Locally 2–arc transitive graphs, homogeneous factorisations and partial linear spaces*

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Abstract

We establish natural bijections between three different classes of combinatorial objects; namely certain families of locally 2–arc transitive graphs, partial linear spaces, and homogeneous factorisations of arc-transitive graphs. Moreover, the bijections intertwine the actions of the relevant automorphism groups. Thus constructions in any of these areas provide examples for the others.

1 Introduction

First we introduce relevant information about the three combinatorial structures in the title, and then we state our main theorem linking them. Let Γ be a finite graph with vertex set $V\Gamma$. An *s–arc* in Γ is an $(s+1)$–tuple $v_0, v_1, \ldots, v_s$ of vertices such that v_i is adjacent to v_{i+1} and $v_i \neq v_{i+2}$ for all i. We refer to 1–arcs as *arcs*. For a subgroup $G \leqslant \mathrm{Aut}(\Gamma)$ we say that Γ is *locally (G,s)–arc transitive* if for each vertex v, the vertex stabiliser G_v is transitive on the set of s–arcs starting at v. If G is vertex-transitive, then G is also transitive on the set of all s–arcs in Γ and we say that Γ is *(G,s)–arc*

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transitive. If G is not transitive on vertices and all vertices have valency at least two, then Γ is bipartite and the two parts of the bipartition are G–orbits. Moreover, a graph Γ is locally $(G, 2)$–arc transitive if and only if for every vertex v, G_v acts 2–transitively on the set $\Gamma(v)$ of vertices adjacent to v.

In [10], the authors introduced a global action analysis of locally (G, s)–arc transitive graphs for which G is vertex intransitive. Let Γ be a locally (G, s)–arc transitive graph with bipartite halves Δ_1 and Δ_2 which is not G–vertex transitive. Let N be a normal subgroup of G and suppose that N is intransitive on both Δ_1 and Δ_2. The quotient graph Γ_N, which has vertex set the set of N–orbits and adjacency inherited from Γ, is locally $(G/N, s)$–arc transitive and Γ is a *cover* of Γ_N, that is, if B_1 and B_2 are N–orbits which are adjacent in Γ_N then for each $v \in B_1$ there is a unique vertex $w \in B_2$ adjacent to v. This provides a useful reduction tool for studying locally s–arc transitive graphs. However, if N is a normal subgroup of G which acts transitively on Δ_1 and has k orbits on Δ_2, then Γ_N is the star $K_{1,k}$ and we have lost the local s–arc transitivity of the original graph. The following example is typical of graphs of this type.

Example 1.1. Let p be a prime and $q = p^e$ for some positive integer $e \geq 1$. Let Δ_1 be the set of points of the affine space $\mathrm{AG}(d, q)$ and Δ_2 be the set of lines of $\mathrm{AG}(d, q)$. Let Γ be the graph with vertex set $\Delta_1 \cup \Delta_2$ and adjacency given by the natural incidence. Let $G = \mathrm{AGL}(d, q)$. Then $G \leqslant \mathrm{Aut}(\Gamma)$ and Γ is locally $(G, 3)$–arc transitive. Let N be the normal subgroup of G given by all translations. Then N acts transitively on Δ_1 while it has $\frac{q^d-1}{q-1}$ orbits on Δ_2, these being the parallel classes of lines.

A *partial linear space* is a set $\mathcal{P}$ of points equipped with a collection $\mathcal{L}$ of subsets of $\mathcal{P}$ called *lines*, such that every pair of points is contained in at most one line. The affine space $\mathrm{AG}(d, q)$ is an example of a partial linear space. We say that a subset S of $\mathcal{L}$ is a *parallel class* if every point of $\mathcal{P}$ lies on a unique line of S, that is, if S is a partition of $\mathcal{P}$. A partial linear space is called a *linear space* if each pair of points lies on a unique line. There are two important graphs associated with a partial linear space $(\mathcal{P}, \mathcal{L})$. One is the *incidence graph*, which is the bipartite graph with vertex set $\mathcal{P} \cup \mathcal{L}$ such that $p \in \mathcal{P}$ and $l \in \mathcal{L}$ are adjacent if and only if $p \in l$. The other is the *point graph* which has vertex set $\mathcal{P}$ and two points are adjacent if and only if they lie on the same line. We say that a partial linear space is *connected* if its point graph is connected (or equivalently, if its incidence graph is connected).

Let Σ be a graph and let $\mathcal{F}$ be a partition of the edge set of Σ. Let $G \leqslant \mathrm{Aut}(\Sigma)$ and suppose that G leaves $\mathcal{F}$ invariant, the group $G^{\mathcal{F}}$ induced by G on $\mathcal{F}$ is transitive, and the kernel M of the action of G on $\mathcal{F}$ is transitive on $V\Sigma$. Then we say that $(M, G, \Sigma, \mathcal{F})$ is a *homogeneous factorisation.* We call each $F \in \mathcal{F}$ a *factor* and we also refer to as factors the subgraphs of Σ with edge sets equal to some $F \in \mathcal{F}$. Homogeneous factorisations of complete graphs were first introduced in [12] as a generalisation of vertex transitive, self-complementary graphs and their study was extended to general graphs and digraphs in [8].

