On limit graphs of finite vertex-primitive graphs*

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Abstract

The class of all connected vertex-transitive graphs forms a metric space under a natural combinatorially defined metric. In this paper we study graphs which are limit points in this metric space of the subset consisting of all finite graphs that admit a vertex-primitive group of automorphisms. A description of these limit graphs provides a useful description of the possible local structures of generic finite graphs that admit a vertex-primitive automorphism group. We give an analysis of the possible types of these limit graphs, and suggest directions for future research. Some of the analysis relies on the finite simple group classification.

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1 Introduction

In this paper we study limit graphs of finite vertex-primitive graphs in a natural metric space consisting of vertex-transitive graphs. Before defining this metric space $(\mathcal{G}, \rho)$, we make a few remarks concerning the terminology used in the paper.

The metric space $(\mathcal{G}, \rho)$ and limit graphs

All graphs in this paper will be undirected graphs without loops or multiple edges. For a graph Γ, the vertex set and edge set of Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$ respectively. For a connected graph Γ, the usual metric on $V(\Gamma)$ is denoted $d_\Gamma(.,.)$, namely $d_\Gamma(x, y)$ denotes the length of a shortest path in Γ between the vertices x and y. For a vertex $x \in V(\Gamma)$ and a non-negative integer r, $B_\Gamma(x, r) = \{y \,|\, d_\Gamma(x, y) \leq r\}$ is the ball of radius r with centre x in Γ. Furthermore, we denote $B_\Gamma(x, 1)\backslash\{x\}$ by $\Gamma(x)$. Thus $\deg_\Gamma(x) := |\Gamma(x)|$ is the *valency* (or degree) of x. If $\deg_\Gamma(x)$ is finite for all $x \in V(\Gamma)$, then the graph Γ is said to be *locally finite*. If $\deg_\Gamma(x)$ is independent of $x \in V(\Gamma)$ (for example, if the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ is vertex-transitive), then $\deg(\Gamma) := \deg_\Gamma(x)$ denotes the valency (or degree) of Γ. If $\mathrm{Aut}(\Gamma)$ is vertex-transitive (respectively edge-transitive, vertex-primitive), then the graph Γ is called *vertex-transitive* (respectively *edge-transitive*, *vertex-primitive*). For a subset $X \subseteq V(\Gamma)$, the induced subgraph on X is denoted $\langle X\rangle_\Gamma$.

Let $\mathcal{G}^*$ denote the set of all pairs (Γ, x), where Γ is a connected locally finite graph and $x \in V(\Gamma)$, that is, the set of all undirected connected locally finite graphs with distinguished vertices. Then $\mathcal{G}^*$ is a metric space under the following metric ρ^*.

For $(\Gamma, x), (\Delta, y) \in \mathcal{G}^*$, $\rho^*((\Gamma, x), (\Delta, y))$ is defined to be zero if there is an isomorphism $\varphi : \Gamma \to \Delta$ such that $\varphi(x) = y$, and otherwise, $\rho^*((\Gamma, x), (\Delta, y))$ is defined as 2^{-k+1}, where k is the least positive integer such that the induced graphs $\langle B_\Gamma(x, k)\rangle_\Gamma \ncong \langle B_\Delta(y, k)\rangle_\Delta$. If Γ, Δ are vertex-transitive, then $\rho^*((\Gamma, x), (\Delta, y))$ is independent of the choice of $x \in V(\Gamma), y \in V(\Delta)$, and we write this value as $\rho(\Gamma, \Delta)$. Thus the function ρ induces a metric (which we also denote by ρ) on the set $\mathcal{G}$ of (isomorphism classes of) undirected connected locally finite vertex-transitive graphs. Studying the limit points in $(\mathcal{G}, \rho)$ of the set of finite vertex-primitive graphs, is the aim of the paper.

For each $d \geq 0$, let $\mathcal{G}_d$ denote the subset of $\mathcal{G}$ consisting of graphs of valency d, so that $\mathcal{G} = \cup_{d\geq 0}\mathcal{G}_d$. Every infinite sequence of graphs in $\mathcal{G}_d$ has a convergent subsequence (see Proposition 1), and hence $\mathcal{G}_d$ is compact for each $d \geq 0$. The basic properties of the metric space $(\mathcal{G}, \rho)$ were first studied in [16] and detailed proofs of several of the results may be found in [5]. In particular, it was shown that $(\mathcal{G}, \rho)$ is a locally compact, complete and totally disconnected metric space [5, Theorem 2.4].

If, in the metric space $(\mathcal{G}^*, \rho^*)$, a sequence $((\Gamma_i, x_i))_{i\geq 0}$ converges to (Γ, x), then, by definition, there exist $\varphi_i : V(\Gamma) \to V(\Gamma_i)$, for $i \geq 0$, such that $\varphi_i(x) = x_i$ for all $i \geq 0$ and, for any positive integer r, the restriction of φ_i to $B_\Gamma(x, r)$

induces an isomorphism from $\langle B_\Gamma(x,r)\rangle_\Gamma$ to $\langle B_{\Gamma_i}(x_i,r)\rangle_{\Gamma_i}$ for all sufficiently large i. If the sequence $(\varphi_i)_{i\geq 0}$ with these properties is given, then the sequence $((\Gamma_i,x_i))_{i\geq 0}$ is said to $(\varphi_i)_{i\geq 0}$-*converge to* (Γ,x).

For a subset $\mathcal{X}$ of $\mathcal{G}^*$ (respectively $\mathcal{G}$), the closure of $\mathcal{X}$ in $(\mathcal{G}^*,\rho^*)$ (respectively $(\mathcal{G},\rho)$) is denoted $\overline{\mathcal{X}}$. Thus $\overline{\mathcal{X}}$ consists of graphs with distinguished vertices from $\mathcal{G}^*$ (respectively graphs from $\mathcal{G}$) to which some sequence from $\mathcal{X}$ converges. Also, for $\mathcal{X}\subseteq(\mathcal{G}^*,\rho^*)$ (respectively $\mathcal{X}\subseteq(\mathcal{G},\rho)$), we define

$$\lim(\mathcal{X})=\{x\in\overline{\mathcal{X}}\mid x\in\overline{\mathcal{X}\setminus\{x\}}\},$$

the set of limit graphs with distinguished vertices (respectively the set of limit graphs) for $\mathcal{X}$. Note that $\lim(\mathcal{X})$ does not contain finite graphs with distinguished vertices (respectively finite graphs). In particular, if $\mathcal{X}$ consists of finite graphs with distinguished vertices (respectively finite graphs), then $\lim(\mathcal{X})=\overline{\mathcal{X}}\setminus\mathcal{X}$.

A description of the limit graphs of a subset $\mathcal{X}$ of $\mathcal{G}$ provides, in some sense, a description of the possible local structures of typical graphs in $\mathcal{X}$. In some situations, to prove that graphs from a set $\mathcal{X}$ have a certain property, it is sufficient to prove an appropriate property for the limit graphs of $\mathcal{X}$ and, most importantly, in these situations it is usually much simpler to work with the limit graphs. The type of arguments (when they work) proceed as follows: under the assumption that our desired result is false, we obtain a convergent sequence of distinct graphs from $\mathcal{X}$ without the appropriate property; further analysis of the corresponding limit graph then shows that the properties of this limit graph are incompatible with those known to hold for graphs in $\overline{\mathcal{X}}$. This type of argument, using limits and convergence in $\mathcal{G}^*$ and $\mathcal{G}$ (as well as convergence of automorphisms of graphs as defined below in Section 2) was used intensively in [16] for investigating the behaviour of automorphisms of graphs with vertex-primitive groups of automorphisms. The results in [16] motivated the following problem.

Problem 1 [9, Problem 12.89] *Describe the limit graphs for the subset $\mathcal{FP}$ of $\mathcal{G}$, where $\mathcal{FP}$ consists of all finite vertex-primitive graphs.*

In this paper we begin a systematic study of the set $\lim(\mathcal{FP})$. As remarked above, each graph in $\lim(\mathcal{FP})$ is infinite and vertex-transitive. However, not all graphs in $\lim(\mathcal{FP})$ admit a vertex-primitive group of automorphisms. For example, the infinite path is not vertex-primitive, and yet it is the unique limit graph of the subset of $\mathcal{FP}$ consisting of all cycles of prime order. More examples of vertex-imprimitive limit graphs are given in Example 3.

Our results involve application of the O'Nan-Scott Theorem for finite primitive permutation groups. This theorem, see for example [3, Chapter 4], partitions the family of finite primitive permutation groups into a number of disjoint types. The precise number of types varies across different statements of the theorem. However, three types are of particular importance in this paper, and these types are denoted by HA, AS and PA. Let G be a primitive permutation group on a finite set V, and let $v\in V$. Then G has *type* HA if it has a transitive abelian normal subgroup; *type* AS if it is an almost simple group, that is, if $T \lhd G \leq \mathrm{Aut}(T)$ and T is a finite nonabelian simple group; and, finally, G has *type* PA if G has a unique minimal normal subgroup $N = T^k$, where T is a finite nonabelian simple group and $k \geq 2$, and the stabiliser N_v projects onto nontrivial proper subgroups of the simple direct factors of N. For each of these three types X, let $\mathcal{FP}_{\mathrm{X}}$ denote the subset of $\mathcal{FP}$ consisting of all finite graphs admitting a vertex-primitive group of automorphisms of type X. Note however, see Example 1, a graph in $\mathcal{FP}$ may lie in more than one of the subsets $\mathcal{FP}_{\mathrm{X}}$. Our first result shows that the subsets $\mathcal{FP}_{\mathrm{X}}$, for $\mathrm{X} \in \{\mathrm{HA}, \mathrm{AS}, \mathrm{PA}\}$, in a sense control the limit graphs of $\mathcal{FP}$.

Theorem 1 $\lim(\mathcal{FP}) = \lim(\mathcal{FP}_{\mathrm{HA}}) \cup \lim(\mathcal{FP}_{\mathrm{AS}}) \cup \lim(\mathcal{FP}_{\mathrm{PA}})$*, and moreover,* $\lim(\mathcal{FP} \setminus (\mathcal{FP}_{\mathrm{HA}} \cup \mathcal{FP}_{\mathrm{AS}} \cup \mathcal{FP}_{\mathrm{PA}})) = \emptyset$*.*

This theorem, proved in Section 3, depends on the classification of the finite simple groups since, in particular, its proof depends on the Sims Conjecture [2] (and see Section 2). It leads us naturally to study the three sets $\lim(\mathcal{FP}_{\mathrm{HA}}), \lim(\mathcal{FP}_{\mathrm{AS}})$ and $\lim(\mathcal{FP}_{\mathrm{PA}})$. Our most decisive result concerns the first of these.

Theorem 2 $\lim(\mathcal{FP}_{\mathrm{HA}})$ *is disjoint from* $\lim(\mathcal{FP}_{\mathrm{AS}}) \cup \lim(\mathcal{FP}_{\mathrm{PA}})$*, and each graph in* $\lim(\mathcal{FP}_{\mathrm{HA}})$ *is a Cayley graph of a free abelian group of finite rank.*

This theorem is proved in Section 4. Although it gives a satisfactory general description of graphs in $\lim(\mathcal{FP}_{\mathrm{HA}})$, it is not clear precisely which Cayley graphs arise. Thus we pose the following problem.

Problem 2 *Determine which Cayley graphs of free abelian groups of finite rank lie in* $\lim(\mathcal{FP}_{\mathrm{HA}})$*.*

In Section 4 we give several examples of graphs in $\lim(\mathcal{FP}_{\mathrm{HA}})$, showing in particular that examples exist for free abelian groups of each finite rank (see Examples 2 and 3). However the most important open problem concerning $\lim(\mathcal{FP})$ is the following one.

Problem 3 *Give a useful description of the limit graphs in* $\lim(\mathcal{FP}_{\mathrm{AS}})$ *and* $\lim(\mathcal{FP}_{\mathrm{PA}})$*.*

In Sections 5 and 6 we begin work on this problem. In Section 5, we show that the graphs in $\lim(\mathcal{FP}_{\mathrm{AS}})$ are limits of finite graphs admitting almost simple primitive groups from the same family of Lie type groups, and having very restricted vertex stabilisers. Section 6 contains a construction of a family of graphs in $\lim(\mathcal{FP}_{\mathrm{PA}})$ that are Cartesian products of graphs in $\lim(\mathcal{FP}_{\mathrm{AS}})$.

