0 = 1$ for all $l \in M$. Note that, for all $\alpha, \beta \in \mathrm{GF}(p^m)$, we have $l^{\alpha+\beta} = l^\alpha l^\beta$ (since using additive notation in $(\mathrm{GF}(p^m), +)$, $l^\alpha l^\beta$ corresponds to $l\alpha + l\beta = l(\alpha+\beta)$). Also, from the homomorphism property, $l_1^\alpha l_2^\alpha = (l_1 l_2)^\alpha$ for all $l_1, l_2 \in M$. With this notation we do indeed have $l^{-1} = l^{-1}$, where the left hand side denotes the image of l under the automorphism induced by -1 and the right hand side denotes the inverse of l in M. Note that $\mathrm{GF}(p^m)^*$ and P induce the same group of automorphisms of M. Thus as M is abelian and multiplication in $\mathrm{GF}(p^m)$ is commutative, it follows that for all $h \in M \rtimes P \cong C_p^m \rtimes C_{p^m-1}$, $\alpha \in \mathrm{GF}(p^m)$ and $l \in M$, we have

$$(l^\alpha)^h = (l^h)^\alpha. \tag{3}$$

In addition,

$$(l^\alpha)^{t_2} = (l^{t_2})^{\alpha^p} \tag{4}$$

for all $l \in M$, where, recall that $B = \langle t_2 \rangle \cong C_m$ induces field automorphisms.

For each $t \in A$ and $l \in M$ define the function

$$\begin{aligned} f_{l,t} : \mathrm{GF}(p^m) &\to A \\ \beta &\mapsto l^\beta t \end{aligned} \tag{5}$$

Note that $f_{l,t}$ is the product of a linear function and a constant function, where $f : \mathrm{GF}(p^m) \to M$ is said to be *linear* if

$$f(\alpha+\beta) = f(\alpha)f(\beta) \text{ and } f(\alpha\beta) = f(\beta)^\alpha \text{ for all } \alpha, \beta \in \mathrm{GF}(p^m) \tag{6}$$

and f is *constant* if $f(\beta)$ is independent of β. To be explicit we have $f_{l,t}(\beta) = f_{l,1}(\beta)f_{1,t}(\beta)$ and $f_{l,1}$ is linear (by the property noted above) and $f_{1,t}$ is constant.

Now, for each $l \in M, t \in A$ and $\alpha \in \mathrm{GF}(p^m)$, we have

$$\begin{aligned}(f_{l,t})^{\sigma_\alpha}(\beta) &= f_{l,t}(\beta - \alpha)\\ &= l^{\beta-\alpha}t\\ &= l^{\beta}l^{-\alpha}t\\ &= f_{l,l^{-\alpha}t}(\beta).\end{aligned} \tag{7}$$

If $\alpha \neq 0$ then we also have

$$\begin{aligned}(f_{l,t})^{\tau_\alpha}(\beta) &= f_{l,t}(\beta\alpha^{-1})\\ &= l^{\beta\alpha^{-1}}t\\ &= (l^{\alpha^{-1}})^{\beta}t\\ &= f_{l^{\alpha^{-1}},t}(\beta).\end{aligned} \tag{8}$$

Now ρ centralises the constant functions $f_{1,t}$ for all $t \in A$. Let

$$S = \langle f_{1,t_2}\rho\rangle = \{f_{1,t_2^i}\rho^i \mid 0 \leq i \leq m-1\} \cong C_m.$$

Note that S normalises N, and as t_2 normalises P we also have that S normalises $\langle f_{1,t_1}\rangle$. Hence S normalises $\langle N, f_{1,t_1}\rangle$. When $p = 2$, $\langle N, f_{1,t_1}\rangle = N$ while when p is odd, $\langle N, f_{1,t_1}\rangle \cong T^{p^m}.2$. Furthermore, let $\sigma_\alpha\tau_\gamma \in K$ (defined in (2)). Then

$$\begin{aligned}(f_{1,t_2}\rho)^{-1}\sigma_\alpha\tau_\gamma f_{1,t_2}\rho &= \rho^{-1}\sigma_\alpha\tau_\gamma\rho \qquad \text{as } K \text{ and } \rho \text{ centralise } f_{1,t_2}\\ &= \rho^{-1}\sigma_\alpha\rho\rho^{-1}\tau_\gamma\rho\\ &= \sigma_{\alpha^p}\tau_{\gamma^p} \in K\end{aligned}$$

and so S normalises K. Thus the group G generated by N, f_{1,t_1}, K and S is the following semidirect product.

$$G = ((\langle N, f_{1,t_1}\rangle \rtimes K) \rtimes S. \tag{9}$$

Then G is isomorphic to the group in (1).

Note that when $(p,m) \neq (2,1),(3,1)$ the group T is simple. Furthermore, we have $N = \mathrm{soc}(G)$ and N is a minimal normal subgroup of G since K permutes the p^m simple direct factors transitively. However, if $(p,m) = (2,1)$ then $\mathrm{soc}(G) \cong C_3^2$ and if $(p,m) = (3,1)$, then $\mathrm{soc}(G) \cong C_2^6$.