Example 1.2. Let $q = p^e$ for some prime p and positive integer $e \geq 1$ and let Σ be the point graph of the affine space $\mathrm{AG}(d, q)$. Note that in $\mathrm{AG}(d, q)$ each pair of points lies on a unique line and so Σ is the complete graph on q^d vertices. Each line l in $\mathrm{AG}(d, q)$ provides $\binom{q}{2}$ edges between the q points of l, thus giving a clique of Σ. For each parallel class P of lines in $\mathrm{AG}(d, q)$ let F_P be the set of edges induced in Σ. The subgraph of Σ induced by each F_P is a vertex disjoint union of complete graphs, with each connected component corresponding to a line in P. Now $G = \mathrm{AGL}(d, q) \leqslant \mathrm{Aut}(\Sigma)$ and maps parallel classes to parallel classes. Hence if we let $\mathcal{F}$ be the set of all the F_P, then G leaves $\mathcal{F}$ invariant and $G^{\mathcal{F}}$ is a transitive group. Furthermore, the normal subgroup N given by all translations fixes each F_P setwise and hence $(N, G, \Sigma, \mathcal{F})$ is a homogeneous factorisation.

We have just seen that the partial linear space given by the affine space $\mathrm{AG}(d, q)$ determines on the one hand a locally 2–arc transitive graph with a star normal quotient (namely its incidence graph) and on the other a homogeneous factorisation such that each factor is a vertex disjoint union of complete graphs. Our main theorem shows that these links are not confined to affine spaces. First we define three classes of objects, each equipped with a natural equivalence relation. Given two sets Ω_1, Ω_2, and two groups $G_1 \leqslant \mathrm{Sym}(\Omega_1)$, $G_2 \leqslant \mathrm{Sym}(\Omega_2)$, we say that a bijection $\varphi : \Omega_1 \to \Omega_2$ *intertwines* G_1 and G_2 if $G_2 = \varphi \circ G_1 \circ \varphi^{-1}$.

$\mathcal{C}_1$ Triples (N, G, Γ), where Γ is a connected locally $(G, 2)$–arc transitive graph of girth greater than four such that G has two orbits Δ_1 and Δ_2 on vertices, and a normal subgroup N which is transitive on Δ_1 but intransitive on Δ_2. We say that $(N_1, G_1, \Gamma_1) \cong (N_2, G_2, \Gamma_2)$ if there exists a graph isomorphism $\varphi : V\Gamma_1 \to V\Gamma_2$ which intertwines G_1 and G_2, and N_1 and N_2.

$\mathcal{C}_2$ Triples $(N, G, \mathcal{S})$, where $\mathcal{S}$ is a connected partial linear space $(\mathcal{P}, \mathcal{L})$ with a group G of automorphisms such that for each point p and line l, G_p acts 2-transitively on the set of lines containing p and G_l acts 2–transitively on the set of points lying on l. Moreover, G has a normal subgroup N which acts transitively on points but intransitively on lines such that each orbit of N is a parallel class. We say that $(N_1, G_1, \mathcal{S}_1) \cong (N_2, G_2, \mathcal{S}_2)$ if there exists a partial linear space isomorphism $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ which intertwines G_1 and G_2, and N_1 and N_2.

$\mathcal{C}_3$ Homogeneous factorisations $(N, G, \Sigma, \mathcal{F})$ such that G acts 2-transitively on $\mathcal{F}$ and transitively on the set of arcs of the connected graph Σ, and each factor is a vertex disjoint union of complete graphs. We say that $(N_1, G_1, \Sigma_1, \mathcal{F}_1) \cong (N_2, G_2, \Sigma_2, \mathcal{F}_2)$ if there exists a graph isomorphism $\varphi : V\Sigma_1 \to V\Sigma_2$ which maps $\mathcal{F}_1$ to $\mathcal{F}_2$ and intertwines G_1 and G_2, and N_1 and N_2.

Theorem 1.3. *There are natural one-to-one correspondences between* $\mathcal{C}_1$*,* $\mathcal{C}_2$ *and* $\mathcal{C}_3$*, which preserve equivalence.*

There are several important observations to be made about this result.

1. We define an explicit one-to-one correspondence between each pair of these families, such that, for corresponding objects, the same group G is involved, and an explicit structure-preserving bijection is given that intertwines the actions of G. The correspondences use simple combinatorial constructions such as incidence graphs, point graphs and distance two graphs.

2. All the partial linear spaces in $\mathcal{C}_2$ are flag-transitive, that is, the group G acts transitively on the set of point-line incident pairs. A classification of the subfamily consisting of the flag-transitive linear spaces, apart from the one-dimensional affine case, was announced in [3] with proofs in [1, 2, 5, 6, 7, 11, 13, 14].