Finite edge-transitive graphs admitting vertex-primitive automorphism groups are important in both Combinatorics and Algebra, and this is a major reason for studying the limits of such graphs in this paper. An additional motivation for studying the limits of edge-transitive graphs in $\mathcal{FP}$, which we discuss in the next paragraph, is that they shed light on the structure of general graphs in $\lim(\mathcal{FP})$. Thus we define $\mathcal{FP}^{e-trans}$ as the set of all finite vertex-primitive, edge-transitive graphs. Also, for each type $\mathrm{X} \in \{\mathrm{HA}, \mathrm{AS}, \mathrm{PA}\}$, we define $\mathcal{FP}_{\mathrm{X}}^{e-trans}$ to be the subset of $\mathcal{FP}$ of all finite graphs admitting an edge-transitive group of automorphisms that is also vertex-primitive of type X.

For each type X, the connection between $\mathcal{FP}_{\mathrm{X}}$ and $\mathcal{FP}_{\mathrm{X}}^{e-trans}$ is well understood and may be described as follows. If $\Gamma \in \mathcal{FP}_{\mathrm{X}}$ with a vertex-primitive subgroup G of automorphisms of type X, and if $E_1, \ldots, E_r$ are the G-orbits on $E(\Gamma)$, then for each i, the graph Σ_i on the same vertex set as Γ, and with edge set E_i, admits G as a vertex-primitive and edge-transitive group of automorphisms, and so Σ_i is connected (since G is primitive) and lies in $\mathcal{FP}_{\mathrm{X}}^{e-trans}$. Moreover Γ is the edge-disjoint union of $\Sigma_1, \ldots, \Sigma_r$. A similar connection exists between $\lim(\mathcal{FP}_{\mathrm{X}})$ and $\lim(\mathcal{FP}_{\mathrm{X}}^{e-trans})$. Let $\Gamma \in \lim(\mathcal{FP}_{\mathrm{X}})$, with vertex-transitive group G of automorphisms and G-edge orbits $E_1, \ldots, E_r$, and let the Σ_i be defined as before, so that Γ is the edge-disjoint union of $\Sigma_1, \ldots, \Sigma_r$. We cannot guarantee that the group G can be chosen to be vertex-primitive, so some of the Σ_i may be unconnected. However, (see Proposition 7), G may be chosen such that each connected component of each of the Σ_i lies in $\lim(\mathcal{FP}_{\mathrm{X}}^{e-trans})$.

Thus the graphs in $\lim(\mathcal{FP}^{e-trans})$ play a central role in our understanding of $\lim(\mathcal{FP})$. It follows from Theorem 1 that $\lim(\mathcal{FP}^{e-trans}) = \lim(\mathcal{FP}_{\mathrm{HA}}^{e-trans}) \cup \lim(\mathcal{FP}_{\mathrm{AS}}^{e-trans}) \cup \lim(\mathcal{FP}_{\mathrm{PA}}^{e-trans})$, and thus Problem 3 in the case of edge-transitive graphs may be formulated as follows.

Problem 4 *Describe the graphs in* $\lim(\mathcal{FP}_{\mathrm{AS}}^{e-trans})$ *and* $\lim(\mathcal{FP}_{\mathrm{PA}}^{e-trans})$.

We give several families of examples of limit graphs of $\mathcal{FP}_{\mathrm{AS}}^{e-trans}$ in Section 5, and in Section 7 we outline an approach for studying $\mathcal{FP}_{\mathrm{PA}}^{e-trans}$. However, even the subset $\lim(\mathcal{FP}^{e-trans})$ of $\lim(\mathcal{FP})$ seems difficult to describe in a satisfactory manner. Thus we restrict our current investigation further to studying the subset $\mathcal{FP}^{min}$ of $\mathcal{FP}^{e-trans}$ consisting of all graphs which are graphs of minimal valency for some finite vertex-primitive group. Here a connected graph Γ is a *graph of minimal valency* for a vertex-transitive group G of automorphisms, if the valency of Γ is minimal among all connected graphs Δ with $V(\Delta) = V(\Gamma)$ and $G \leq \mathrm{Aut}(\Delta)$. For such a graph Γ and group G, if G is vertex-primitive then it must act edge-transitively on Γ. Graphs of minimal valency are of special interest as being natural ones admitting primitive permutation groups. At the same time (we repeat) a description of limit graphs for $\mathcal{FP}^{min}$ provides in some sense, a description of the possible local structures of typical graphs in $\mathcal{FP}^{min}$, so $\lim(\mathcal{FP}^{min})$ is also of interest.

For each of the types $\mathrm{X} \in \{\mathrm{HA}, \mathrm{AS}, \mathrm{PA}\}$, we define $\mathcal{FP}_{\mathrm{X}}^{min}$ as the subset of $\mathcal{FP}_{\mathrm{X}}^{e-trans}$ of all finite graphs that are of minimal valency for some vertex-primitive group of type X. By Theorem 1, $\lim(\mathcal{FP}^{min}) = \lim(\mathcal{FP}_{\mathrm{HA}}^{min}) \cup \lim(\mathcal{FP}_{\mathrm{AS}}^{min}) \cup \lim(\mathcal{FP}_{\mathrm{PA}}^{min})$. In Section 6 we prove the following result about the graphs in $\lim(\mathcal{FP}_{\mathrm{PA}}^{min})$.

Theorem 3 $\Gamma \in \lim(\mathcal{FP}_{\mathrm{PA}}^{min})$ *if and only if* Γ *is the* n^{th} *Cartesian power of* Δ, *for some* $n > 1$ *and* $\Delta \in \lim(\mathcal{FP}_{\mathrm{AS}}^{min})$.

For a definition of Cartesian powers of graphs see the paragraph after Theorem 7. Consequently, the main open problem concerning graphs in $\lim(\mathcal{FP}^{min})$ is the following.

Problem 5 *Describe the graphs in* $\lim(\mathcal{FP}^{min}_{\mathrm{AS}})$.

2 Preliminary results on limit graphs and limit automorphisms

We recall some definitions and results from [5] and [16]. Let (Γ, x) be a limit graph with a distinguished vertex in $\mathcal{G}^*$. Fix a sequence $((\Gamma_i, x_i))_{i\geq 0}$ in $\mathcal{G}^*$ that $(\varphi_i)_{i\geq 0}$-converges to (Γ, x). Note that, if this holds, then for any integer sequence $0 \leq i_0 < i_1 < \dots$, the subsequence $((\Gamma_{i_j}, x_{i_j})_{j\geq 0}$ also $(\varphi_{i_j})_{j\geq 0}$-converges to (Γ, x).

A sequence $(h_i)_{i\geq 0}$ of automorphisms $h_i \in \mathrm{Aut}(\Gamma_i)$ is said to $(\varphi_i)_{i\geq 0}$-*converge to* $h \in \mathrm{Aut}(\Gamma)$ if, for any $y \in V(\Gamma)$, $\varphi_i(h(y)) = h_i(\varphi_i(y))$ holds for all sufficiently large i. For each i, let $G_i \subseteq \mathrm{Aut}(\Gamma_i)$. Then an automorphism g of Γ is a *limit automorphism with respect to* $(G_i)_{i\geq 0}$ provided there exists an increasing sequence $(i_j)_{j\geq 0}$ of non-negative integers such that, for some $g_j \in G_{i_j}$ $(j \geq 0)$, the sequence $(g_j)_{j\geq 0}$ is $(\varphi_{i_j})_{j\geq 0}$-convergent to g. The subgroup of $\mathrm{Aut}(\Gamma)$ generated by all automorphisms which are limit automorphisms with respect to $(G_i)_{i\geq 0}$ is called the *limit group with respect to* $(G_i)_{i\geq 0}$.

The next four propositions follow immediately from [16, Propositions 2.1-2.3], and are fundamental to our investigation. Detailed proofs can be found in [5]. They will often be used without reference throughout the paper. In the fourth we require the notion of an s-*arc* in a graph Γ: this is an $(s+1)$-tuple $(v_0, v_1, \dots, v_s)$ of vertices such that, for each i, v_i is adjacent to v_{i+1} and $v_i \neq v_{i+2}$.

Proposition 1 *Let* $((\Gamma_i, x_i))_{i\geq 0}$ *be a sequence in* $\mathcal{G}^*$ *such that the valencies of all vertices of all graphs* Γ_i, $i \geq 0$, *are bounded above by a constant. Then there exists an increasing sequence* $(i_j)_{j\geq 0}$ *of non-negative integers such that* $((\Gamma_{i_j}, x_{i_j}))_{i_j\geq 0}$ *is* $(\varphi_{i_j})_{i_j\geq 0}$*-convergent to some* $(\Gamma, x) \in \mathcal{G}^*$.

Proposition 2 *Let* $((\Gamma_i, x_i))_{i\geq 0}$ *be a sequence in* $\mathcal{G}^*$ *that* $(\varphi_i)_{i\geq 0}$*-converges to* (Γ, x)*, and let* r *be a non-negative integer. If, for each* $i \geq 0$, $g_i \in \mathrm{Aut}(\Gamma_i)$ *is such that* $d_{\Gamma_i}(x_i, g_i(x_i)) \leq r$*, then there exists an increasing sequence* $(i_j)_{j\geq 0}$ *of non-negative integers such that the sequence* $(g_{i_j})_{j\geq 0}$ *is* $(\varphi_{i_j})_{j\geq 0}$*-convergent to an automorphism of* Γ.

Proposition 3 *Let* $((\Gamma_i, x_i))_{i\geq 0}$ *be a sequence in* $\mathcal{G}^*$ *that* $(\varphi_i)_{i\geq 0}$*-converges to* (Γ, x)*. For each* $i \geq 0$*, let* $G_i \leq \mathrm{Aut}(\Gamma_i)$*, and let* G *be the limit group with respect to* $(G_i)_{i\geq 0}$*. Then, for any non-negative integers* r_1, r_2*, there exists a non-negative integer* $i(r_1, r_2)$ *such that, for any* $i \geq i(r_1, r_2)$ *and any* $g' \in G_i$ *with* $d_{\Gamma_i}(x_i, g'(x_i)) \leq r_1$*, there exists* $g \in G$ *such that* $\varphi_i(g(y)) = g'(\varphi_i(y))$ *for all* $y \in B_\Gamma(x, r_2)$.

Proposition 4 *Let $((\Gamma_i, x_i))_{i\geq 0}$ be a sequence in $\mathcal{G}^*$ that $(\varphi_i)_{i\geq 0}$-converges to (Γ, x), let r be a non-negative integer, for each $i \geq 0$ let $G_i \leq \mathrm{Aut}(\Gamma_i)$, and let G be the limit group with respect to $(G_i)_{i\geq 0}$. Then:*

(1) If X is a finite block of imprimitivity of G on $V(\Gamma)$, then, for each sufficiently large i, $\varphi_i(X)$ is a block of imprimitivity of G_i on $V(\Gamma_i)$.

(2) For all sufficiently large i, the restriction of φ_i to $B_\Gamma(x, r)$ induces a graph isomorphism from $\langle B_\Gamma(x, r)\rangle_\Gamma$ to $\langle B_{\Gamma_i}(x_i, r)\rangle_{\Gamma_i}$ and a permutation isomorphism from the group induced by $(G_i)_{x_i}$ on $B_{\Gamma_i}(x_i, r)$ to a subgroup of the group induced by G_x on $B_\Gamma(x, r)$.

(3) If, for all sufficiently large i, the group G_i is s-arc-transitive, where $s \geq 0$, then G is s-arc-transitive. If, for all sufficiently large i, the group G_i is edge-transitive, then G is edge-transitive.

It is a corollary of Proposition 2 that graphs in $\lim(\mathcal{FP})$ are vertex-transitive. They may not be vertex-primitive, but by Proposition 4 (1) their automorphism groups preserve no nontrivial system of imprimitivity with finite blocks. Proposition 4 (3) implies that graphs in $\lim(\mathcal{FP}^{e-trans})$ are indeed edge-transitive.