2.2 The subgroups L and R of G

We define core-free subgroups L, R of G as below and discuss the coset actions of G on $\Delta_1 = [G : L]$ and $\Delta_2 = [G : R]$.

$$L \cong (\mathrm{PGL}(2, p^m) \times \mathrm{AGL}(1, p^m)) \rtimes C_m \tag{10}$$

$$R \cong ((C_p^{2m} \rtimes C_{p^m-1}) \rtimes \mathrm{AGL}(1, p^m)) \rtimes C_m \tag{11}$$

$$L \cap R \cong (\mathrm{AGL}(1, p^m) \times \mathrm{AGL}(1, p^m)) \rtimes C_m \tag{12}$$

Let

$$L = (\{f_{1,t} \mid t \in \mathrm{PGL}(2, p^m)\} \times K) \rtimes S \leqslant G. \tag{13}$$

Then L satisfies (10). For the action of G on the set $\Delta_1 = [G : L]$, the stabiliser in $N = \mathrm{soc}(G)$ of the point L is

$$N \cap L = \{f_{1,t} \mid t \in T\} \cong T.$$

When $(p, m) \neq (2, 1)$ or $(3, 1)$ we have T is simple and so the action of G on Δ_1 is quasiprimitive of type SD. In fact, since K acts primitively on $\mathrm{GF}(p^m)$, it follows that L is a maximal subgroup of G and so the action of G on Δ_1 is primitive in this case. However, if $(p, m) = (2, 1)$ or $(3, 1)$ then $|\,\mathrm{soc}(G)| = 9$ or 64 respectively while $|\Delta_1| = 6$ and 24^2. Thus in these cases $\mathrm{soc}(G)$ is an intransitive normal subgroup of G and so G is not quasiprimitive.

For all $l, l_1 \in M$ and $h, h_1 \in M \rtimes P$ we have

$$\begin{aligned} f_{l,h} f_{l_1,h_1}(\beta) &= f_{l,h}(\beta) f_{l_1,h_1}(\beta) \\ &= l^\beta h l_1^\beta h_1 \\ &= l^\beta (l_1^\beta)^{h^{-1}} h h_1 \\ &= l^\beta (l_1^{h^{-1}})^\beta h h_1 \qquad \text{by (3)} \\ &= (l l_1^{h^{-1}})^\beta h h_1 \\ &= f_{l l_1^{h^{-1}}, h h_1}(\beta). \end{aligned} \tag{14}$$

Hence $f_{l,h} f_{l_1,h_1} = f_{l l_1^{h^{-1}}, h h_1}$, and so $\{f_{l,h} \mid l \in M, h \in M \rtimes P\}$ is a subgroup of $\langle N, f_{1,t_1} \rangle$ which, by (7) and (8), is normalised by K.

Now for all $l \in M$ and $t \in A$,

$$\begin{aligned}
(f_{l,t})^{f_{1,t_2}\rho}(\beta) &= (f_{l,t})^{f_{1,t_2}}(\beta^{p^{m-1}}) \\
&= f_{1,t_2^{-1}}(\beta^{p^{m-1}}) f_{l,t}(\beta^{p^{m-1}}) f_{1,t_2}(\beta^{p^{m-1}}) \\
&= t_2^{-1} l^{\beta^{p^{m-1}}} t t_2 \\
&= (l^{\beta^{p^{m-1}}})^{t_2} t^{t_2} \\
&= (l^{t_2})^{\beta^{p^m}} t^{t_2} \qquad \text{by (4)} \\
&= f_{l^{t_2},t^{t_2}}(\beta) \qquad \text{as } \beta^{p^m} = \beta. \qquad (15)
\end{aligned}$$

Then as t_2 normalises M and P, it follows that $\{f_{l,h} \mid l \in M, h \in M \rtimes P\}$ is normalised by S.

Now let

$$R = (\{f_{l,h} \mid l \in M, h \in M \rtimes P\} \rtimes K) \rtimes S \leqslant G. \qquad (16)$$

Note that $R \cap F$ is the group generated by all constant functions with values in $M \rtimes P$ and all linear functions $f : \mathrm{GF}(p^m) \to M$. Thus $R \cap F \cong C_p^{2m} \rtimes C_{p^m-1}$. Then R satisfies (11). Further, we have

$$L \cap R = (\{f_{1,h} \mid h \in M \rtimes P\} \times K) \rtimes S$$

and so L and R satisify (12). For the action of G on the set $\Delta_2 = [G : R]$, the stabiliser in N of the point R is

$$N \cap R = \{f_{l,h} \mid l \in M, h \in H \cap T\}.$$

Hence $N \cap R$ is a subgroup of the group of all functions $f : \mathrm{GF}(p^m) \to H \cap T$ and for each $\beta \in \mathrm{GF}(p^m)$ the set of images of β under $N \cap R$ is equal to $H \cap T$. When $(p, m) \neq (2, 1), (3, 1)$, T is simple and so the action of G on Δ_2 is quasiprimitive of type PA. Also, when $(p, m) = (2, 1)$, $\mathrm{soc}(G)$ is elementary abelian and acts regularly on Δ_2. Hence G is primitive on Δ_2 but not of type PA. However, when $(p, m) = (3, 1)$ we have $|\mathrm{soc}(G)| = 64$ and $|\Delta_1| = 192$. Hence $\mathrm{soc}(G)$ is intransitive on Δ_2 and so G is not quasiprimitive in this case.

We have shown that the group G defined in (9) has core-free subgroups L and R such that when $(p, m) \neq (2, 1), (3, 1)$:

(1) the action of G on $\Delta_1 = [G : L]$ is primitive of type SD, and
(2) the action of G on $\Delta_2 = [G : R]$ is quasiprimitive of type PA.

Construction 2.2 *With G as in (9), L as in (13) and R as in (16), we define $\mathcal{G}(p,m) = Cos(G,L,R)$.*

Let $\Gamma = \mathcal{G}(p,m)$. The vertices in $\Delta_1 = [G : L]$ have valency $|L : L \cap R| = p^m + 1$ while the vertices in $\Delta_2 = [G : R]$ have valency $|R : L \cap R| = p^m$. Furthermore,

$$|\Delta_1| = |G : L| = |T|^{p^m - 1}$$

and

$$|\Delta_2| = |G : R| = |T|^{p^m - 1}(p^m + 1)/p^m.$$

When $(p,m) \neq (2,1)$ or $(3,1)$, L is maximal in G, and it follows that $\langle L, R \rangle = G$. Thus by Lemma 2.1, Γ is connected. When $(p,m) = (2,1)$, R is maximal in G so $\langle L, R \rangle = G$ and again Γ is connected. We will prove in Section 4 that $\mathcal{G}(3,1)$ has 3 connected components. The following lemma summarises these basic properties of $\mathcal{G}(p,m)$.