3. For partial linear spaces in $\mathcal{C}_2$, the partition of the set of lines into parallel classes given by the orbits of N is a *resolution* (an equivalence relation on the set of lines such that each point is contained in precisely one line in each equivalence class). Hence the partial linear spaces which arise in $\mathcal{C}_2$ are special types of resolvable incomplete block designs.

4. All vertex-intransitive locally 2–arc transitive connected graphs are either covers of complete bipartite graphs or of locally $(G, 2)$–arc transitive graphs for which G acts faithfully on both vertex orbits and quasiprimitively on at least one [10, Theorem 1.1]. Locally $(G, 2)$–arc transitive graphs for which G acts quasiprimitively on only one orbit lie in $\mathcal{C}_1$ and are investigated in [9].

We will prove Theorem 1.3 in Section 3, after supplying some more examples in Section 2.

In this paper we also prove the following theorem concerning the case where $(N, G, \Gamma) \in \mathcal{C}_1$ and N is regular on Δ_1, that is, N is transitive on Δ_1 and only the identity fixes a vertex. In this case, standard permutation group theory (see, for example, [4, Section 1.7]) allows us to identify Δ_1 with N such that N acts on Δ_1 by right multiplication, and if $v = 1_N$, then $G = N \rtimes G_v$ and G_v acts on Δ_1 by conjugation. The statement involves the following notion of an incidence graph: for a set N and a collection $\mathcal{B}$ of subsets of N, the *incidence graph relative to* $(N, \mathcal{B})$ has vertex set $N \cup \mathcal{B}$ with edges $\{x, B\}$ whenever $x \in B \in \mathcal{B}$.

Theorem 1.4. *Let $(N, G, \Gamma) \in \mathcal{C}_1$ and suppose that N acts regularly on Δ_1. Then after identifying Δ_1 with N as above and letting $v = 1_N$, the following all hold:*

(1) for each $w \in \Gamma(v)$, the subset $\Gamma(w)$ is an elementary abelian subgroup of N of order p^m, say, for some prime p,

(2) if we let $\mathcal{N} = \{\Gamma(w) : w \in \Gamma(v)\}$ and let $\mathcal{B}$ be the set of right cosets in N of the subgroups in $\mathcal{N}$, then Γ is the incidence graph relative to $(N, \mathcal{B})$.

(3) G_v acts 2–transitively on $\mathcal{N}$ and, for $M \in \mathcal{N}$, $(G_v)_M$ acts transitively by conjugation on the nontrivial elements of M.

(4) for $w \in \Delta_2$, $G_w^{\Gamma(w)}$ is a 2–transitive subgroup of $\mathrm{AGL}(m, p)$.

We prove Theorem 1.4 in Section 4. By Theorem 1.3, Theorem 1.4 has consequences for partial linear spaces $(N, G, \mathcal{S}) \in \mathcal{C}_2$ where N acts regularly on the set of points, and for homogeneous factorisations $(N, G, \Sigma, \mathcal{F}) \in \mathcal{C}_3$ where N acts regularly on $V\Sigma$. In the partial linear space case, the points of $\mathcal{S}$ can be identified with the elements of N such that each parallel class is the set of cosets in N of some elementary abelian subgroup. Given a nonempty self-inverse subset S of $N \backslash \{1\}$, the *Cayley graph* $\mathrm{Cay}(N, S)$ is the graph with vertex set N and for $x, y \in N$, $\{x, y\}$ is an edge if and only if $xy^{-1} \in S$. In the homogeneous factorisation case, $\Sigma = \mathrm{Cay}(N, S)$, where $S = \bigcup_{M \in \mathcal{N}} M \backslash \{1\}$ with $\mathcal{N}$ as given in Theorem 1.4(2). Moreover, $\mathcal{N}$ is a set of cliques containing 1_N in Σ and $\mathcal{B}$ is the set of images of these cliques under N.

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2 Some more examples

In this section we give two more families of locally 2–arc transitive graphs which lie in $\mathcal{C}_1$.

Example 2.1. Let $H(k,m)$ be the Hamming graph whose vertex set is the set of k–tuples of elements from a set of size m and two k–tuples are adjacent if and only if they differ in precisely one coordinate. The maximal cliques in $H(k,m)$ are subsets of m vertices such that there exists some i for which the vertices only differ in the i^{th} coordinate. Let Δ_1 be the set of vertices of $H(k,m)$ and Δ_2 be the set of maximal cliques. Define the graph Γ to be the incidence graph relative to (Δ_1,Δ_2). Then $G=S_m \operatorname{wr} S_k$ is a group of automorphisms of Γ such that Γ is locally $(G,3)$–arc transitive. Furthermore, $N=S_m^k \lhd G$ and acts transitively on Δ_1 while having k orbits on Δ_2. Thus $\Gamma_N=K_{1,k}$.