The Sims Conjecture states that there exists a function f on the natural numbers such that, if G is a primitive permutation group on a finite set and, for a point v, the stabiliser G_v of v has an orbit of length d on the remaining points, then $|G_v| \leq f(d)$. This conjecture was proved in [2], using the finite simple group classification. Moreover, according to [8], if G is a primitive group of automorphisms of a finite connected graph Γ and $v \in V(\Gamma)$, then G_v acts faithfully on the ball of radius 6 with centre v. Since the size of a ball of radius six is bounded, we can replace any convergent sequence in $\mathcal{FP}^{e-trans}$ by an infinite subsequence in which the permutation groups induced on the balls of radius six are permutationally isomorphic. This gives us the following result which is Proposition 2.7 of [5].

Proposition 5 *Let $((\Gamma_i, x_i))_{i\geq 0}$ be a sequence in $\mathcal{G}^*$, with each Γ_i finite, that $(\varphi_i)_{i\geq 0}$-converges to (Γ, x). Suppose that, for each $i \geq 0$, G_i is a vertex-primitive subgroup of $\mathrm{Aut}(\Gamma_i)$. Then there exists an infinite subsequence $(i_j)_{j\geq 0}$ and a finite group H such that $(G_{i_j})_{x_{i_j}} \cong H$ for all $j \geq 0$.*

Proposition 5 relies on the classification of finite simple groups since it relies on the Sims Conjecture. However, the following result was proved in [16, Theorem 1] without use of the finite simple group classification. It will be used in the proof of Theorem 5.

Proposition 6 *Let d and r be positive integers, and let f be a mapping on the non-negative integers such that $f(n) = o(n)$ as $n \to \infty$. Then there exists a positive integer $c(d, f, r)$ such that, if H is a vertex-primitive group of automorphisms of a graph Δ of valency d, and if $v \in V(\Delta)$ and $h \in H$ are such that h does not fix $B_\Delta(v, r)$ pointwise and $d_\Delta(u, h(u)) \leq f(d_\Delta(u, v))$ for all $u \in B_\Delta(v, c(d, f, r))$, then either $\langle h^H\rangle$ is abelian, or the diameter of Δ is at most $c(d, f, r)$.*

We remark that this result has the following interesting consequence: for any fixed finite valency $d \geq 1$, there are only finitely many graphs of valency d admitting a vertex-primitive group of automorphisms that contains an abelian non-normal vertex-transitive subgroup. Finally in this section we prove the observation made in the introduction concerning $\lim(\mathcal{FP}_{\mathrm{X}})$ and $\lim(\mathcal{FP}_{\mathrm{X}}^{e-trans})$.

Proposition 7 *Let $X \in \{\mathrm{HA}, \mathrm{AS}, \mathrm{PA}\}$, and let $\Gamma \in \lim(\mathcal{FP}_{\mathrm{X}})$. Then there exists a vertex-transitive subgroup G of $\mathrm{Aut}(\Gamma)$ such that the following holds. Let $E_1, \ldots, E_r$ be the G-orbits on $E(\Gamma)$, and for each s let Σ_s be the graph with $V(\Sigma_s) = V(\Gamma)$ and $E(\Sigma_s) = E_s$, so that Γ is the edge-disjoint union of $\Sigma_1, \ldots, \Sigma_r$. Then each connected component of each Σ_s lies in $\lim(\mathcal{FP}_{\mathrm{X}}^{e-trans})$.*

Proof Let $((\Gamma_i, x_i))_{i \geq 0}$ be a sequence in $\mathcal{G}^*$ with $\Gamma_i \in \mathcal{FP}_{\mathrm{X}}$ that $(\varphi_i)_{i\geq 0}$-converges to (Γ, x). For each $i \geq 0$, fix a vertex-primitive group G_i of automorphisms of Γ_i of type X and set $M_i = \{g \in G_i \mid d_{\Gamma_i}(x_i, g(x_i)) \leq 1\}$. It is straightforward to check that $\langle M_i \rangle$ is vertex-transitive and contains the stabiliser in G_i of x_i, and hence M_i generates G_i. Then, taking Proposition 5 into account, replacing the sequence $((\Gamma_i, x_i))_{i \geq 0}$ by a proper subsequence if necessary, we may assume that

(i) for each $i \geq 0$, the restriction of φ_i to $B_\Gamma(x, 1)$ is a bijection onto $B_{\Gamma_i}(x_i, 1)$;

(ii) there exist a finite subset M of $\mathrm{Aut}(\Gamma)$ and, for each $i \geq 0$, a bijection χ_i of M onto M_i such that for any $g \in M$ the sequence $(\chi_i(g))_{i\geq 0}$ is $(\varphi_i)_{i\geq 0}$-convergent to g. Note that M is the set of limit automorphisms with respect to $(M_i)_{i\geq 0}$.

For each $x' \in \Gamma(x)$, there exists an element in M mapping x to x' (since, for each $i \geq 0$, there exists an element in M_i mapping x_i to $\varphi_i(x')$). Thus the group $G = \langle M \rangle \leq \mathrm{Aut}(\Gamma)$ is vertex-transitive. In addition, $M = \{g \in G | d_\Gamma(x, g(x)) \leq 1\}$. In fact, if $g = g_1 \ldots g_k$ for some $g_1, \ldots, g_k \in M$ and $d_\Gamma(x, g(x)) \leq 1$, then the sequence $(\chi_i(g_1) \ldots \chi_i(g_k))_{i \geq 0}$ is $(\varphi_i)_{i\geq 0}$-convergent to g. Hence $\chi_i(g_1) \ldots \chi_i(g_k) \in M_i$ for all sufficiently large i, and, as a result, $g \in M$.

By Proposition 4(2), for all sufficiently large i, say for all $i \geq i_0$, the mapping φ_i realizes a bijection from the set of G_x-orbits on $\Gamma(x)$ onto the set of $(G_i)_{x_i}$-orbits on $\Gamma_i(x_i)$ such that paired orbits correspond to paired orbits. Let $O_1 = diag(V(\Gamma) \times V(\Gamma)), O_2, \ldots, O_r$ be the orbitals of G on $V(\Gamma)$ that contain a pair (x, x') with $x' \in \{x\} \cup \Gamma(x)$. For each $i \geq i_0$, let $O_{i,1}, \ldots, O_{i,r}$ be the corresponding orbitals of G_i (on $V(\Gamma_i)$), where $O_{i,s}(x_i) = \varphi_i(O_s(x))$, $1 \leq s \leq r$.

Fix an arbitrary s such that $1 \leq s \leq r$. We show that $(y, z) \in O_s$ implies $(\varphi_i(y), \varphi_i(z)) \in O_{i,s}$ for all sufficiently large i, with $i \geq i_0$. In fact, $x = g'_1 \ldots g'_{k'}(y)$ and $g'_1 \ldots g'_{k'}(z) \in O_s(x)$ for some $g'_1, \ldots, g'_{k'} \in M$. Since $(\chi_i(g'_1) \ldots \chi_i(g'_{k'}))_{i\geq 0}$ is $(\varphi_i)_{i\geq 0}$-convergent to the limit automorphism $g'_1 \ldots g'_{k'}$, it follows that $\chi_i(g'_1) \ldots \chi_i(g'_{k'})(\varphi_i(y)) = x_i$ and $\chi_i(g'_1) \ldots \chi_i(g'_{k'})(\varphi_i(z)) \in O_{i,s}(x_i)$ for all sufficiently large i, with $i \geq i_0$. It follows that $(\varphi_i(y), \varphi_i(z)) \in O_{i,s}$ for all sufficiently large i, with $i \geq i_0$.

Fix an arbitrary s such that $2 \leq s \leq r$. Let Γ' be the underlying graph (without multiple edges) of the directed graph with vertex set $V(\Gamma)$ and edge set O_s, and let Γ'' be the subgraph of Γ' generated by the connected component of Γ' containing x. For each $i \geq i_0$, let Γ'_i be the underlying graph (without multiple edges) of the directed graph with vertex set $V(\Gamma_i)$ and edge set $O_{i,s}$. Note that, for each $i \geq i_0$, the graph Γ'_i is connected, and G_i is an edge-transitive group of automorphisms of Γ'_i that is vertex-primitive of type X. Thus $\Gamma'_i \in \mathcal{FP}_{\mathrm{X}}^{e-trans}$. Moreover, since $(y,z) \in O_s$ implies that $(\varphi_i(y), \varphi_i(z)) \in O_{i,s}$ for all $i \geq i_0$, it follows that for all $i \geq i_0$

$$(\varphi_i)_{|_{V(\Gamma'')}} : V(\Gamma'') \to V(\Gamma'_i).$$

Hence $(\Gamma'_i)_{i\geq i_0}$ is $((\varphi_i)_{|_{V(\Gamma'')}})_{i\geq i_0}$-convergent to Γ''. Moreover, as G is vertex transitive, the connected components of Γ' are all isomorphic to Γ'' and so each connected component of Γ' lies in $\lim(\mathcal{FP}_{\mathrm{X}}^{e-trans})$. Varying s we get the result. $\square$

We also have the following proposition which we state without proof.

Proposition 8 *Let $\Gamma \in \lim(\mathcal{FP}_{\mathrm{X}})$ for some type* X *and let $O_1, \ldots, O_k$ be some orbitals of* $\mathrm{Aut}(\Gamma)$*. Then the graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma) \cup \bar{O}_1 \cup \ldots \cup \bar{O}_k$ is also in $\lim(\mathcal{FP}_{\mathrm{X}})$, where*

$$\bar{O}_i := \{\{x,y\} | (x,y) \in O_i \text{ or } (y,x) \in O_i\}.$$

3 Proof of Theorem 1

In Section 1 we defined subfamilies $\mathcal{FP}_{\mathrm{X}}$ of $\mathcal{FP}$ corresponding to various types X of finite primitive permutation groups identified by the O'Nan-Scott Theorem. The examples below demonstrate that the subfamilies $\mathcal{FP}_{\mathrm{X}}$ are not mutually disjoint.

Example 1 The complete graph K_5 lies in both $\mathcal{FP}_{\mathrm{HA}}$ and $\mathcal{FP}_{\mathrm{AS}}$ since both Z_5 and S_5 are vertex-primitive subgroups of automorphisms. Similarly, the Cartesian product $K_6 \square K_6$ (see Section 6 for a definition of the Cartesian product of graphs) lies in $\mathcal{FP}_{\mathrm{AS}} \cap \mathcal{FP}_{\mathrm{PA}}$ since both $\mathrm{Aut}(A_6)$ and $S_6 \wr S_2$ are vertex-primitive subgroups of automorphisms; and $K_5 \square K_5 \in \mathcal{FP}_{\mathrm{HA}} \cap \mathcal{FP}_{\mathrm{PA}}$ since it admits both $\mathrm{AGL}(1,5) \wr S_2$ and $S_5 \wr S_2$ acting primitively on vertices. Moreover, $K_{25} \in \mathcal{FP}_{\mathrm{HA}} \cap \mathcal{FP}_{\mathrm{AS}} \cap \mathcal{FP}_{\mathrm{PA}}$ because it admits the automorphism groups $\mathrm{AGL}(2,5)$, S_{25} and $S_5 \wr S_2$.

Defining properties for the primitive types HA, AS and PA were given in Section 1. In this section we prove Theorem 1, showing that each limit graph of $\mathcal{FP}$ is a limit of $\mathcal{FP}_{\mathrm{X}}$ for X = HA, AS or PA, and that the set $\mathcal{FP} \setminus (\mathcal{FP}_{\mathrm{HA}} \cup \mathcal{FP}_{\mathrm{AS}} \cup \mathcal{FP}_{\mathrm{PA}})$ has no limit graphs. We apply the O'Nan-Scott Theorem, a proof of which can be found in [3, Chapter 4]. Let G be a finite primitive permutation group on a set V. The information from the O'Nan-Scott Theorem that we need is related to the *socle* $\mathrm{Soc}(G)$ of G, that is, the product of the minimal normal subgroups of G. The first important fact is that $\mathrm{Soc}(G)$ is either (i) an elementary abelian p-group (for some prime p) acting regularly on V, or (ii) $\mathrm{Soc}(G) = T^k$ for some nonabelian simple group T and positive integer k. Here a permutation group S acts *regularly* on V if S is transitive on V, and the stabiliser $S_v = 1$ for $v \in V$. Groups satisfying (i) are of type HA; groups in case (ii) with $k = 1$ are of type AS. Moreover in case (ii) with $k > 1$, (see [3, Theorems 4.3B, 4.6A and 4.7B]), if $v \in V$, then G_v permutes the simple direct factors of $\mathrm{Soc}(G)$ by conjugation having at most two orbits in this action, and either some subgroup of G_v has T as a quotient, or $\mathrm{Soc}(G)$ is the unique minimal normal subgroup of G and G has type PA.