Lemma 2.3 $\Gamma = \mathcal{G}(p,m)$ *is a bipartite graph of valencies $p^m + 1$ and p^m, and $\mathcal{G}(p,m)$ which is connected if and only if $(p,m) \neq (3,1)$. When $(p,m) \neq (2,1), (3,1)$, Γ is of type* $\{\mathrm{SD}, \mathrm{PA}\}$.

For all p and m, Lemma 2.1 implies that G is edge transitive and $L \cap R$ is the stabiliser of the edge between the vertices corresponding to L and R. Now $G = N(L \cap R)$ and so N is edge transitive on $\mathcal{G}(p,m)$. Thus by Lemma 2.1 we have the following result.

Lemma 2.4 *Let v and w be adjacent vertices in $\mathcal{G}(p,m)$. Then $\mathcal{G}(p,m) \cong Cos(N, N_v, N_w)$.*

3 Proof of Theorem 1.1

Let p be a prime and m a positive integer. Let $\Gamma = \mathcal{G}(p,m)$. We start by determining the largest value of s for which Γ is locally (G,s)–arc transitive, where G is as in (9).

Let v be the vertex of Γ corresponding to L and w be the vertex corresponding to R. Then $G_v = L$, $G_w = R$ and v is adjacent to w. Also

$$G_{vw} = L \cap R = (\{f_{1,h} \mid h \in M \rtimes P\} \times K) \rtimes S.$$

We define $G_v^{[1]}$ to be the kernel of the action of G_v on $\Gamma(v)$ and $G_w^{[1]}$ to be the kernel of the action of G_w on $\Gamma(w)$.

Thus $G_v^{[1]}$ is equal to the group K defined in (2) and $G_v^{\Gamma(v)} \cong \mathrm{P\Gamma L}(2, p^m)$ in its 3–transitive action on $p^m + 1$ points.

Now $\{f_{l,1} \mid l \in M\} \leqslant R$ is a set of coset representatives for $L \cap R$ in R. Hence $\Gamma(w) = \{Lf_{l,1} \mid l \in M\}$ and $\{f_{l,1} \mid l \in M\}$ induces the additive group of $\mathrm{GF}(p^m)$ on $\Gamma(w)$. To determine the action of G_w on $\Gamma(w)$ it remains to determine the action of G_{vw}. For each $h \in M \rtimes P$,

$$Lf_{l,1}f_{1,h} = Lf_{1,h^{-1}}f_{l,1}f_{1,h} = Lf_{l^h,1}$$

by (14). Also,

$$\begin{aligned} Lf_{l,1}\sigma_\alpha &= L\sigma_\alpha^{-1}f_{l,1}\sigma_\alpha \\ &= Lf_{l,l^{-\alpha}} && \text{by (7)} \\ &= Lf_{1,l^{-\alpha}}f_{l,1} && \text{as } M \text{ is abelian} \\ &= Lf_{l,1}, \end{aligned}$$

$$Lf_{l,1}\tau_\gamma = L\tau_\gamma^{-1}f_{l,1}\tau_\gamma = Lf_{l^{\gamma^{-1}},1} \text{ by (8)}$$

and

$$Lf_{l,1}f_{1,t_2}\rho = L(f_{1,t_2}\rho)^{-1}f_{l,1}f_{1,t_2}\rho = Lf_{l^{t_2},1}. \qquad \text{by (15)}$$

Now for each $h \in P$, there is a unique $\alpha \in \mathrm{GF}(p^m)^*$ such that $(l^h)^{\alpha^{-1}} = l$ for all $l \in M$. Let

$$Q = \Big\{ f_{1,h}\tau_\alpha \mid h \in P, \alpha \in \mathrm{GF}(p^m)^* \text{ and } (l^h)^{\alpha^{-1}} = l \text{ for all } l \in M \Big\}.$$

Then for all $f_{1,h_1}\tau_\alpha, f_{1,h_2}\tau_\gamma \in Q$,

$$f_{1,h_1}\tau_\alpha f_{1,h_2}\tau_\gamma = f_{1,h_1h_2}\tau_{\alpha\gamma} \in Q$$

and so $Q \cong P \cong C_{p^m-1}$. Furthermore,

$$G_w^{[1]} = (\{f_{1,h} \mid h \in M\} \times U) \rtimes Q, \tag{17}$$

where we recall that $U = \{\sigma_\alpha \mid \alpha \in \mathrm{GF}(p^m)$. Hence $G_w^{\Gamma(w)} \cong \mathrm{A\Gamma L}(1, p^m)$ in its 2–transitive action of degree p^m. Thus Γ is locally $(G, 2)$–arc transitive.

We now look at 3–arcs. We can pick $u \in \Gamma(v)\backslash\{w\}$ such that

$$G_{uvw} = (\{f_{1,h} \mid h \in P\} \times K) \rtimes S.$$

Then the kernel of the action of G_{uvw} on $\Gamma(w)$ is equal to $U \rtimes Q$ and

$$(G_{uvw})^{\Gamma(w)} \cong \Gamma\mathrm{L}(1, p^m)$$

which fixes v and acts transitively on $\Gamma(w)\backslash\{v\}$. Hence G_u acts transitively on the set of 3–arcs starting at u.