Example 2.2. Let $T=\mathrm{PSL}(2,p^f)$ for some prime p and positive integer f. Let S be the set of all elements of order p in T and let $\Sigma=\mathrm{Cay}(T,S)$. There are p^f+1 Sylow p–subgroups of T, each is elementary abelian of order p^f and all pairwise intersections are trivial. The maximal cliques of Σ correspond to the cosets of the subgroups of T which are maximal with respect to being contained in $S\cup\{1\}$, that is, the cosets of the Sylow p–subgroups. Thus Σ has $(p^f+1)(p^{2f}-1)/(gcd(2,p^f-1))$ maximal cliques, each of size p^f. Let $\Delta_1=T$, let Δ_2 be the set of maximal cliques of Σ, and Γ be the incidence graph relative to (Δ_1,Δ_2). When $p=2$ these are the graphs constructed in [10, Example 4.1].

Since S is a union of conjugacy classes, the group $G=T\rtimes\mathrm{PGL}(2,p^f)$ is a group of automorphisms of Σ, where T acts regularly by right multiplication and $\mathrm{PGL}(2,p^f)$ acts by conjugation. Also $G\leqslant\mathrm{Aut}(\Gamma)$ and the normal subgroup T acts regularly on Δ_1 while having p^f+1 orbits on Δ_2. Let $v=1_T$. Then $G_v=\mathrm{PGL}(2,p^f)$ and $\Gamma(v)$ is the set of Sylow p–subgroups of T. Thus $G_v^{\Gamma(v)}$ is 2–transitive. Let P be a Sylow p–subgroup of T. Then $(G_v)_P=N_T(P)=P\rtimes C_{p^f-1}$, which acts transitively on the set of nontrivial elements of P. Thus G_P acts 2–transitively on P and so Γ is locally $(G,2)$–arc transitive.

We now look at 3–arcs. Let Q be a Sylow p–subgroup of T distinct from P. Then (Q,v,P) is a 2–arc in Γ. Now $G_{Q,v,P}=N_T(P)\cap N_T(Q)\cong C_{p^f-1}$ which acts transitively on the set of nontrivial elements of P. Thus G_Q acts transitively on the set of 3–arcs emerging from Q. Let u be a nontrivial element of P. Then (u,P,v) is a 2–arc of Γ. Now $G_{u,P,v}=P$ which acts transitively on the set of Sylow p–subgroups of T distinct from P. Thus

G_u acts transitively on the set of 3–arcs starting at u and so Γ is locally $(G,3)$–arc transitive.

3 Proof of Theorem 1.3

A graph Γ is *G–locally primitive* if for every vertex v, G_v acts primitively on the set $\Gamma(v)$ of vertices adjacent to v. If Γ is a locally $(G,2)$–arc transitive graph and all vertices have valency at least two then, for all $v \in V\Gamma$, G_v acts 2–transitively on $\Gamma(v)$ and so Γ is G–locally primitive.

First we state Lemma 5.5 from [10] which is crucial to our results.

Lemma 3.1. *Let Γ be a connected G–locally primitive bipartite graph with G–orbits Δ_1 and Δ_2 on $V\Gamma$. Suppose there exists $N \lhd G$ such that N is transitive on Δ_1 but has $k > 1$ orbits on Δ_2. Then Γ_N is the star $K_{1,k}$. Furthermore, for each vertex $v \in \Delta_1$ and N–orbit B in Δ_2, $|B \cap \Gamma(v)| = 1$ and the vertex stabiliser N_v acts trivially on $\Gamma(v)$.*

We will use (and prove) a reformulation and extension of this result, Proposition 3.3, in terms of incidence graphs. It relies also on the following simple lemma which we state without proof.

Lemma 3.2. *Let Γ be a bipartite graph with the two parts of the bipartition being Δ_1, Δ_2 such that for distinct $w, w' \in \Delta_2$ the sets $\Gamma(w), \Gamma(w')$ are distinct, and let $\mathcal{B} = \{\Gamma(w) : w \in \Delta_2\}$. Then Γ is isomorphic to the incidence graph relative to $(\Delta_1, \mathcal{B})$.*

Proposition 3.3. *Let Γ be a connected G–locally primitive bipartite graph such that G has a nontrivial normal subgroup N which acts transitively on Δ_1 but intransitively on Δ_2. Suppose that Γ is not a complete bipartite graph and let $v \in \Delta_1$. Then the following all hold.*

- *(a) The set $\mathcal{N} = \{\Gamma(w) : w \in \Gamma(v)\}$ is a set of N_v–invariant subsets of Δ_1, each containing v, upon which G_v acts primitively, and given any set $W \in \mathcal{N}$ the set of images of W under N partitions Δ_1.*
- *(b) Γ is isomorphic to the incidence graph relative to $(\Delta_1, \mathcal{B})$, where $\mathcal{B}$ is the set of images of the elements of $\mathcal{N}$ under N.*
- *(c) Γ is locally $(G,2)$–arc transitive if and only if G_v acts 2–transitively on $\mathcal{N}$ and, for some $W \in \mathcal{N}$, G_W acts 2–transitively on the elements of W.*