We now prove Proposition 9, a technical result from which Theorem 1 follows immediately. We note that the proof of Proposition 9 depends on the finite simple group classification. The reason is two-fold. Firstly it uses Proposition 5 that depends on the Sims' Conjecture, and secondly it uses certain detailed information from the O'Nan-Scott Theorem that depends on the 'Schreier Conjecture' (specifically the assertions about case (ii) above with $k > 1$) which, in turn, relies on the simple group classification.

Proposition 9 *Let* $((\Gamma_i, x_i))_{i\geq 0}$ *be a sequence in* $\mathcal{G}^*$*, with each* Γ_i *finite, that* $(\varphi_i)_{i\geq 0}$*-converges to some infinite graph* (Γ, x)*. Suppose that, for each* $i \geq 0$*,* G_i *is a vertex-primitive subgroup of* $\mathrm{Aut}(\Gamma_i)$*. Then*

(i) only finitely many of the Γ_i *lie in* $\mathcal{FP} \setminus (\mathcal{FP}_{\mathrm{HA}} \cup \mathcal{FP}_{\mathrm{AS}} \cup \mathcal{FP}_{\mathrm{PA}})$*, and in particular* $\lim(\mathcal{FP} \setminus (\mathcal{FP}_{\mathrm{HA}} \cup \mathcal{FP}_{\mathrm{AS}} \cup \mathcal{FP}_{\mathrm{PA}}))$ *is empty;*

(ii) there exists a subsequence $(i_j)_{j\geq 0}$ *such that the* G_{i_j} *are all of the same primitive type, and this type is* HA, AS *or* PA*.*

Proof Suppose that an infinite number of the Γ_i lie in $\mathcal{FP} \setminus (\mathcal{FP}_{\mathrm{HA}} \cup \mathcal{FP}_{\mathrm{AS}} \cup \mathcal{FP}_{\mathrm{PA}})$ and replace $((\Gamma_i, x_i))_{i\geq 0}$ by the subsequence consisting of these (Γ_i, x_i). By Proposition 5, again replacing the sequence by a subsequence if necessary, we may suppose further that, for each $i \geq 0$, the stabiliser $(G_i)_{x_i}$ of x_i in G_i is isomorphic to a fixed finite group H. Then, for each $i \geq 0$, it follows from the remarks preceding the statement that, since G_i is not of type HA, AS or PA, we have $\mathrm{Soc}(G_i) \cong T_i^{k_i}$, where T_i is a nonabelian simple group, $k_i \geq 2$, and some subgroup of $(G_i)_{x_i}$ has T_i as a quotient. Moreover $(G_i)_{x_i}$, acting by conjugation, has at most two orbits on the k_i simple direct factors of $\mathrm{Soc}(G_i)$. Thus both $|T_i| \leq |H|$ and $k_i/2 \leq |H|$, whence $|G_i| = |\mathrm{Soc}(G_i)(G_i)_{x_i}| \leq |T_i|^{k_i}|H| \leq |H|^{2|H|+1}$ is bounded above independently of i. This is a contradiction, and so the first assertion of part (i) is proved. This implies immediately that $\lim(\mathcal{FP} \setminus (\mathcal{FP}_{\mathrm{HA}} \cup \mathcal{FP}_{\mathrm{AS}} \cup \mathcal{FP}_{\mathrm{PA}}))$ is empty.

Now $(G_i)_{i\geq 0}$ is an infinite sequence of primitive groups and as there are only finitely many types, there exists a subsequence $(i_j)_{j\geq 0}$ such that the G_{i_j} are all of the same type. By part (i) this type is one of HA, AS or PA. □

4 Proof of Theorem 2: limit graphs of type HA

In this section we first show in Theorem 4 that any limit graph of $\mathcal{FP}_{\mathrm{HA}}$ is a Cayley graph of a finitely generated free abelian group. For a group H and a subset $S \subset H \setminus \{1\}$ such that $S^{-1} = \{s^{-1} | s \in S\}$ equals S, the *Cayley graph* $\mathrm{Cay}(H, S)$ of H relative to S is the graph with vertex set H such that $\{u, v\}$ is an edge if and only if $u^{-1}v \in S$. The group H induces a group of automorphisms that is regular on vertices in its action by left multiplication ($h : u \to h^{-1}u$, for $h, u \in H$). Thus $\mathrm{Cay}(H, S)$ is vertex-transitive; and $\mathrm{Cay}(H, S)$ is connected if and only if S generates H.

Moreover a graph Γ is isomorphic to a Cayley graph of a group H if and only if $\mathrm{Aut}(\Gamma)$ contains a subgroup isomorphic to H that acts regularly on vertices. In particular, a connected graph Γ admitting a vertex-transitive abelian group A of automorphisms is a Cayley graph of A (with A acting naturally by left multiplication). We also have the following proposition. It uses the concept of a bounded automorphism h of a graph Γ: this is an element $h \in \mathrm{Aut}(\Gamma)$ such that there is some constant c for which $d(x, h(x)) \leq c$ for all vertices x of Γ. The set $\mathrm{Aut}_0(\Gamma)$ of all bounded automorphisms of Γ is a normal subgroup of $\mathrm{Aut}(\Gamma)$.

Proposition 10 *Let Γ be a connected Cayley graph of the abelian group A and suppose that Γ is locally finite and the only imprimitivity system of* $\mathrm{Aut}(\Gamma)$ *on $V(\Gamma)$ with finite blocks is the trivial imprimitivity system with singleton blocks. Then A is a finitely generated free abelian normal subgroup of* $\mathrm{Aut}(\Gamma)$*.*

Proof The group A satisfies condition (ii) in [15, Proposition 2.2], so by (iii) of [15, Proposition 2.2], $\mathrm{Aut}_0(\Gamma)'$ has finite orbits, where $\mathrm{Aut}_0(\Gamma)$ is the group of all bounded automorphisms and $\mathrm{Aut}_0(\Gamma)'$ is its derived subgroup. Since $\mathrm{Aut}_0(\Gamma)'$ is normal in $\mathrm{Aut}(\Gamma)$, its orbits are blocks of imprimitivity; so by assumption the orbits of $\mathrm{Aut}_0(\Gamma)'$ are singletons and therefore $\mathrm{Aut}_0(\Gamma)$ is abelian. Since Γ is locally finite, $\Gamma = \mathrm{Cay}(A, S)$ where S is finite. Moreover, for all $s \in S$, the left multiplication by s lies in $A \cap \mathrm{Aut}_0(\Gamma)$. Since Γ is connected, $A = \langle S \rangle$ and so $A \subseteq \mathrm{Aut}_0(\Gamma)$ and A is finitely generated. Since A is vertex-transitive and $\mathrm{Aut}_0(\Gamma)$ is abelian it follows that $A = \mathrm{Aut}_0(\Gamma)$. Thus $\mathrm{Aut}_0(\Gamma)$ is finitely generated and so also the torsion part of $\mathrm{Aut}_0(\Gamma)$ is normal in $\mathrm{Aut}(\Gamma)$, and is finitely generated and hence finite. Its orbits form a system of imprimitivity for $\mathrm{Aut}(\Gamma)$ and hence they must be singletons. It follows that $A = \mathrm{Aut}_0(\Gamma)$ is a finitely generated free abelian normal subgroup of $\mathrm{Aut}(\Gamma)$. $\square$

We now prove the following theorem.

Theorem 4 *Let $((\Gamma_i, x_i))_{i \geq 0}$ be a sequence in $\mathcal{G}^*$ that $(\varphi_i)_{i \geq 0}$-converges to (Γ, x), and let $G_i \leq \mathrm{Aut}(\Gamma_i)$, for each $i \geq 0$. If each G_i contains a vertex-transitive abelian subgroup, then the limit group of automorphisms of Γ with respect to $(G_i)_{i \geq 0}$ also contains a vertex-transitive abelian subgroup. If, in addition, the groups G_i are vertex-primitive for all sufficiently large i and Γ is infinite, then Γ is a Cayley graph of a free abelian group of rank at most* $\deg(\Gamma)/2$.

Proof Set $d = \deg(\Gamma)$. For each $i \geq 0$, let A_i be a vertex-transitive abelian subgroup of G_i, and let $\Gamma(x) = \{y_1, \ldots, y_d\}$. Then it follows from Proposition 2 that there exist an increasing sequence of non-negative integers $(i_j)_{j\geq 0}$ and, for each $j \geq 0$, elements $a_{j,1}, \ldots, a_{j,d}$ of A_{i_j} such that, for $1 \leq k \leq d$, the sequence $(a_{j,k})_{j\geq 0}$ is $(\varphi_{i_j})_{j\geq 0}$-convergent to an automorphism a_k of Γ mapping x to y_k. If $1 \leq k_1, k_2 \leq d$, then for all j we have $[a_{j,k_1}, a_{j,k_2}] = 1$ (since A_{i_j} is abelian), and hence $[a_{k_1}, a_{k_2}] = 1$. Thus $\langle a_1, \ldots, a_d\rangle$ is an abelian subgroup of G and is vertex-transitive since Γ is connected. Finally, if the groups G_i are vertex-primitive for all sufficiently large i, then, by Propositions 4 (1) and 10, $\langle a_1, \ldots, a_d\rangle$ is a free abelian (normal) subgroup of $\mathrm{Aut}(\Gamma)$ of finite rank, k say. Thus $\Gamma \cong \mathrm{Cay}(\mathbb{Z}^k, S)$, where S is an inverse-closed generating set for the abelian group $\mathbb{Z}^k$ of size d. It follows that $k \leq d/2$. □

It was seen in [5, Example 3.1] that, for any integer $k \geq 1$, the k-dimensional grid is contained in $\lim(\mathcal{FP}_{\mathrm{HA}})$. In fact, all of these grids lie in $\lim(\mathcal{FP}_{\mathrm{HA}}^{min})$, and in particular, the upper bound in Theorem 4 on the valency is sharp. The construction also demonstrates that $\lim(\mathcal{FP}_{\mathrm{HA}})$ is infinite, and indeed that $\lim(\mathcal{FP}_{\mathrm{HA}}^{min})$ is infinite, since the k-dimensional grid has valency $2k$, for each positive integer k. Our next construction shows that there are infinitely many graphs of the same valency in $\lim(\mathcal{FP}_{\mathrm{HA}})$ that are Cayley graphs of $\mathbb{Z}$.

Example 2 For a prime p and an increasing sequence $u_1, \ldots, u_k$ of positive integers, let $\Gamma_{p;u_1,\ldots,u_k} = \mathrm{Cay}(\mathbb{Z}_p, S)$ where $S = \{\pm 1, \pm u_1, \ldots, \pm u_k\}$ modulo p. The graph $\Gamma_{p;u_1,\ldots,u_k}$ admits D_{2p} as a subgroup of automorphisms acting vertex-primitively, so $\Gamma_{p;u_1,\ldots,u_k} \in \mathcal{FP}_{\mathrm{HA}}$.