Choose $l_1 \in M\backslash\{1\}$ such that l_1 is centralised by B (which is defined in the first paragraph of subsection 2.1). This is possible as B centralises p elements of M. Let $x = Lf_{l_1,1} \in \Gamma(w)$. Then

$$G_{xwv} = ((\{f_{1,h} \mid h \in M\} \times U) \rtimes Q) \rtimes S.$$

Note that for $f_{1,h}\tau_\alpha \in Q$,

$$\rho^{-1} f_{1,t_2^{-1}} f_{1,h}\tau_\alpha f_{1,t_2}\rho = \rho^{-1} f_{1,h^{t_2}}\tau_\alpha\rho = f_{1,h^{t_2}}\tau_{\alpha^p}.$$

Now $l^h = l^\alpha$ for all $l \in M$ and using (4) we see that $l^{h^{t_2}} = l^{\alpha^p}$. Hence $f_{1,h^{t_2}}\tau_{\alpha^p} \in Q$ and so S normalises Q. The kernel of the action of G_{xwv} on $\Gamma(v)$ is equal to U and $(G_{xwv})^{\Gamma(v)} \cong \mathrm{A\Gamma L}(1, p^m)$ which fixes w and acts transitively on $\Gamma(v)\backslash\{w\}$. Hence G_x acts transitively on the set of 3–arcs starting at x and so Γ is locally $(G, 3)$–arc transitive.

Next we investigate the action of $G_v^{[1]} \cap G_w^{[1]}$ on the set of vertices at distance two from v or w.

Lemma 3.1 *Let v and w be adjacent vertices such that v has valency $p^m + 1$ and w has valency p^m. Then $G_v^{[1]} \cap G_w^{[1]} = U$ which has order p^m and acts trivially on the set of vertices at distance two from v while it has $p^m - 1$ orbits of length p^m on the set of those vertices at distance two from w which are not adjacent to v. Furthermore, the orbits of length p^m are the sets $\Gamma(x')\backslash\{w\}$ for each $x' \in \Gamma(w)\backslash\{v\}$.*

PROOF. We have already seen that $G_v^{[1]} = K$ and $G_w^{[1]}$ was given in (17). Hence $G_v^{[1]} \cap G_w^{[1]} = U$, which has order p^m and so it remains to prove the assertions about the action of U on the set of vertices at distance two from v or w.

Let u' be an arbitrary element of $\Gamma(v)\backslash\{w\}$. Then there exists $t \in T$ such that $w^{f_{1,t}} = u'$. Thus

$$\Gamma(u') = \Gamma(w)^{f_{1,t}} = \{Lf_{l,1}f_{1,t} \mid l \in M\} = \{Lf_{l,t} \mid l \in M\}.$$

For each $\alpha \in \mathrm{GF}(p^m)$ and $l \in M$ we have

$$\begin{aligned} Lf_{l,t}\sigma_\alpha &= L\sigma_\alpha^{-1} f_{l,t}\sigma_\alpha \\ &= Lf_{l,l^{-\alpha}t} \qquad \text{by (7)} \\ &= Lf_{1,l^{-\alpha}} f_{l,t} \quad \text{as } M \text{ is abelian} \\ &= Lf_{l,t}. \end{aligned}$$

Hence U acts trivially on each $\Gamma(u')$ and so acts trivially on the set of vertices at distance two from v.

Let $\{a_1, a_2, \ldots, a_{p^m+1}\}$, where $a_1 = 1$, be a set of coset representatives for $T \cap H$ in T. Then $\{f_{1,a_i} \mid i \in \{1, 2, \ldots, p^m + 1\}\}$ is a set of coset representatives for G_{vw} in G_v. Hence

$$\Gamma(v) = \{Rf_{1,a_i} \mid i \in \{1, 2, \ldots, p^m + 1\}\}.$$

For each $x' = Lf_{l,1} \in \Gamma(w) \backslash \{v\}$ we have

$$\Gamma(x') = \Gamma(v)^{f_{l,1}} = \{Rf_{1,a_i} f_{l,1} \mid i \in \{1, 2, \ldots, p^m + 1\}\}$$

and note that $Rf_{1,a_1} f_{l,1} = R = w$. Hence the set of all vertices at distance two from w which are not adjacent to v is

$$\{Rf_{1,a_i} f_{l,1} \mid l \in M \backslash \{1\}, i \in \{2, \ldots, p^m + 1\}\}.$$

Let $i \in \{2, 3, \ldots, p^m + 1\}$. Then for all $\alpha \in \mathrm{GF}(p^m)$ and $l \in M$,

$$\begin{aligned} Rf_{1,a_i} f_{l,1}\sigma_\alpha &= R\sigma_\alpha^{-1} f_{1,a_i} f_{l,1}\sigma_\alpha \\ &= Rf_{1,a_i} f_{l,l^{-\alpha}} \qquad \text{by (7)} \\ &= Rf_{1,a_i} f_{l,1} f_{1,l^{-\alpha}} \\ &= Rf_{1,a_i} f_{1,l^{-\alpha}} f_{l,1} \quad \text{as } M \text{ is abelian} \\ &= Rf_{1,a_i l^{-\alpha}} f_{l,1}. \end{aligned}$$

For a fixed $l \in M$, we have $\{l^{-\alpha} \mid \alpha \in \mathrm{GF}(p^m)\} = M$. Then as M acts transitively on $\Omega \backslash \{\omega_1\}$, for each $l \in M$ we have

$$\{Rf_{1,a_i l^{-\alpha}} f_{l,1} \mid i \in \{2, \ldots, p^m + 1\}\} = \{Rf_{1,a_i} f_{l,1} \mid i \in \{2, \ldots, p^m + 1\}\}.$$

Thus U acts transitively on each set $\Gamma(x') \backslash \{w\}$.

Next we consider 4–arcs. As above let $u \in \Gamma(v) \backslash \{w\}$ and $x = Lf_{l_1,1} \in$

$\Gamma(w)\backslash\{v\}$ such that

$$G_{uvwx} = (U \rtimes Q) \rtimes S \cong \mathrm{A\Gamma L}(1, p^m),$$

where $U \cong C_p^m$ is the unique minimal normal subgroup of G_{uvwx}. Since $x = Lf_{l_1,1}$,

$$\Gamma(x) = \Gamma(v)^{f_{l_1,1}} = \{Rf_{1,a_i}f_{l_1,1} \mid i \in \{1, 2, \ldots, p^m + 1\}\}.$$

By Lemma 3.1, U acts transitively on $\Gamma(x)$. Then as U is the unique minimal normal subgroup of G_{uvwx}, the group G_{uvwx} acts faithfully on $\Gamma(x)$ and fixes w. Hence $(G_{uvwx})^{\Gamma(x)} \cong \mathrm{A\Gamma L}(1, p^m)$ and G_u acts transitively on the set of 4–arcs starting at u.