Proof. Let $\mathcal{B} = \{\Gamma(w) : w \in \Delta_2\}$. Let $w, w' \in \Delta_2$ with $w \neq w'$ and suppose that $\Gamma(w) = \Gamma(w')$. Let $u \in \Gamma(w)$ and let $A = \{x \in \Gamma(u) : \Gamma(x) = \Gamma(w)\}$. Then A contains $\{w, w'\}$ and A is a block of imprimitivity for the primitive group $G_u^{\Gamma(u)}$. Hence $A = \Gamma(u)$, that is, $\Gamma(x) = \Gamma(w)$ for all $x \in \Gamma(u)$. It follows that $\Gamma(u') = \Gamma(u)$ for all $u' \in A$ and since Γ is connected, we conclude that $\Delta_1 = A$, $\Delta_2 = \Gamma(u)$ and Γ is a complete bipartite graph, which is a contradiction. Thus $\Gamma(w) \neq \Gamma(w')$ for distinct $w, w' \in \Delta_2$. Hence by Lemma 3.2, Γ is isomorphic to the incidence graph relative to $(\Delta_1, \mathcal{B})$. Furthermore, for all $g \in G$ and $w \in \Gamma(v)$, we have $w^g \in \Gamma(v^g)$ and $\Gamma(w^g) = \Gamma(w)^g$. Thus the action of G on $\mathcal{B}$ is equivalent to the action of G on Δ_2. Let $\mathcal{N} = \{\Gamma(w) : w \in \Gamma(v)\} \subset \mathcal{B}$. Then each subset in $\mathcal{N}$ contains v and $G_v^{\mathcal{N}} \cong G_v^{\Gamma(v)}$. Furthermore, as $N_v^{\Gamma(v)} = 1$ (by Lemma 3.1), it follows that each subset in $\mathcal{N}$ is N_v–invariant.

Let B be an orbit of N on Δ_2. Then by Lemma 3.1, every element of Δ_1 is adjacent to a unique element of B. Therefore, the set $\{\Gamma(w) : w \in B\}$ partitions Δ_1. Thus if w is the unique element of B adjacent to v, the set $\{\Gamma(w)^n : n \in N\}$ partitions Δ_1. This completes the proof of part (a).

By Lemma 3.1, the set $\mathcal{N}$ consists of a subset $\Gamma(w)$ for exactly one point w from each N–orbit on Δ_2. It follows that $\mathcal{B}$ is the set of images of elements of $\mathcal{N}$ under N. This completes the proof of part (b) and part (c) follows immediately from part (a). □

We now give three propositions, which together prove Theorem 1.3.

Proposition 3.4. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a partial linear space with $N \lhd G \leqslant \mathrm{Aut}(\mathcal{S})$ such that $(N, G, \mathcal{S}) \in \mathcal{C}_2$, and let Γ be the incidence graph relative to $(\mathcal{P}, \mathcal{L})$. Then $(N, G, \Gamma) \in \mathcal{C}_1$. Moreover, let $(N_1, G_1, \mathcal{S}_1)$, $(N_2, G_2, \mathcal{S}_2) \in \mathcal{C}_2$ with respective incidence graphs Γ_1, Γ_2. Then $(N_1, G_1, \Gamma_1) \cong (N_2, G_2, \Gamma_2)$ if and only if $(N_1, G_1, \mathcal{S}_1) \cong (N_2, G_2, \mathcal{S}_2)$.*

Proof. Let $p \in \mathcal{P}$ and $l \in \mathcal{L}$. Then as G_p acts 2–transitively on the set of lines containing the point p and G_l acts 2–transitively on the set of points contained in the line l, it follows that Γ is locally $(G, 2)$–arc transitive. As each pair of points is on at most one line the girth of Γ is greater than four, and as $\mathcal{S}$ is connected so is Γ. Furthermore, $N \lhd G$ acts transitively on one G–orbit (the set $\mathcal{P}$ of points) and intransitively on the other G–orbit (the set $\mathcal{L}$ of lines). Hence $(N, G, \Gamma) \in \mathcal{C}_1$.

Let $(N_1, G_1, \mathcal{S}_1)$, $(N_2, G_2, \mathcal{S}_2) \in \mathcal{C}_2$ such that $(N_1, G_1, \Gamma_1) \cong (N_2, G_2, \Gamma_2)$, where each Γ_i is the incidence graph of $\mathcal{S}_i$. Then there exists a graph isomorphism $\varphi : V\Gamma_1 \to V\Gamma_2$ which intertwines the actions of G_1 and G_2, and N_1 and N_2 on $V\Gamma_1$ and $V\Gamma_2$. Now $V\Gamma_1 = \Delta_1 \cup \Delta_2$, where $\Delta_1 = \mathcal{P}_1$ and