Now let $(p_i)_{i\geq 0}$ be an increasing sequence of primes each greater than $2u_k$, let $\Gamma_i = \Gamma_{p_i;u_1,\ldots,u_k}$ and let $\Gamma = \mathrm{Cay}(\mathbb{Z}, S)$. Then, for each r, provided p_i is large enough, the induced subgraph $\langle B_{\Gamma_i}(0, r)\rangle$ is isomorphic to the subgraph $\langle B_\Gamma(0, r)\rangle$ of Γ. Thus, the sequence $(\Gamma_i)_{i\geq 0}$ of graphs from $\mathcal{FP}_{\mathrm{HA}}$ converges to Γ and hence $\Gamma \in \lim(\mathcal{FP}_{\mathrm{HA}})$. It is not difficult to show that distinct values of $u_1, \ldots, u_k$ lead to nonisomorphic limit graphs $\Gamma \in \lim(\mathcal{FP}_{\mathrm{HA}})$ and all these graphs have valency 2k.

Now let $(p_i)_{i\geq 0}$ be an increasing sequence of primes and, for each $1 \leq j \leq k$, let $(u_{i,j})_{i\geq 0}$ be an increasing sequence of positive integers. Suppose that all numbers $u_{i,1}, u_{i,2}/u_{i,1}, \ldots, u_{i,k}/u_{i,k-1}, p_i/u_{i,k}$ tend to ∞ as $i \to \infty$. Then the sequence $(\Gamma_{p_i;u_{i,1},\ldots,u_{i,k}})_{i\geq 0}$ from $\mathcal{FP}_{\mathrm{HA}}$ converges to the grid $\mathbb{Z}^k$. This gives a different sequence of graphs in $\mathcal{FP}_{\mathrm{HA}}$ from that given in [5, Example 3.1] that converges to the k-dimensional grid.

Kostousov [10] has shown that for each $k \geq 2$ there are infinitely many Cayley graphs of $\mathbb{Z}^k$ of a fixed valency, depending on k, contained in the set $\lim(\mathcal{FP}_{\mathrm{HA}}^{e-trans})$. In particular, he constructs infinitely many Cayley graphs of $\mathbb{Z}^2$ of valency 8 in $\lim(\mathcal{FP}_{\mathrm{HA}}^{e-trans})$. He has also shown that, for $k < 4$, there are only finitely many Cayley graphs of $\mathbb{Z}^k$ which are in $\lim(\mathcal{FP}_{\mathrm{HA}}^{min})$.

The finite dimensional grids are precisely those Cayley graphs of finitely generated free abelian groups for which the valency is equal to twice the rank. The next family of examples shows that $\lim(\mathcal{FP}_{\mathrm{HA}}^{min})$ also contains many Cayley graphs of free abelian groups having valency greater than twice the rank. The construction used in this example is a special case of a more general construction that can be applied to certain permutation groups and composition factors of their permutation modules.

Example 3 Let H be the symmetric group S_k, where $k \geq 3$. Let $V = \mathbb{Z}^k$ be the natural permutation $\mathbb{Z}H$-module, and let W be the $\mathbb{Z}H$-submodule of V given by $\{(v_1, \ldots, v_k) : \sum_{i=1}^{k} v_i = 0\}$. Let $w = (w_1, \ldots, w_k) \in W \setminus \{0\}$, and let O be the H-orbit containing w. For any prime $p > \max\{k, |w_1|, \ldots, |w_k|\}$, let $V_p = \psi_p(V)$, where ψ_p is the natural map $\mathbb{Z}^k \to \mathbb{Z}_p^k$ which replaces each entry of $x = (x_1, \ldots, x_k)$ with its value modulo p, and let $W_p = \psi_p(W)$. Then V_p is the natural permutation $\mathbb{Z}_p H$-module, $W_p \cong \mathbb{Z}_p^{k-1}$, and since $p > k$, W_p is an irreducible $\mathbb{Z}_p H$-submodule. Let $w_p = \psi_p(w) \in W_p$. Then $O_p = \psi_p(O)$ is the H-orbit in W_p containing w_p.

Now $w = (w_1, \ldots, w_k) \in O \subseteq W \setminus \{0\}$ and without loss of generality $w_1 \neq 0$. Let K be the stabiliser of w in H. If K were transitive on $\{1, \ldots, k\}$ then we would have $w = w_1(1, \ldots, 1)$, but since $w \in W$ this would imply that $k.1 = 0$ which is not the case. Thus K is intransitive on $\{1, \ldots, k\}$. Therefore, applying [3, Theorem 5.2B], one of the following holds: (i) $|O| = |O_p| = k$ and $O \cap (-O) = \emptyset$ (and in this case $k - 1$ of the entries of w are equal), or (ii) $|O| = |O_p| \geq 2k$, or (iii) $k = 4$ and K is an intransitive subgroup of order 4, so, $|O| = |O_p| = 6$, w is an H-image of (u, u, u', u'), for distinct $u, u' \in \mathbb{Z}$, $u' = -u \neq 0$ and $O = -O$. In all cases, the group $G_p = W_p \rtimes H$ is a vertex-primitive and edge-transitive group of automorphisms of the graph $\Gamma_p := \mathrm{Cay}(W_p, O_p \cup (-O_p))$. Thus $\Gamma_p \in \mathcal{FP}_{\mathrm{HA}}^{e-trans}$. In addition, if $k \neq 4$ and O is as in case (i), then $\Gamma_p \in \mathcal{FP}_{\mathrm{HA}}^{min}$ and $\deg(\Gamma_p) = 2k$; while if $k = 4$ and O is as in case (iii), then $\Gamma_p \in \mathcal{FP}_{\mathrm{HA}}^{min}$ and $\deg(\Gamma_p) = 6$.

Let $(p_i)_{i \geq 0}$ be any increasing sequence of primes with each p_i greater than $\max\{k, |w_1|, \ldots, |w_k|\}$, and let $(\Gamma_{p_i})_{i \geq 0}$ be a sequence of graphs from $\mathcal{FP}_{\mathrm{HA}}^{e-trans}$ (or from $\mathcal{FP}_{\mathrm{HA}}^{min}$ with either $k \neq 4$ and O as in (i), or $k = 4$ and O as in (iii)). Then it is easy to see that $(\Gamma_{p_i})_{i \geq 0}$ converges to the graph $\Gamma = \mathrm{Cay}(\langle O \rangle, O \cup (-O))$, where the subgroup $\langle O \rangle$ of W is of rank $k - 1$. Thus there is a Cayley graph Γ of a free abelian group of rank $k - 1$ in $\lim(\mathcal{FP}_{\mathrm{HA}}^{e-trans})$ having valency $2|O| \geq 2k$. In the case where $|O| = k \neq 4$, we also have $\Gamma \in \lim(\mathcal{FP}_{\mathrm{HA}}^{min})$ and $\deg(\Gamma) = 2k$. For example, if $|O| = k = 3$, then the limit graph Γ is the 'tessellation of the plane by triangles', demonstrated in Figure 1. If $|O| = k = 4$ then the limit graph is the 'tessellation of 3-space by tetrahedra'.

The limit graphs obtained in Example 3 with $|O| = k$ are the root lattices of type A_{k-1}. These examples, along with the k-dimensional grids are part of a more general construction, our investigation of which was motivated by a question of Neil Sloane.

Example 4 Let L be a lattice, that is the set of all $\mathbb{Z}$-linear combinations of a basis $\Delta \subset \mathbb{R}^k$ and let $B : \mathbb{R}^k \times \mathbb{R}^k \to R$ be the usual dot product. Let $A \cong \mathbb{Z}^k$ be the additive group of L and suppose that there exists a finite group

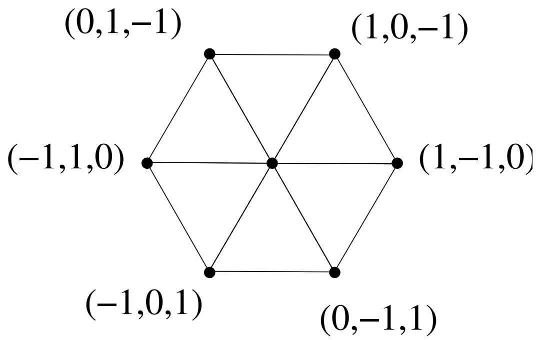

Figure 1: Tessellation of the plane by triangles

$H \leqslant \mathrm{GL}(k, \mathbb{R})$ which preserves B, fixes L setwise and acts irreducibly on $\mathbb{R}^k$. Let $\Phi = \Delta^H$ and let $\Gamma = \mathrm{Cay}(A, \Phi)$, the graphical representation of L.

Each element of Φ is a $\mathbb{Z}$-linear combination of the elements of Δ. Let $t = \max\{|\lambda_\delta| \mid \sum_{\delta\in\Delta} \lambda_\delta \delta \in \Phi\}$ and let p be a prime such that $p > \max\{t, |H|\}$. Let $A_p = \psi_p(A)$, where ψ_p is the natural map $\sum_{\delta\in\Delta} \lambda_\delta\delta \mapsto \sum_{\delta\in\Delta} \overline{\lambda_\delta}\delta$ that replaces each λ_δ by its value modulo p. (Recall that $|\lambda_\delta| \leq t < p$.) Then $|\psi_p(\Phi)| = |\Phi|$. Now H permutes Φ and so we obtain an embedding of H in $\mathrm{GL}(k, p)$. Moreover, as H acts irreducibly on $\mathbb{R}^k$ and p is coprime to $|H|$ it follows that H acts irreducibly on A_p. Thus $G_p = A_p \rtimes H$ is a primitive permutation group on the set A_p. Let $\Gamma_p = \mathrm{Cay}(A_p, \psi_p(\Phi))$. Then $\Gamma_p \in \mathcal{FP}_{\mathrm{HA}}$. Let $(p_i)_{i\geq 0}$ be an infinite sequence of distinct primes satisfying $p_i > \max\{t, |H|\}$. Then $(\Gamma_{p_i})_{i\geq 0}$ is $(\varphi_{p_i})_{i\geq 0}$-convergent to Γ.

Any root lattice of a crystallographic finite reflection group H arises in this way as a limit graph with Φ being the root system and Δ a simple system. The Leech lattice also arises with H being the double cover of Co_1.

Next we prove that no limit graph for $\mathcal{FP}_{\mathrm{HA}}$ is a limit graph for $\mathcal{FP}_{\mathrm{X}}$ for any type $X \neq HA$. Proposition 6 is crucial to the proof of this result.

Theorem 5 *Let $((\Gamma_i, x_i))_{i\geq 0}$ be a sequence in $\mathcal{G}^*$ that $(\varphi_i)_{i\geq 0}$-converges to (Γ, x), where Γ is a connected Cayley graph of a finitely generated free abelian group, and, for each $i \geq 0$, suppose there exists a subgroup $G_i \leq \mathrm{Aut}(\Gamma_i)$ that is vertex-primitive on $V(\Gamma_i)$. Then, for all sufficiently large i, the graph Γ_i is finite and G_i is of type HA.*

Proof Set $G = \mathrm{Aut}(\Gamma)$, and let $d = \deg(\Gamma)$. The assertion obviously holds in the case where Γ is the Cayley graph of the free abelian group of rank 0. Thus we may assume that Γ is a connected Cayley graph for $\mathbb{Z}^k$ for some positive integer $k \geq 1$, and as in the proof of Theorem 4, $k \leq d/2$. Note that a primitive permutation group containing a nontrivial normal finitely generated abelian subgroup is finite. For if, H were an infinite primitive group with nontrivial normal finitely generated abelian subgroup N then N is regular and hence infinite, and so $N = \mathbb{Z}^l \times \mathbb{Z}_{p_1}^{r_1} \times \cdots \times \mathbb{Z}_{p_s}^{r_s}$ with $l \geq 1$. The torsion part of N is a characteristic subgroup, and hence normal in H. However, such a normal subgroup is intransitive and so $N = \mathbb{Z}^l$. Thus $(2\mathbb{Z})^l$ provides a system of imprimitivity for H, a contradiction.

Suppose that the conclusion to the theorem is false. Thus there must exist infinitely many i such that G_i does not contain a non-trivial normal abelian subgroup. We may therefore assume without loss of generality that, for each $i \geq 0$, the group G_i contains no non-trivial abelian normal subgroup.