We now look at 4–arcs beginning at x. Since $N_v^{\Gamma(v)}$ is 2–transitive, there exists $g \in N_v$ such that $w^g = u$ and $u^g = w$. Furthermore, we can take $g = f_{1,t_3}$ for any $t_3 \in T\backslash(T \cap H)$ such that $H \cap H^{t_3} = P \rtimes B$. Now t_3 normalises $H \cap H^{t_3}$ and $N_{\mathrm{PGL}(2,p^m)}(P \rtimes B) \cong D_{2(p^m-1)}$. Thus t_3 inverts every element of P and has order 2. Furthermore, since we may take $t_3 \in \mathrm{PSL}(2, p)$ it follows that B centralises t_3. Now

$$\Gamma(u) = \Gamma(w)^g = \{Lf_{l,1}f_{1,t_3} \mid l \in M\} = \{Lf_{l,t_3} \mid l \in M\}$$

and so $\{f_{l,t_3} \mid l \in M\}$ is a set of representatives for the L–cosets in $\Gamma(u)$. Since the elements of $\Gamma(u)$ are at distance two from v, Lemma 3.1 implies that U acts trivially on $\Gamma(u)$.

We now investigate how $Q \leqslant G_{uvwx}$ acts on $\Gamma(u)$. Let $f_{1,h}\tau_\alpha \in Q$ and recall that $h \in P \leqslant H \cap H^{t_3}$. Then

$$\begin{aligned}
Lf_{l,t_3}f_{1,h}\tau_\alpha &= Lf_{l,t_3h}\tau_\alpha \\
&= L\tau_\alpha^{-1}f_{l,t_3h}\tau_\alpha \\
&= Lf_{l^{\alpha^{-1}},t_3h} && \text{by (8)} \\
&= Lf_{l^{h^{-1}},t_3h} && \text{as } l^h = l^\alpha \text{ for all } l \in M \\
&= Lf_{l^{h^{-1}},h^{t_3^{-1}}t_3} \\
&= Lf_{l^{h^{-1}},h^{-1}t_3} && \text{as } t_3 \text{ inverts } h \text{ and } t_3 = t_3^{-1} \\
&= Lf_{1,h}f_{l^{h^{-1}},h^{-1}}f_{1,t_3} \\
&= Lf_{l^{h^{-2}},1}f_{1,t_3} && \text{by (14)} \\
&= Lf_{l^{h^{-2}},t_3}.
\end{aligned}$$

Hence Q induces the group $\{h^{-2} \mid h \in P\}$ on $\Gamma(u)\backslash\{v\}$. Recall that $S =$

$\langle f_{1,t_2}\rho\rangle$ where $t_2 \in B$. Then

$$\begin{aligned} Lf_{l,t_3}f_{1,t_2}\rho &= L\rho^{-1}f_{1,t_2^{-1}}f_{l,t_3}f_{1,t_2}\rho \\ &= Lf_{l^{t_2},(t_3)^{t_2}} && \text{by (15)} \\ &= Lf_{l^{t_2},t_3} && \text{as } B \text{ centralises } t_3. \end{aligned}$$

Thus $(G_{uvwx})^{\Gamma(u)\backslash\{v\}} \cong \{h^{-2} \mid h \in P\} \rtimes \{h \mid h \in B\}$. Hence when p is odd, $(G_{uvwx})^{\Gamma(u)\backslash\{v\}} \cong C_{(p^m-1)/2} \rtimes C_m$ which has two orbits of equal length and so Γ is not locally $(G,4)$–arc transitive. However, when $p = 2$ we have $(G_{uvwx})^{\Gamma(u)\backslash\{v\}} \cong C_{2^m-1} \rtimes C_m \cong \Gamma\mathrm{L}(1,2^m)$ and so in this case Γ is locally $(G,4)$–arc transitive.

In the case where Γ is locally $(G,4)$–arc transitive, that is, when $p = 2$, we now look at 5–arcs. Since $(G_{uvwx})^{\Gamma(x)} \cong \mathrm{A}\Gamma\mathrm{L}(1,2^m)$ we can choose $y \in \Gamma(x)\backslash\{w\}$ such that

$$G_{uvwxy} = Q \rtimes S \cong \Gamma\mathrm{L}(1,2^m).$$

This still induces $\Gamma\mathrm{L}(1,2^m)$ on $\Gamma(u)\backslash\{v\}$ and so G_y is transitive on the set of 5–arcs starting at y. Recall that t_2 centralises l_1 and t_3 and let $z = Lf_{l_1,t_3} \in \Gamma(u)\backslash\{v\}$. Then by (15) and Lemma 3.1,

$$G_{zuvwx} = U \rtimes S,$$

which acts transitively on $\Gamma(x)\backslash\{w\}$. Thus Γ is locally $(G,5)$–arc transitive. Finally $G_{zuvwxy} = S$, which does not act transitively on a set of size p^m or of size $p^m - 1$ as $|S| = m$. Thus Γ is not locally $(G,6)$–arc transitive. Hence we have proved the following proposition.

Proposition 3.2 *Let $\Gamma = \mathcal{G}(p,m)$ and G be as in (9).*

(1) If $p = 2$ then Γ is locally $(G,5)$–arc transitive but not locally $(G,6)$–arc transitive.
(2) If p is odd then Γ is locally $(G,3)$–arc transitive. Moreover, G_w is transitive on the set of 4–arcs starting at w, while G_v has two orbits of equal size on the set of 4–arcs beginning at v.