$\Delta_2 = \mathcal{L}_1$, with N_1 acting transitively on Δ_1 and intransitively on Δ_2. Similarly, $V\Gamma_2 = \Delta_1' \cup \Delta_2'$ with N_2 acting transitively on Δ_1' and intransitively on Δ_2'. Since φ intertwines the actions of N_1 and N_2 it follows that φ maps Δ_1 to Δ_1' and Δ_2 to Δ_2'. Hence $\phi = \varphi_{|\Delta_1}$ is a map from $\mathcal{P}_1$ to $\mathcal{P}_2$ and as φ preserves graph incidence, ϕ preserves partial linear space incidence. Hence ϕ is a partial linear space isomorphism, and it intertwines that actions of G_1 and G_2, and N_1 and N_2, on $\mathcal{P}_1$ and $\mathcal{P}_2$. Thus $(N_1, G_1, \mathcal{S}_1) \cong (N_2, G_2, \mathcal{S}_2)$.

Conversely, suppose $(N_1, G_1, \mathcal{S}_1) \cong (N_2, G_2, \mathcal{S}_2)$. Then there exists a partial linear space isomorphism $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ which intertwines both G_1 and G_2, and N_1 and N_2. Now ϕ induces a map from $\mathcal{L}_1$ to $\mathcal{L}_2$ which preserves incidence and so induces a graph isomorphism φ from Γ_1 to Γ_2. Moreover, ϕ intertwines the actions of G_1 and G_2, and N_1 and N_2, on $\mathcal{L}_1$ and $\mathcal{L}_2$. Hence φ intertwines the actions of G_1 and G_2, and N_1 and N_2, on $V\Gamma_1$ and $V\Gamma_2$. Thus $(N_1, G_1, \Gamma_1) \cong (N_2, G_2, \Gamma_2)$. □

The next proposition involves the notion of the *distance two graph* of a graph Γ: this has the same vertex set as Γ with $\{x, y\}$ an edge if and only if x, y are at distance two in Γ.

Proposition 3.5. *Let $(N, G, \Gamma) \in \mathcal{C}_1$ and let Σ be the connected component of the distance two graph of Γ containing a vertex of Δ_1. Then there exists a partition $\mathcal{F}$ of the edge set of Σ such that $(N, G, \Sigma, \mathcal{F}) \in \mathcal{C}_3$. Moreover, suppose that $(N_1, G_1, \Sigma_1, \mathcal{F}_1)$ and $(N_2, G_2, \Sigma_2, \mathcal{F}_2)$ are obtained from Γ_1 and Γ_2 respectively. Then $(N_1, G_1, \Sigma_1, \mathcal{F}_1) \cong (N_2, G_2, \Sigma_2, \mathcal{F}_2)$ if and only if $(N_1, G_1, \Gamma_1) \cong (N_2, G_2, \Gamma_2)$.*

Proof. As Γ is connected it follows that $V\Sigma = \Delta_1$. Let $v \in \Delta_1$ and $\mathcal{N} = \{\Gamma(w) : w \in \Gamma(v)\}$. Then $\mathcal{N}$ is a set of cliques in Σ, each containing v. Furthermore, as Γ does not have girth four, we have $W_1 \cap W_2 = \{v\}$ for all $W_1, W_2 \in \mathcal{N}$. For each $W_i \in \mathcal{N}$, Proposition 3.3(a) implies that the set of images of W_i under N partitions Δ_1. Thus the set of images of W_i under N partitions Σ into a set of vertex disjoint cliques in Σ. We denote the set of edges of Σ arising from this set of cliques by F_i. As the girth of Γ is greater than four, the F_i are pairwise disjoint and the set $\mathcal{F}$ of all the F_i forms a partition of the edge set of Σ. As G_v acts 2–transitively on $\Gamma(v)$ it acts 2–transitively on $\mathcal{F}$. By definition of the F_i, N fixes each F_i setwise, and as N is transitive on Δ_1, $G = NG_v$. Thus $G^{\mathcal{F}}$ is 2-transitive. Given $W \in \mathcal{N}$, Proposition 3.3(c) implies that G_W acts 2–transitively on the set of elements of W. Thus G_W acts transitively on the set of arcs in the clique of Σ corresponding to W. Then as N acts transitively on the set of cliques in each F_i and G is transitive on $\mathcal{F}$, it follows that G is transitive on the set of arcs of Σ. Also N is transitive on $V\Sigma$ and fixes each F_i setwise, so $(N, G, \Sigma, \mathcal{F})$ is a homogeneous factorisation which lies in $\mathcal{C}_3$.