By Proposition 4 (1), G has no non-trivial finite blocks of imprimitivity in $V(\Gamma)$, and so, by Proposition 10, G has a normal abelian subgroup $V \cong \mathbb{Z}^k$ acting regularly on $V(\Gamma)$. Thus the group G is a semidirect product of V and the stabiliser G_x of the vertex x such that G_x acts by conjugation on V. Since Γ is a Cayley graph, $\Gamma(x)$ corresponds to a generating set for V and so the pointwise stabiliser in G_x of $\Gamma(x)$ centralises V. Thus G_x acts faithfully on $\Gamma(x)$, and so $|G_x| \leq d!$.

Now the limit group of $(G_i)_{i\geq 0}$ is transitive on $V(\Gamma)$. Thus, replacing $(G_i)_{i\geq 0}$ by a subsequence if necessary, we may assume that, for each $y \in B_\Gamma(x, d!)$ and each $i \geq 0$, there are elements $g_{y,i} \in G_i$, and $g_y \in G$ such that $g_y(x) = y$ and the sequence $(g_{y,i})_{i\geq 0}$ is $(\varphi_i)_{i\geq 0}$-convergent to g_y. Since $|B_\Gamma(x, d!)| > d! \geq |G_x| = |G : V|$, there exist distinct $y_1, y_2 \in B_\Gamma(x, d!)$ such that g_{y_1}, g_{y_2} lie in the same left V-coset, so $1 \neq g_{y_1}^{-1} g_{y_2} \in V$. Note that $0 < d_\Gamma(x, g_{y_1}^{-1} g_{y_2}(x)) = d_\Gamma(g_{y_1}(x), g_{y_2}(x)) = d_\Gamma(y_1, y_2) \leq 2d!$. On the other hand, since $g_{y_1}^{-1} g_{y_2}$ lies in the transitive abelian group V, it follows that $d_\Gamma(x', g_{y_1}^{-1} g_{y_2}(x')) = d_\Gamma(x, g_{y_1}^{-1} g_{y_2}(x))$ for all $x' \in V(\Gamma)$. In addition, the sequence $(g_{y_1,i}^{-1} g_{y_2,i})_{i\geq 0}$ is $(\varphi_i)_{i\geq 0}$-convergent to $g_{y_1}^{-1} g_{y_2}$.

We shall now apply Proposition 6 with $r = d!$, and f the constant map $f(n) = 2d!$ for all n. Let c denote the integer $c(d, f, r)$ given by Proposition 6 for these values of r, f. Now for all sufficiently large i, the valency of Γ_i is equal to d, and the diameter of Γ_i is greater than c. Moreover, for all sufficiently large i, the element $h_i = g_{y_1,i}^{-1} g_{y_2,i}$ in the primitive group G_i does not fix $B_{\Gamma_i}(x_i, d!)$ pointwise, and $d_{\Gamma_i}(x', h_i(x')) = d_\Gamma(x, g_{y_1}^{-1} g_{y_2}(x))$ for all $x' \in B_{\Gamma_i}(x_i, c)$, and this value is at most $2d!$. Thus by Proposition 6, $\langle h_i^{G_i} \rangle$ is a non-trivial abelian normal subgroup of G_i, which is a contradiction. □

Now Theorem 2 follows immediately from Theorems 4 and 5.

5 Limit graphs of type AS

In this section we use the finite simple group classification to obtain more information about the limit graphs of $\mathcal{FP}_{\mathrm{AS}}$. In the first part of our analysis we use only the fact that the finite simple groups can be divided into various classes: the finite alternating groups, the finite simple Lie type groups, and a finite number of sporadic simple groups. The Lie type simple groups comprise a finite number of infinite families. For the next steps in the analysis we divide the Lie type simple groups into the classical groups and the exceptional Lie type groups. To study the classical groups we use detailed information from Aschbacher's Theorem [1] that divides the maximal subgroups of these groups into a number of families. Roughly equivalent information about maximal subgroups of the exceptional Lie type groups is provided by a theorem of Liebeck and Seitz in [12, Theorem 2]. To describe these maximal subgroups, we need some definitions. A finite group H is *quasisimple* if H is perfect (that is $H = H'$) and $H/Z(H)$ is simple. The *components* of H are its subnormal quasisimple subgroups, and $E(H)$ is the subgroup generated by the components of H. The *Fitting subgroup* $F(H)$ of a finite group H is the largest nilpotent normal subgroup of H; and the *generalized Fitting subgroup* $F^*(H)$ of H is the (central) product $F^*(H) = E(H)F(H)$.

Definition 1 *Let A be an almost simple group of Lie type. We define five classes $\mathcal{C}_1$, $\mathcal{C}_2, \ldots, \mathcal{C}_5$ of maximal subgroups of A as follows.*

- $\mathcal{C}_1$ *consists of centralisers of field automorphisms of prime order,*
- $\mathcal{C}_2$ *consists of the normalisers of elementary abelian r-groups, for a prime r different from the characteristic,*
- $\mathcal{C}_3$ *occurs only when A is a classical group and consists of normalisers of absolutely irreducible quasisimple groups,*
- $\mathcal{C}_4$ *occurs only when A is an orthogonal group and consists of stabilisers of direct sum decompositions of the underlying vector space into isometric nonsingular* 1*-dimensional subspaces,*
- $\mathcal{C}_5$ *occurs only when A is an exceptional group of Lie type and consists of groups H for which $F^*(H)$ is either a simple group, or (only if A is of type E_8) $A_5 \times A_6$.*

Detailed information about the subgroups identified in $\mathcal{C}$ may be found in [7] and [12].

We can now give the main result in this section. It gives a preliminary classification of the graphs in $\lim(\mathcal{FP}_{\mathrm{AS}})$.

Theorem 6 *Let $((\Gamma_i, x_i))_{i\geq 0}$ be a sequence of finite graphs with distinguished vertices in $\mathcal{G}^*$ that $(\varphi_i)_{i\geq 0}$-converges to an infinite graph (Γ, x) with a distinguished vertex. Moreover, suppose that, for each $i \geq 0$, there is a subgroup $G_i \leqslant \mathrm{Aut}(\Gamma_i)$ that is vertex-primitive and almost simple with socle T_i. Then there exists a subsequence $(i_j)_{j\geq 0}$, and a finite group H, such that the T_{i_j} are pairwise distinct finite simple groups of the same Lie type and rank, $H \in \mathcal{C}_1, \ldots, \mathcal{C}_4$ or $\mathcal{C}_5$ and, for each $j \geq 0$, $(G_{i_j})_{x_{i_j}} \cong H$.*

Proof For each simple group T, there are only finitely many almost simple groups with socle T and finitely many graphs which are T-vertex-transitive. Thus we may choose a subsequence $(i_j)_{j\geq 0}$ such that the simple groups T_{i_j} are pairwise distinct. Furthermore, since there are only a finite number of sporadic simple groups, we may assume that none of the T_{i_j} are sporadic. The remaining finite simple groups are divided into 17 infinite families and so we may choose the subsequence $(i_j)_{j\geq 0}$ so that all the T_{i_j} are from the same family, and so that the order $|G_{i_j}|$ increases (unboundedly) with j. By Proposition 5, we may assume further that the stabiliser in G_{i_j} of the point x_{i_j} (a maximal subgroup of G_{i_j}) is isomorphic to a fixed finite group H, say, for all $j \geq 0$. We need to show that the T_{i_j} are not finite alternating groups. Assume to the contrary that this is the case, say $T_{i_j} \cong A_{n_j}$ with n_j increasing unboundedly with j. We may assume therefore that $n_j/2 > |H|$ for each j. Since each maximal intransitive subgroup of S_{n_j} or A_{n_j} has a subgroup isomorphic to A_m with $m \geq n_j/2$, and since $|H| < n_j/2$, it follows that the maximal subgroup H of G_{i_j} is transitive in its natural action of degree n_j. This however implies that n_j divides $|H|$ which is a contradiction. Thus the T_{i_j} are simple groups of Lie type.

Next we assume that the T_{i_j} are all finite simple classical groups of the same type (linear, symplectic, unitary or orthogonal), and that the central extension $\overline{G}_{i_j}$ of G_{i_j} by scalar matrices acts naturally on a d_j-dimensional vector space V_j over a field of order q_j. We recall that $(G_{i_j})_{x_{i_j}} \cong H$ is a maximal subgroup of G_{i_j} for each j, and let $\overline{H} \leq \overline{G}_{i_j}$ be the central extension of H. Aschbacher's Theorem ([1], or see [7]) divides the maximal subgroups of $\overline{G}_{i_j}$ into nine types. Subgroups of the first eight types preserve various kinds of geometric structures on V_j, while those of the last type are normalisers of quasisimple groups with an absolutely irreducible action on V_j. Suppose first that $\overline{H}$ is the normaliser of an absolutely irreducible quasisimple group. Since the dimensions of the irreducible representations of $\overline{H}$ over any finite field are bounded from above by a constant independent of the field size, we can choose the subsequence $(i_j)_{j\geq 0}$ in such a way that the dimensions d_j are all the same, and $H \in \mathcal{C}_3$, and the result is proved in this case.

Now suppose that $\overline{H}$, as a subgroup of $\overline{G}_{i_j}$, belongs to one of the eight "geometric families". Then looking through the tables in [7, Section 3.5] we see that in each case the order of H can be expressed as a function of the field size and dimension, or as a function of the size of a subfield and the dimension, or as a function of the dimension. In each case, if the dimension tends to infinity then the order of the maximal subgroup tends to infinity. Hence d_j is bounded, and again we can choose the subsequence so that the d_j are all equal. This implies that the field size q_j tends to infinity as $j \to \infty$. However, in all cases when the field size occurs in the order of the maximal subgroup, if the field size tends to infinity then the order also tends to infinity. Hence only a (bounded) subfield size and the dimension can occur in the order formula, which yields that H is either the centraliser of a field automorphism of prime order, that is $H \in \mathcal{C}_1$, or $\overline{H}$ is the normaliser of a symplectic-type r-group leading to $H \in \mathcal{C}_2$, or the G_{i_j} are orthogonal groups and H is the stabiliser of a direct sum decomposition of V_j into isometric 1-spaces, that is $H \in \mathcal{C}_4$. (The structures of such groups H are given in [7, Proposition 4.2.15].) Thus the result holds if the T_{i_j} are classical groups.

Finally we assume that the T_{i_j} are all exceptional Lie type simple groups of the same type, and T_{i_j} is defined over a field of order q_j with $q_j \to \infty$ as $j \to \infty$.

Now [12, Theorem 2] classifies the maximal subgroups of G_{i_j} into five families, and looking through the list of possible maximal subgroups given in that result we see the only maximal subgroups whose order does not tend to infinity as the field size grows are the ones belonging to our families $\mathcal{C}_1$, $\mathcal{C}_2$ or $\mathcal{C}_5$. $\square$

It was proved in [5, Example 3.2] that the infinite trivalent tree is a member of $\lim(\mathcal{FP}_{\mathrm{AS}}^{e-trans})$. Also [5, Example 3.4] provides an infinite family of examples in $\lim(\mathcal{FP}_{\mathrm{AS}}^{min})$. Before giving more examples we need to introduce the concept of a coset graph. Let G be a group with a core-free subgroup H. Let $g \in G$ such that g does not normalise H and $g^{-1} \in HgH$. We define the *coset graph* $\Gamma = \mathrm{Cos}(G, H, HgH)$ to be the graph with vertex set $[G : H] = \{xH \mid x \in G\}$ and two cosets xH and yH are adjacent if and only if $x^{-1}y \in HgH$. Then G acts on $V\Gamma$ by left multiplication and $G \leqslant \mathrm{Aut}(\Gamma)$. Furthermore, Γ is connected if and only if $\langle H, g\rangle = G$ and $\deg(\Gamma) = |H : H \cap H^g|$. Recall that an *s-arc* in a graph Γ is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that for each i, v_i is adjacent to v_{i+1} and $v_i \neq v_{i+2}$.

Our first example shows that $\lim(\mathcal{FP}_{\mathrm{AS}}^{min})$ contains infinitely many graphs of the same valency, namely of valency 12. (Distinct graphs in the family of examples in $\lim(\mathcal{FP}_{AS}^{min})$ given in [5, Example 3.4] had different valencies.)