Next we determine $\mathrm{Aut}(\Gamma)$ for $(p,m) \neq (3,1),(2,1)$.

Proposition 3.3 $\mathrm{Aut}(\mathcal{G}(p,m)) = G$ *for* $(p,m) \neq (3,1),(2,1)$.

PROOF. Let $\Gamma = \mathcal{G}(p,m)$ and assume that $(p,m) \neq (3,1),(2,1)$. By Lemma 2.3, Γ is connected and by Proposition 3.2, Γ is locally 2–arc transitive, and is not a complete bipartite graph. Also, since Γ is not regular, the two sets Δ_1 and Δ_2 are fixed setwise by $\mathrm{Aut}(\Gamma)$. Hence by [1, Lemma 5.2], $\mathrm{Aut}(\Gamma)$

acts faithfully on both Δ_1 and Δ_2. Thus $G \leqslant \mathrm{Aut}(\Gamma) < \mathrm{Sym}(\Delta_1)$ and $G \leqslant \mathrm{Aut}(\Gamma) < \mathrm{Sym}(\Delta_2)$.

Now G acts primitively on Δ_1 of type SD (see Lemma 2.3) and so by [5, Proposition 8.1], $\mathrm{Aut}(\Gamma)$ is also primitive of type SD and has the same socle as G. Thus

$$\mathrm{Aut}(\Gamma) \leqslant N.(\mathrm{Out}(T) \times \mathrm{Sym}(\mathrm{GF}(p^m))$$

and

$$\mathrm{Aut}(\Gamma)_v \leqslant \{f : \mathrm{GF}(p^m) \to A \mid f \text{ constant}\} \times \mathrm{Sym}(\mathrm{GF}(p^m))$$

where $T = \mathrm{PSL}(2, p^m)$ and $A = \mathrm{P\Gamma L}(2, p^m)$. Also

$$\mathrm{Aut}(\Gamma)_{vw} = \mathrm{Aut}(\Gamma)_v \cap \Big(\{f : \mathrm{GF}(p^m) \to H \mid f \text{ constant}\} \times \mathrm{Sym}(\mathrm{GF}(p^m))\Big)$$

where $H \cong \mathrm{A\Gamma L}(1, p^m)$.

Let $f\sigma \in \mathrm{Aut}(\Gamma)_{vw}$. Then $f\sigma$ normalises $N_w = \{f_{l,h} \mid l \in M, h \in T \cap H\}$. Suppose first that $f = 1$, that is $\sigma \in \mathrm{Aut}(\Gamma)_{vw}$. Let $f_{l,h} \in N_w$. Then $(f_{l,h})^\sigma \in N_w$ and so $(f_{l,h})^\sigma = f_{l',h'}$ for some $l' \in L$ and $h' \in T \cap H$. Suppose further that σ fixes two elements of $\mathrm{GF}(p^m)$. Since

$$K = \mathrm{AGL}(1, p^m) \leqslant \mathrm{Sym}(\mathrm{GF}(p^m)) \cap \mathrm{Aut}(\Gamma)_{vw}$$

and is 2–transitive on $\mathrm{GF}(p^m)$ we may assume that σ fixes 0 and 1. Then $(f_{l,h})^\sigma(0) = f_{l,h}(0) = h$ and $(f_{l,h})^\sigma(1) = f_{l,h}(1) = lh$. Hence $l' = l$, $h' = h$ and $(f_{l,h})^\sigma = f_{l,h}$. By (5), $f_{l,h}(\beta) = l^\beta h$ and so distinct $\beta, \beta' \in \mathrm{GF}(p^m)$ correspond to distinct values $f_{l,h}(\beta) \neq f_{l,h}(\beta')$. It follows that σ fixes each element of $\mathrm{GF}(p^m)$, that is $\sigma = 1$. Hence letting

$$\phi : \mathrm{Aut}(\Gamma)_{vw} \longrightarrow \{f : \mathrm{GF}(p^m) \to H \mid f \text{ constant}\}$$

be the projection homomorphism we have $\ker(\phi) = K$. In particular, $K \lhd \mathrm{Aut}(\Gamma)_{vw}$. Let

$$\psi : \mathrm{Aut}(\Gamma)_{vw} \to \mathrm{Sym}(\mathrm{GF}(p^m))$$

be the other projection homomorphism of $\mathrm{Aut}(\Gamma)_{vw}$. Then as $K \lhd \mathrm{Aut}(\Gamma)_{vw}$ it follows that $K \lhd \psi(\mathrm{Aut}(\Gamma)_{vw})$. Hence $\psi(\mathrm{Aut}(\Gamma)_{vw}) \leqslant \mathrm{A\Gamma L}(1, p^m)$.

Suppose now that $f \in \mathrm{Aut}(\Gamma)_{vw}$. Then there exists $t \in H$ such that $f(\beta) = t$ for all $\beta \in \mathrm{GF}(p^m)$. Thus for all $l \in M$ and $h \in T \cap H$ we have $(f_{l,h})^f(\beta) = t^{-1}l^\beta ht = (l^\beta)^t h^t$. Now $(f_{l,h})^f(0) = h^t$ and $(f_{l,h})^f(1) = l^t h^t$. Since f normalises N_w, it follows that $(f_{l,h})^f = f_{l^t,h^t}$. However, $f_{l^t,h^t}(\beta) = (l^t)^\beta h^t$. Hence for all $l \in M$, $(l^t)^\beta = (l^\beta)^t$. Thus the automorphisms of M induced by β and t commute and this holds for all $\beta \in \mathrm{GF}(p^m)^*$. Since M is self-centralising in H, it follows that $t \in M \rtimes P$. Therefore

$$\ker(\psi) = \{f : \mathrm{GF}(p^m) \to M \rtimes P \mid f \text{ constant}\}.$$