Suppose that $(N_1, G_1, \Sigma_1, \mathcal{F}_1) \cong (N_2, G_2, \Sigma_2, \mathcal{F}_2)$. Then there exists a graph isomorphism $\varphi : V\Sigma_1 \to V\Sigma_2$ which maps $\mathcal{F}_1$ to $\mathcal{F}_2$ and intertwines the actions of G_1 and G_2, and N_1 and N_2, on $V\Sigma_1$ and $V\Sigma_2$. Let $V\Gamma_1 = \Delta_1 \cup \Delta_2$ and $V\Gamma_2 = \Delta_1' \cup \Delta_2'$ such that $V\Sigma_1 = \Delta_1$ and $V\Sigma_2 = \Delta_1'$. Also each $w \in \Delta_2$ corresponds to a connected component of some $F \in \mathcal{F}_1$. Similarly, each $w' \in \Delta_2'$ corresponds to a connected component of some $F' \in \mathcal{F}_2$. Then as φ is a graph isomorphism from Σ_1 to Σ_2 which maps $\mathcal{F}_1$ to $\mathcal{F}_2$, φ induces a graph isomorphism ϕ from Γ_1 to Γ_2. Moreover, as φ intertwines the actions of G_1 and G_2, and N_1 and N_2 on $V\Sigma_1$ and $V\Sigma_2$, it follows that ϕ intertwines the actions of G_1 and G_2, and N_1 and N_2, on $V\Gamma_1$ and $V\Gamma_2$. Hence $(N_1, G_1, \Gamma_1) \cong (N_2, G_2, \Gamma_2)$.

Conversely, suppose that $(N_1, G_1, \Gamma_1) \cong (N_2, G_2, \Gamma_2)$. Then there exists a graph isomorphism $\phi : V\Gamma_1 \to V\Gamma_2$ which intertwines G_1 and G_2, and N_1 and N_2. Then as N_1 acts transitively on Δ_1 and intransitively on Δ_2, and N_2 acts transitively on Δ_1' and intransitively on Δ_2', it follows that ϕ maps Δ_1 to Δ_1', and Δ_2 to Δ_2'. Hence ϕ induces a graph isomorphism φ from Σ_1 to Σ_2. Moreover, as each factor of $\mathcal{F}_1$ is an N_1–orbit on Δ_2 and each factor of $\mathcal{F}_2$ is an N_2–orbit on Δ_2' it follows that φ maps $\mathcal{F}_1$ to $\mathcal{F}_2$. Since ϕ intertwines the actions of G_1 and G_2, and N_1 and N_2, on $V\Gamma_1$ and $V\Gamma_2$, it follows that ϕ intertwines their actions on $V\Sigma_1$ and $V\Sigma_2$. Hence $(N_1, G_1, \Sigma_1, \mathcal{F}_1) \cong (N_2, G_2, \Sigma_2, \mathcal{F}_2)$. $\square$

Proposition 3.6. *Let $(N, G, \Sigma, \mathcal{F})$ be a homogeneous factorisation in $\mathcal{C}_3$. Then Σ is the point graph of a partial linear space $\mathcal{S}$ whose set of lines is the set of connected components of the factors of $\mathcal{F}$, and $(N, G, \mathcal{S}) \in \mathcal{C}_2$. Moreover, let $(N_1, G_1, \Sigma_1, \mathcal{F}_1), (N_2, G_2, \Sigma_2, \mathcal{F}_2) \in \mathcal{C}_3$ with associated partial linear spaces $\mathcal{S}_1$ and $\mathcal{S}_2$, Then $(N_1, G_1, \mathcal{S}_1) \cong (N_2, G_2, \mathcal{S}_2)$ if and only if $(N_1, G_1, \Sigma_1, \mathcal{F}_1) \cong (N_2, G_2, \Sigma_2, \mathcal{F}_2)$.*

Proof. Let $(N, G, \Sigma, \mathcal{F})$ be a homogeneous factorisation in $\mathcal{C}_3$. Let $\mathcal{P}$ be the set of vertices of Σ and $\mathcal{L}$ be the set of subsets of $\mathcal{P}$ that form the connected components of the $F_i \in \mathcal{F}$. Then as $\mathcal{F}$ is a partition of the edge set of Σ, each pair of vertices lies in at most one element of $\mathcal{L}$ and so $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a partial linear space with point graph Σ. Since Σ is connected, so is $\mathcal{S}$. Clearly G is admitted as a group of automorphisms of $\mathcal{S}$. Since $G = NG_v$ and $G^{\mathcal{F}}$ is 2-transitive while N acts trivially on $\mathcal{F}$, it follows that $G_v^{\mathcal{F}}$ is 2–transitive. Furthermore, G_v fixes setwise the set L of lines containing v and $G_v^L \cong G_v^{\mathcal{F}}$. Thus G_v acts 2–transitively on the set of lines containing v. Let l be a line in $\mathcal{L}$. Then l is a clique of Σ and is a block of imprimitivity for G on $E\Sigma$. As G is arc transitive, G_l acts transitively on the set of arcs in the clique of Σ

corresponding to l and so G_l acts 2–transitively on the points of l. Finally, N is transitive on the point set $\mathcal{P}$ and intransitive on lines, and each N–orbit on lines is the set of connected components from some $F_i \in \mathcal{F}$, and hence forms a parallel class of lines of $\mathcal{L}$. Thus $(N, G, \mathcal{S}) \in \mathcal{C}_2$.