Example 5 Let $r > 5$ be a fixed prime, and let $(p_i)_{i\geq 0}$ be a sequence of primes such that $5r$ divides $p_i - 1$. Let $G_{p_i} = \mathrm{PSL}(2, p_i)$. Then G_{p_i} has a maximal subgroup $H_{p_i} \cong A_5$ and contains an element g_{p_i} of order $5r$ which lies in a maximal subgroup of G_{p_i} isomorphic to D_{p_i-1} such that $g_{p_i}^r \in H_{p_i}$. Let Γ_{p_i} be the digraph with vertex set $[G_{p_i} : H_{p_i}]$, the set of left cosets of H_{p_i} in G_{p_i}, such that (yH_{p_i}, zH_{p_i}) is an arc if and only if $y^{-1}z \in H_{p_i}g_{p_i}H_{p_i}$. Since $g_{p_i} \in D_{p_i-1}$ and H_{p_i} contains a subgroup of D_{p_i-1} isomorphic to D_{10}, there exists an element $h_{p_i} \in H_{p_i}$ such that $h_{p_i}^{-1} g_{p_i} h_{p_i} = g_{p_i}^{-1}$. Thus $g_{p_i}^{-1} \in H_{p_i}g_{p_i}H_{p_i}$ and so if (yH_{p_i}, zH_{p_i}) is an arc then (zH_{p_i}, yH_{p_i}) is as well. Hence Γ_{p_i} may be viewed as an undirected graph and is the coset graph $\mathrm{Cos}(G_{p_i}, H_{p_i}, H_{p_i}g_{p_i}H_{p_i})$. Since H_{p_i} is a maximal subgroup of G_{p_i}, we have $\langle H_{p_i}, g_{p_i}\rangle = G_{p_i}$ and so Γ_{p_i} is connected. Now G_{p_i} is vertex-primitive and arc-transitive on Γ_{p_i} (acting by left multiplication). Furthermore, $\deg(\Gamma_{p_i}) = |H_{p_i} : (H_{p_i})^{g_{p_i}} \cap H_{p_i}|$, and since $(H_{p_i})^{g_{p_i}} \cap H_{p_i} \cong C_5$ we have $\deg(\Gamma_{p_i}) = 12$. Now $H_{p_i}, g_{p_i}H_{p_i}, g_{p_i}^2 H_{p_i}, \ldots, g_{p_i}^{r-1}H_{p_i}, g_{p_i}^r H_{p_i} = H_{p_i}$ is a cycle of length r in Γ_{p_i}. Denoting the vertex H_{p_i} of Γ_{p_i} by x_{p_i}, by Proposition 1 the sequence $((\Gamma_{p_i}, x_{p_i}))_{i\geq 0}$ has a convergent subsequence. The limit graph $(\Gamma^{[r]}, x^{[r]})$ of this sequence has degree 12, contains a cycle $C^{[r]}$ of length r through x, and has a limit automorphism $g^{[r]}$ of order $5r$ such that $(g^{[r]})^5$ rotates $C^{[r]}$ and has the following property:

(†) $(g^{[r]})^r$ fixes $C^{[r]}$ pointwise and for all $y \in C^{[r]}$, $(g^{[r]})^r$ acts fixed-point-freely on $\Gamma(y) \setminus C^{[r]}$.

We note that each $\Gamma_{p_i} \in \mathcal{FP}_{\mathrm{AS}}^{e-trans}$ and from [14], when $p_i \equiv \pm 91 \pmod{120}$, the graph $\Gamma_{p_i} \in \mathcal{FP}_{\mathrm{AS}}^{min}$. Thus we obtain graphs of valency 12 lying in both $\lim(\mathcal{FP}_{\mathrm{AS}}^{e-trans})$ and $\lim(\mathcal{FP}_{\mathrm{AS}}^{min})$ that are not trees.

Furthermore, we claim that there are infinitely many pairwise non-isomorphic graphs among the $\Gamma^{[r]}$. Suppose, on the contrary, that the set of limit graphs is finite. Then there exists an infinite sequence of primes $(r_i)_{i\geq 0}$ and a limit graph $(\Gamma, x) \cong (\Gamma^{[r_i]}, x^{[r_i]})$ for all $i \geq 0$, such that Γ contains cycles $C^{[r_i]}$ of length r_i through x and limit automorphisms $g^{[r_i]}$ satisfying (†). Let $G = \mathrm{Aut}(\Gamma)$. Since $(g^{[r_i]})^5$ has order r_i, these elements are pairwise distinct, and each of them maps x to one of the 12 vertices of $\Gamma(x)$. It follows that at least one of the 12 cosets of G_x contains infinitely many of the elements $(g^{[r_i]})^5$, and in particular that G_x is infinite. Since $|\Gamma(x)| = 12$, we must have $C^{[r_i]} \cap \Gamma(x) = C^{[r_j]} \cap \Gamma(x)$ for some $i \neq j$. Now $A_5 \leqslant G_x^{\Gamma(x)}$ and A_5 acts with stabiliser C_5. Since the only intermediate subgroup is D_{10}, the only block system for A_5 in $\Gamma(x)$ has blocks of length two. Hence if $G_x^{\Gamma(x)}$ is imprimitive, then the blocks of imprimitivity must be of size two and $C^{[r_i]} \cap \Gamma(x) = C^{[r_j]} \cap \Gamma(x)$ is one of the blocks. In this case, $G_y^{\Gamma(y)}$ is imprimitive for all $y \in V(\Gamma)$ and, for $y \in C^{[r_i]}$, $C^{[r_i]} \cap \Gamma(y)$ is one of the blocks for G_y on $\Gamma(y)$. This implies, since $C^{[r_i]} \cap \Gamma(x) = C^{[r_j]} \cap \Gamma(x)$, that $C^{[r_i]} = C^{[r_j]}$, which is a contradiction since $|C^{[r_i]}| = r_i$ and $|C^{[r_j]}| = r_j$. Thus $G_x^{\Gamma(x)}$ is a primitive group of degree 12. There are 6 such groups: $\mathrm{PSL}(2,11), \mathrm{PGL}(2,11), M_{11}, M_{12}, A_{12}$, and S_{12}. All of these have the property that for any $y \in \Gamma(x)$, any nontrivial subnormal subgroup of $G_{xy}^{\Gamma(x)\setminus\{y\}}$ acts transitively on $\Gamma(x) \setminus \{y\}$. By [17, Proposition 3.1], this property implies that either G_x is finite or Γ is a tree. Thus there are infinitely many nonisomorphic graphs among the $\Gamma^{[r_i]}$.

The action of $\mathrm{PSL}(2,p)$ on the cosets of a maximal subgroup isomorphic to A_5 is actually a good source of examples of limit graphs as the following additional example shows.

Example 6 For each prime $p \equiv \pm 1 \pmod{10}$ the simple group $G_p = \mathrm{PSL}(2,p)$ has a maximal subgroup H_p isomorphic to A_5. Thus G_p acts primitively by left multiplication on the set V_p of left cosets of H_p in G_p. Let $x_p \in V(\Gamma_p)$ denote the trivial coset H_p. The orbits of H_p on V_p were determined in [14], where it was shown that, if in addition $p \equiv \pm 1 \pmod 8$, then H_p has a unique orbit of length 5 and this orbit is self-paired. Hence an analogous coset graph construction to that in Example 5 gives, if $p \equiv \pm 1 \pmod 8$, a graph Γ_p of valency 5 with vertex set V_p. As H_p acts 2-transitively on its orbit of length 5, it follows that G_p is 2-arc transitive on Γ_p. Let $(p_i)_{i\geq 0}$ be an infinite increasing sequence of primes such that $p_i \equiv \pm 1 \pmod{10}$ and $p_i \equiv \pm 1 \pmod 8$. Then by Proposition 1 the sequence $((\Gamma_{p_i}, x_{p_i}))_{i\geq 0}$ has a convergent subsequence. Based on computations in *GAP* [4], we conjecture that the girth of Γ_{p_i} tends to infinity as p_i tends to infinity. This conjecture would imply that the limit is the infinite 5-valent tree.

For suitable primes $p_i \equiv \pm 1 \pmod{10}$, similar constructions yield graphs of valencies 6, 10, 12, 20, 30 or 60 in $\lim(\mathcal{FP}_{\mathrm{AS}}^{e-trans})$. The primes can also be chosen so that we can get graphs of valencies 5, 6, 10 or 12 in $\lim(\mathcal{FP}_{\mathrm{AS}}^{min})$.

The final family of examples involves the finite orthogonal groups and contains limit graphs of unboundedly large valencies.

Example 7 Let V be a $2m$-dimensional vector space over $\mathrm{GF}(p)$, where $p \equiv \pm 1 \pmod 8$, equipped with a quadratic form Q polarising to the symmetric bilinear form B. Let $O^+(2m,p)$ be the group of all invertible linear transformations which preserve Q, and let $G_p = PO^+(2m,p)$ be the quotient of $O^+(2m,p)$ modulo scalars. Let $\langle e_1\rangle\oplus\langle e_2\rangle\oplus\ldots\oplus\langle e_{2m}\rangle$, where each $Q(e_i)=1$ and $B(e_i,e_j)=0$ for $i\neq j$, be a direct sum decomposition of V into isometric nonsingular 1-dimensional subspaces. Then $H_p = (C_2^{2m}\rtimes S_{2m})/\{\pm I\}\cong C_2^{2m-1}\rtimes S_{2m}$ is the stabiliser of this decomposition in G_p and by [7, Proposiiton 7.2.2], H_p is a maximal subgroup of G_p.

Since $p\equiv\pm1 \pmod 8$, there exists $\lambda\in\mathrm{GF}(p)$ such that $\lambda^2=2$. Now let $f_1=\lambda^{-1}(e_1+e_2)$ and $f_2=\lambda^{-1}(e_1-e_2)$. Then $Q(f_1)=Q(f_2)=1$ and $B(f_1,f_2)=0$. Furthermore, $B(f_i,e_j)=0$ for $i=1,2$ and $j\geq3$. Thus $\langle f_1\rangle\oplus\langle f_2\rangle\oplus\langle e_3\rangle\oplus\ldots\oplus\langle e_{2m}\rangle$ is also a direct sum decomposition of V into isometric nonsingular 1-dimensional subspaces. Furthermore, there exists $g_p\in PO^+(2m,p)$ mapping the first decomposition to the second and we can choose g_p to have order 2. Then $(H_p)^{g_p}\cap H_p$ is the stabiliser of both decompositions and hence $(H_p)^{g_p}\cap H_p=(C_2^{2m-2}\rtimes S_{2m-2})\times(C_2^2))/\{\pm I\}$. Note that $|H_p : H_p\cap(H_p)^{g_p}| = 2m(2m-1)$. Then $\Gamma_p=\mathrm{Cos}(G_p,H_p,H_pg_pH_p)$ is a vertex-primitive graph of valency $2m(2m-1)$. Let x_p be the vertex corresponding to the trivial coset H_p.

Thus for each positive integer m at least 4, and infinite increasing sequence $(p_i)_{i\geq0}$ of primes $p_i\equiv\pm1 \pmod 8$, we obtain a sequence $((\Gamma_{p_i},x_{p_i}))_{i\geq0}$ of G_{p_i}-vertex-primitive graphs of valency $2m(2m-1)$ in $\mathcal{FP}_{\mathrm{AS}}$. By Proposition 1, there is a subsequence that converges to a graph $\Gamma\in\lim(\mathcal{FP}_{\mathrm{AS}})$ of valency $2m(2m-1)$. As there is an infinite number of choices for m we obtain infinitely many graphs in $\lim(\mathcal{FP}_{\mathrm{AS}})$.

6 Limit graphs of type PA

The first result of this section, Theorem 7, is an analogue of Theorem 6 for the almost simple case. It shows that the almost simple components of primitive groups of type PA involved in limiting sequences for graphs in $\lim(\mathcal{FP}_{\mathrm{PA}})$ correspond to maximal subgroups of Lie type almost simple groups given in Definition 1. This result suggests a strong connection between the AS and PA cases. Exploring this possibility we obtained a characterisation in Theorem 8 of the limit graphs of $\mathcal{FP}_{\mathrm{PA}}^{min}$ as Cartesian powers of limit graphs for $\mathcal{FP}_{\mathrm{AS}}^{min}$.