Now suppose that $f : \mathrm{GF}(p^m) \to H$ such that $f(\beta) = t$ for all $\beta \in \mathrm{GF}(p^m)$ and $t \in H \backslash (M \rtimes P)$. Let $\sigma \in \mathrm{Sym}(\mathrm{GF}(p^m))$ be such that $f\sigma \in \mathrm{Aut}(\Gamma)_{vw}$. We have just shown that $\sigma \in \mathrm{A\Gamma L}(1, p^m)$. Also since $f \notin \mathrm{Aut}(\Gamma)_{vw}$ and $K \leqslant \mathrm{Aut}(\Gamma)_{vw}$, we have $\sigma \in \mathrm{A\Gamma L}(1, p^m) \backslash \mathrm{AGL}(1, p^m)$. As K is 2–transitive we may assume (by multiplying on the right by an appropriate element of K if necessary) that σ fixes 0 and 1. Similarly, we can assume (by multiplying on the left by a suitable element of $M \rtimes P$ if necessary) that $t \in B$. Then for all $f_{l,h} \in N_w$,

$$\begin{aligned}(f_{l,h})^{f\sigma}(\beta) &= (f_{l,h})^f(\beta^{\sigma^{-1}})\\ &= t^{-1} l^{\beta^{\sigma^{-1}}} ht\\ &= (l^{\beta^{\sigma^{-1}}})^t h^t.\end{aligned}$$

Then $(f_{l,h})^{f\sigma}(0) = h^t$ and $(f_{l,h})^{f\sigma}(1) = l^t h^t$. Hence as $f\sigma$ normalises N_w we have $(f_{l,h})^{f\sigma} = f_{l^t,h^t}$. However, $f_{l^t,h^t}(\beta) = (l^t)^\beta h^t$ and so $(l^t)^\beta = (l^{\beta^{\sigma^{-1}}})^t$ for all $l \in M$ and $\beta \in \mathrm{GF}(p^m)$. Now $t = (t_2)^i$ for some i, $0 \le i \le m-1$ and $\beta^\sigma = \beta^{p^s}$ for some s, $0 \le s \le m-1$. Therefore for all $l \in M$ and $\beta \in \mathrm{GF}(p^m)$,

$$\begin{aligned}(l^{(t_2)^i})^\beta &= (l^{\beta^{p^{m-s}}})^{(t_2)^i}\\ &= (l^{(t_2)^i})^{(\beta^{p^{m-s}})^{p^i}} \qquad \text{by (4)}\\ &= (l^{(t_2)^i})^{\beta^{p^{m-s+i}}}\end{aligned}$$

Hence for all $\beta \in \mathrm{GF}(p^m)$, we have $\beta = \beta^{p^{m-s+i}}$. Therefore $-s + i = 0$ and so $i = s$. Hence $f\sigma = f_{1,(t_2)^i}\rho^i \in S$. Thus

$$\mathrm{Aut}(\Gamma)_{vw} = (\{f : \mathrm{GF}(p^m) \to M \rtimes P \mid f \text{ constant}\} \rtimes K) \rtimes S = G_{vw}$$

Since N is transitive on Δ_1 we have $\mathrm{Aut}(\Gamma) = N \,\mathrm{Aut}(\Gamma)_v$. Furthermore, N_v is transitive on $\Gamma(v)$ and so $\mathrm{Aut}(\Gamma)_v = N_v \,\mathrm{Aut}(\Gamma)_{vw}$. Hence $\mathrm{Aut}(\Gamma) = N \,\mathrm{Aut}(\Gamma)_{vw} = NG_{vw} = G$.

To complete the proof of Theorem 1.1 it remains to show that $\mathrm{Aut}(\mathcal{G}(2,1)) = G$. Note that in this case $G \cong S_3 \,\mathrm{wr}\, S_2$. In the next section we prove our assertion made in the introduction that $\mathcal{G}(2,1)$ is the graph formed by placing a vertex at the midpoint of every edge of $K_{3,3}$. Since the automorphism group of $K_{3,3}$ is $S_3 \,\mathrm{wr}\, S_2$ this will complete the proof of Theorem 1.1.

4 $\mathcal{G}(2,1)$ and $\mathcal{G}(3,1)$

In this section we examine the structure of $\mathcal{G}(2,1)$ and $\mathcal{G}(3,1)$ as here T is not simple and so the graphs are not of type $\{\mathrm{SD}, \mathrm{PA}\}$. Given a graph Γ and group $G \leqslant \mathrm{Aut}(\Gamma)$, we say that Γ is (G, s)*–arc transitive* if G acts transitively on the set of s–arcs in Γ. A locally (G, s)–arc transitive graph for which G is vertex transitive is (G, s)–arc transitive. The *distance two graph* of a graph Γ is the graph with vertex set $V\Gamma$ where two vertices are joined by an edge if they are at distance two in Γ.

First we prove the remark made after the statement of Theorem 1.1 regarding the structure of $\mathcal{G}(2,1)$, namely:

Lemma 4.1 *The distance two graph of* $\mathcal{G}(2,1)$ *has a connected component isomorphic to* $K_{3,3}$ *and* $\mathcal{G}(2,1)$ *may be formed by placing a vertex at the midpoint of every edge of* $K_{3,3}$.

PROOF. Recall that $\mathcal{G}(2,1)$ has six vertices of valency 3 and nine vertices of valency 2. In [1, Section 3.3], locally $(G, 2s-1)$–arc transitive graphs with a vertex of valency 2 were investigated and all such graphs were shown to be formed by placing a vertex at the midpoint of every edge of some (G, s)–arc transitive graph. We apply this theory to $\mathcal{G}(2,1)$.