Suppose that $(N_1, G_1, \mathcal{S}_1) \cong (N_2, G_2, \mathcal{S}_2)$. Then there exists a partial linear space isomorphism $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ which intertwines G_1 and G_2, and N_1 and N_2. Since $V\Sigma_1 = \mathcal{P}_1$ and $V\Sigma_2 = \mathcal{P}_2$ and φ preserves incidence, φ induces a graph isomorphism ϕ from Σ_1 to Σ_2. Moreover, φ takes N_1–orbits on $\mathcal{L}_1$ to N_2–orbits on $\mathcal{L}_2$. As each line in $\mathcal{L}_i$ is a connected component of some $F \in \mathcal{F}_i$, it follows that ϕ maps $\mathcal{F}_1$ to $\mathcal{F}_2$. Thus $(N_1, G_1, \Sigma_1, \mathcal{F}_1) \cong (N_2, G_2, \Sigma_2, \mathcal{F}_2)$.

Conversely, suppose that $(N_1, G_1, \Sigma_1, \mathcal{F}_1) \cong (N_2, G_2, \Sigma_2, \mathcal{F}_2)$. Then there exists a graph isomorphism $\phi : V\Sigma_1 \to V\Sigma_2$ which maps $\mathcal{F}_1$ to $\mathcal{F}_2$ and which intertwines G_1 and G_2, and N_1 and N_2. Since $\mathcal{L}_1$ is the set subsets of $\mathcal{P}_1 = V\Sigma_1$ arising from the connected components of the factors of $\mathcal{F}_1$ and $\mathcal{L}_2$ is the set subsets of $\mathcal{P}_2 = V\Sigma_2$ arising from the connected components of the factors of $\mathcal{F}_2$, it follows that ϕ induces a partial linear space isomorphism φ from $\mathcal{S}_1$ to $\mathcal{S}_2$ which intertwines the actions of G_1 and G_2, and N_1 and N_2, on $\mathcal{P}_1$ and $\mathcal{P}_2$. Hence $(N_1, G_1, \mathcal{S}_1) \cong (N_2, G_2, \mathcal{S}_2)$. □

4 Regular normal subgroups

We now prove Theorem 1.4, which deals with the case where N acts regularly on Δ_1. Recall that when N is regular on Δ_1, we may identify Δ_1 with N such that N acts on Δ_1 by right multiplication. Furthermore, if $v = 1_N$, then $G = N \rtimes G_v$ and G_v acts on Δ_1 by conjugation.

Proof of Theorem 1.4. Let Γ be a connected locally $(G, 2)$–arc transitive graph such that Γ is not a complete bipartite graph and G has two orbits Δ_1 and Δ_2 on vertices. Suppose that G has a normal subgroup N which acts regularly on Δ_1 but intransitively on Δ_2, and identify Δ_1 with N and $v = 1_N$ as above. By Proposition 3.3, the set $\mathcal{N} = \{\Gamma(w) : w \in \Gamma(v)\}$ is a set of N_v–invariant subsets of N, each containing v, such that Δ_2 can be identified with the set of images of the elements of $\mathcal{N}$ under N, and adjacency in Γ is given by incidence. Moreover, given $M \in \mathcal{N}$, the images of M under N form a partition of N.

Let $M \in \mathcal{N}$. Then $v = 1_N \in M$. Now let $m \in M$. Since the images of M under N partition Δ_1 and as N acts on Δ_1 by right multiplication, either $Mm = M$ or $Mm \cap M = \varnothing$. Now $1_N \in M$ and so $m \in Mm \cap M$. Thus $Mm = M$ and hence M is closed under multiplication. Also $1_N \in Mm^{-1} \cap M$. Thus $Mm^{-1} = M$ and as $1_N \in M$ it follows that $m^{-1} \in M$. Hence M is closed under inverses and so is a subgroup of N. Furthermore, the images of M under N are the right cosets of M in N.

Now by Proposition 3.3, G_v acts 2–transitively on $\mathcal{N}$ and G_M acts 2–transitively on M. Thus $(G_v)_M$ acts transitively on the set of nontrivial elements of M. As G_v acts on Δ_1 by conjugation it follows that all nontrivial elements of M have the same order and this order must be a prime, p say. Thus M is a finite p-group of exponent p and so it has a nontrivial centre which is characteristic in M. Since G_v must leave $Z(M)$ invariant and since also G_v is transitive on $M \backslash \{1_N\}$, it follows that $M = Z(M)$ and so M is elementary abelian. Hence we have proved (1) and (2). Part (3) follows from Proposition 3.3(c). As N acts on Δ_1 by right multiplication, if w is the vertex of Δ_2 corresponding to $M \in \mathcal{N}$ then $M = N_w \lhd G_w$. Thus $G_w^{\Gamma(w)}$ is a 2-transitive group with an elementary abelian normal subgroup $M^{\Gamma(w)} \cong M$ of order $|M| = p^m$. Thus $G_w^{\Gamma(w)} \lesssim \mathrm{AGL}(m, p)$ and we have completed the proof. □

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