Let $\Gamma = \mathcal{G}(2,1)$ and $\tilde{\Gamma}$ be a connected component of the distance two graph of Γ which contains a vertex of valency 3. Then by [1, Theorem 3.10(1)], $\tilde{\Gamma}$ contains all of the six vertices of Γ of valency 3 and is $(G, 3)$–arc transitive. Furthermore, by [1, Theorem 3.10(2)], Γ can be formed by placing a vertex at the midpoint of every edge of $\tilde{\Gamma}$. It remains to show that $\tilde{\Gamma} = K_{3,3}$. Let v be a vertex of Γ of valency 3. Since Γ is locally $(G, 5)$–arc transitive, the smallest cycle in Γ has length at least 8 (see for example [12, Proposition 1.1]). Thus the vertices of $\tilde{\Gamma}(v)$ are pairwise nonadjacent. As $\tilde{\Gamma}$ has only 6 vertices and all are of valency 3 it follows that $\tilde{\Gamma} \cong K_{3,3}$.

Corollary 4.2 $\mathrm{Aut}(\mathcal{G}(2,1)) = \mathrm{Aut}(K_{3,3}) = S_3 \operatorname{wr} S_2$.

Next we investigate $\mathcal{G}(3,1)$ using the notation of Section 2. To describe the structure of $\mathcal{G}(3,1)$ we first define a new graph Σ. Let $H(3,4)$ be the Hamming graph whose vertex set is the set of 3-tuples of a set of size 4 and two vertices (x_1, x_2, x_3) and (y_1, y_2, y_3) are adjacent if and only if they differ in exactly one coordinate. Let Σ_1 be the set of vertices of $H(3,4)$ and Σ_2 be the set of maximal cliques of $H(3,4)$, that is, all the subsets of Σ_1 of size 4 such that there exists $i \in \{1,2,3\}$ for which two elements differ only in the i^{th} coordinate. We let Σ be the bipartite graph with vertex set $\Sigma_1 \dot\cup \Sigma_2$ and adjacency given by inclusion. We have the following proposition.

Proposition 4.3 $\mathcal{G}(3,1)$ *has three connected components, each isomorphic to* Σ.

PROOF. Let $\Gamma = \mathcal{G}(3,1)$ and recall that $\Gamma = Cos(G, L, R)$ where G is as in (9), L is as in (13) and R is as in (16). Note that $A = \mathrm{P\Gamma L}(2,3) \cong S_4$ has a normal subgroup Y isomorphic to C_2^2 and $A = YH$. Let X be the set of all functions $f : \mathrm{GF}(3) \to Y$. Then $X \cap R = 1$ and $\langle L, R\rangle = XR$. Note that $|XR| = |G|/3$ and so $\mathcal{G}(3,1)$ has three connected components. As G is edge transitive on Γ, these connected components are pairwise isomorphic. Let v be the vertex given by L and w be the vertex given by R. Let Γ_1 be the connected component of Γ which contains v and w. Then the stabiliser of Γ_1 in G is $\langle L, R\rangle = XR$.

Now $|XR : R| = |X|$ and $|XR : L| = 3.4^2$. Hence Γ_1 is bipartite with bipartite halves $\overline{\Delta}_1$ and $\overline{\Delta}_2$ such that $|\overline{\Delta}_1| = 3.4^2$ and $|\overline{\Delta}_2| = 4^3$. Furthermore, L and R are core free in XR and so by Lemma 2.1, XR acts faithfully on both Δ_1 and Δ_2. Since $X \cap R = 1$, X acts regularly on $\overline{\Delta}_2$ and so we can identify the elements of $\overline{\Delta}_2$ with the elements of X. Now X has three orbits B_1, B_2, B_3 on $\overline{\Delta}_1$. Let $X_1 = X_v = X \cap L$. Then $|X_1| = 4$ and X_1 fixes every vertex in the orbit B_1 containing v. Thus we can identify the elements of B_1 with the cosets of X_1 in X. Furthermore, since X preserves adjacency, it follows that each $x \in X$ is adjacent to the coset X_1x, that is adjacency is given by inclusion. Similarly, let X_2 be the stabiliser of a vertex in B_2 and X_3 be the stabiliser of a vertex in B_3. Then the vertices of B_2 and B_3 are the cosets of X_2 and X_3, respectively, in X and again adjacency between elements of X and of B_2, B_3 is inclusion. Then since Γ_1 is connected it follows that $X = X_1 \times X_2 \times X_3$. Thus the vertices of $\overline{\Delta}_2$ are the vertices of the Hamming graph $H(3,4)$ and the vertices in $\overline{\Delta}_1$ are the maximal cliques of $H(3,4)$, with adjacency given by inclusion. Thus $\Gamma_1 \cong \Sigma$.

Corollary 4.4 $\mathrm{Aut}(\mathcal{G}(3,1)) = \mathrm{Aut}(\Sigma) \,\mathrm{wr}\, S_3$ *and* $\mathrm{Aut}(\Sigma) = S_4 \,\mathrm{wr}\, S_3$. *Furthermore,* $\mathcal{G}(3,1)$ *is locally 3–arc transitive but not locally 4–arc transitive.*

PROOF. The first assertion follows from Proposition 4.3 and we saw in Proposition 3.2 that $\mathcal{G}(3,1)$ is locally 3–arc transitive. All that remains is to determine $\mathrm{Aut}(\Sigma)$ and show that $\mathcal{G}(3,1)$ is not locally 4–arc transitive. Any automorphism of $H(3,4)$ gives an automorphism of Σ. Conversely, given any two adjacent vertices in $H(3,4)$, they belong in a maximal clique. Then since automorphisms of Σ map maximal cliques to maximal cliques they map adjacent vertices of $H(3,4)$ to adjacent vertices of $H(3,4)$. Hence $\mathrm{Aut}(\Sigma) = \mathrm{Aut}(H(3,4)) = S_4 \,\mathrm{wr}\, S_3$. It is straightforward to find two 4–arcs starting at a vertex in Σ_2, (that is, a maximal clique of $H(3,4)$) for which one cannot be mapped to the other and so Σ is not locally 4–arc transitive.

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