Primitive permutation groups of type PA

A finite primitive permutation group G of type PA has a unique minimal normal subgroup $N=T_1\times\cdots\times T_n$ where $n\geq2$ and each $T_i\cong T$ for some finite nonabelian simple group T. The point set can be identified with the Cartesian product $V=U_1\times\cdots\times U_n=U^n$ such that $G\leq\mathrm{Sym}(U)\wr S_n=\mathrm{Sym}(U)^n.S_n$ acting in product action. The product action is given by the following, where $h=(h_1,\ldots,h_n)\in\mathrm{Sym}(U)^n$, $\sigma\in S_n$ and $(u_1,\ldots,u_n)\in V$.

$$\begin{aligned} h &: (u_1,\ldots,u_n)\mapsto(h_1(u_1),\ldots,h_n(u_n))\\ \sigma &: (u_1,\ldots,u_n)\mapsto(u_{\sigma(1)},\ldots,u_{\sigma(n)}). \end{aligned}$$

Thus, in particular, $\sigma^{-1}h\sigma = (h_{\sigma^{-1}(1)}, \ldots, h_{\sigma^{-1}(n)})$. It follows from the primitivity of G on V that G projects onto a transitive subgroup of S_n; thus G acts transitively on $\{1, \ldots, n\}$ in the same way that it acts on the entries of points of V. Let G_1 be the stabiliser of the point 1 in this action, so that $G_1 \leq \mathrm{Sym}(U) \times (\mathrm{Sym}(U) \wr S_{n-1})$, and let H denote the image of G_1 under the natural homomorphism $G_1 \to \mathrm{Sym}(U)$ to the first direct factor of this direct product. Thus G_1 induces the group H on the first entries of points of V. By [11, 2.2], we may (and will) replace G by a conjugate under an element of $\mathrm{Sym}(U) \wr S_n$ if necessary, and thereby assume that $G \leq H \wr S_n$. Moreover, $T \leq H \leq \mathrm{Aut}(T)$, and H acts primitively on U of type AS. This description demonstrates the link between finite primitive groups of type PA and those of type AS. One extra fact that we need about the primitive group H of type AS, that follows from the O'Nan–Scott Theorem, is that, for $u \in U$, $T_u \neq 1$, and moreover u is the only fixed point of T_u in U. (Note that the fact that $T_u \neq 1$ depends on the Finite Simple Group Classification.)

The following result relates the groups involved in convergent sequences of graphs in $\mathcal{FP}_{\mathrm{PA}}$ to the groups involved in convergent sequences of graphs in $\mathcal{FP}_{\mathrm{AS}}$. Recall Definition 1 of the families $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_5$ of maximal subgroups of almost simple Lie type groups.

Theorem 7 *Let $((\Gamma_i, x_i))_{i\geq 0}$ be an infinite sequence in $\mathcal{G}$ which $(\varphi_i)_{i\geq 0}$-converges to an infinite graph (Γ, x) and let $G_i \leqslant \mathrm{Aut}(\Gamma_i)$ be vertex-primitive groups of type PA with $G_i \leqslant A_i \operatorname{wr} S_{n_i}$ for some almost simple group A_i with socle T_i. Then there exist a subsequence $(i_j)_{j\geq 0}$, a positive integer n, and a finite group M such that each $n_{i_j} = n$, the T_{i_j} are pairwise distinct finite simple groups of the same Lie type and rank, $M \in \mathcal{C}_1, \ldots, \mathcal{C}_4$ or $\mathcal{C}_5$ as a subgroup of A_{ij}, and $(G_{i_j})_{x_{i_j}} = G_{i_j} \cap (M \operatorname{wr} S_n)$.*

Proof Let $i \geq 0$. Since the action of G_i on $V(\Gamma_i)$ is primitive of type PA, there exists a set U_i such that $V(\Gamma_i) = U_i^{n_i}$. Furthermore, there exists a maximal subgroup M_i of A_i such that $(G_i)_{x_i} = G_i \cap (M_i \operatorname{wr} S_{n_i})$.

By Proposition 5, there exists a subsequence $(i_j)_{j\geq 0}$ such that for each $j \geq 0$, the stabiliser $(G_{i_j})_{x_{i_j}}$ is isomorphic to a fixed finite group H. Let $N_i = \mathrm{Soc}(G_i) = T^{n_i}$. Then as $G_i = N_i(G_i)_{x_i}$ it follows that $(G_i)_{x_i}$ acts transitively on the set of n_i simple direct factors of N_i. Thus n_{i_j} divides $|H|$ and so $(n_{i_j})_{j\geq 0}$ is bounded. Hence, by taking a subsequence if necessary, we may assume that $n_{i_j} = n$ for all $j \geq 0$. Thus $H \cong (G_{i_j})_{x_{i_j}} \leqslant M_{i_j} \operatorname{wr} S_n$. Moreover, since the stabiliser in $(G_{i_j})_{x_{i_j}}$ of the first simple direct factor of $\mathrm{Soc}(G_{i_j})$ projects onto M_{i_j}, it follows that $|M_{i_j}|$ divides $|H|$. Thus, by restricting to a subsequence if necessary, we may assume that there exists a group M such that $M_{i_j} \cong M$ for all $j \geq 0$. Hence we have a sequence $(A_{i_j})_{j\geq 0}$ of primitive almost simple groups with socles T_{i_j} and point stabilisers $M_{i_j} \cong M$. Thus arguing as in the proof of Theorem 6, by restricting to a subsequence if necessary, we may assume that the T_{i_j} are pairwise distinct Lie type simple groups of the same type and rank, and $M \in \mathcal{C}_1, \ldots, \mathcal{C}_4$ or $\mathcal{C}_5$ and the result follows. □

Cartesian products of graphs and direct products of groups

The *Cartesian product* $\Delta_1\Box\Delta_2\Box\dots\Box\Delta_n$ of graphs $\Delta_1, \Delta_2, \dots, \Delta_n$, is the graph with vertex set $V(\Delta_1) \times V(\Delta_2) \times \cdots \times V(\Delta_n)$ and edge set the set of all pairs $\{(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)\}$ such that there exists $1 \leq i \leq n$ with $x_j = y_j$ for all $j \in \{1, 2, \dots, n\} \setminus \{i\}$ and $\{x_i, y_i\} \in E(\Delta_i)$. In the case where $\Delta_1 = \Delta_2 = \cdots = \Delta_n = \Delta$, we write $\Delta^{\Box n}$ for $\Delta_1\Box\Delta_2\Box\dots\Box\Delta_n$ and call $\Delta^{\Box n}$ the n^{th} *Cartesian power* of Δ. Note that $\mathrm{Aut}(\Delta) wr S_n \leq \mathrm{Aut}(\Delta^{\Box n})$ with $\mathrm{Aut}(\Delta) wr S_n$ acting on $V(\Delta^{\Box n})$ in product action.

First we characterize, as Cartesian products, graphs that are limits of a sequence of graphs admitting subgroups of automorphisms that are direct products. For a direct product $H = H_1 \times \cdots \times H_n$, define, for each $i \leq n$, $\widetilde{H}_i = \langle H_j | j \in \{1, \dots, n\} \setminus \{i\}\rangle$, so that $H = H_i \times \widetilde{H}_i$. For a permutation group H and a point x, we denote by $H(x)$ the H-orbit containing x. Finally, for a graph Δ and a subset $X \subseteq V(\Delta)$, the subgraph of Δ induced on X is denoted $\langle X\rangle_\Delta$

Proposition 11 *Let Δ be an undirected connected graph that admits a vertex-transitive group of automorphisms $H = H_1 \times \cdots \times H_n$, where $n > 1$, and let $x \in V(\Delta)$. If $\Delta(x) \subseteq \cup_{1\leq i\leq n} H_i(x)$, and $H_i(x) \cap \widetilde{H}_i(x) = \{x\}$ for each $i = 1, \dots, n$, then $\Delta \cong \Delta_1\Box\dots\Box\Delta_n$, where $\Delta_i = \langle H_i(x)\rangle_\Delta$ for each $i = 1, \dots, n$.*

Proof First we claim that $H_x = (H_1)_x \times \cdots \times (H_n)_x$. Clearly $(H_1)_x \times \cdots \times (H_n)_x \leqslant H_x$ so suppose that there exists $h \in H_x \setminus ((H_1)_x \times \cdots \times (H_n)_x)$. Thinking of H as an internal direct product we have $h = h_1 h_2 \dots h_n$ where each $h_i \in H_i$, and $h_j \notin (H_j)_x$ for some j. Then as the H_i commute, we have $x = h_1 h_2 \dots h_n(x) = h_j h_1 h_2 \dots h_{j-1} h_{j+1} \dots h_n(x)$. It follows that $h_j^{-1}(x) = h_1 h_2 \dots h_{j-1} h_{j+1} \dots h_n(x) \in H_j(x) \cap \widetilde{H}_j(x) = \{x\}$, which is a contradiction since $h_j(x) \neq x$. Thus the claim is proved.

Let $y \in V(\Delta)$. Then there exists $g \in H$ such that $g(x) = y$. Now g can be written uniquely as $g = g_1 g_2 \dots g_n$ where $g_i \in H_i$ for each i. Define

$$\begin{array}{rccl} \phi: & V(\Delta) & \to & V(\Delta_1\Box\dots\Box\Delta_n) \\ & y & \mapsto & (g_1(x), \dots, g_n(x)). \end{array}$$

To show that ϕ does not depend on the choice of g, suppose that $g, h \in H$ satisfy $y = g(x) = h(x)$. Then $h \in gH_x$. If $g = g_1 g_2 \dots g_n$ and $h = h_1 h_2 \dots h_n$, where $g_i, h_i \in H_i$ for each i, then it follows from $H_x = (H_1)_x \times \cdots \times (H_n)_x$ that, for each i, we have $h_i = g_i u_i$ for some $u_i \in (H_i)_x$. Hence

$$(g_1(x), \dots, g_n(x)) = (g_1 u_1(x), \dots, g_n u_n(x)) = (h_1(x), \dots, h_n(x)).$$

Thus ϕ is well defined.

Let $(x_1, \dots, x_n) \in V(\Delta_1\Box\dots\Box\Delta_n)$. Then for each i, $x_i \in H_i(x)$ and so there exists $g_i \in H_i$ such that $g_i(x) = x_i$. Let $y = g_1 g_2 \dots g_n(x)$. Then it follows from the previous paragraph that $\phi(y) = (x_1, \dots, x_n)$, so ϕ is onto.

Next suppose that $\phi(y_1) = \phi(y_2)$. Let $y_1 = g(x)$, where $g = g_1 g_2 \dots g_n$, and $y_2 = h(x)$ where $h = h_1 h_2 \dots h_n$. Then

$$(g_1(x), \dots, g_n(x)) = \phi(y_1) = \phi(y_2) = (h_1(x), \dots, h_n(x))$$

and so for each $i = 1, \dots, n$ we have $h_i \in g_i(H_i)_x$. Thus $h_1 h_2 \dots h_n \in g_1 g_2 \dots g_n H_x$ and so $y_1 = g(x) = h(x) = y_2$. Hence ϕ is one-to-one.

It remains to prove that ϕ is a graph isomorphism. Suppose first that z and y are adjacent in Δ. Then there exists $g \in H$ such that $g(x) = z$ and if $g = g_1 g_2 \dots g_n$ then $\phi(z) = (g_1(x), \dots, g_n(x))$. Since y is adjacent to z in Δ, there exists $w \in \Delta(x)$ such that $g(w) = y$, and since $\Delta(x) \subseteq \cup_{1 \le i \le n} H_i(x)$, there exist $i \in \{1, \dots, n\}$ and $h_i \in H_i$ such that $h_i(x) = w$. Thus $gh_i(x) = y$. Furthermore, $gh_i = g_1 g_2 \dots g_{i-1}(g_i h_i) g_{i+1} \dots g_n$. Thus

$$\begin{aligned}\phi(y) &= (g_1(x), \dots, g_{i-1}(x), g_i h_i(x), g_{i+1}(x), \dots, g_n(x)) \\ &= (g_1(x), \dots, g_{i-1}(x), g_i(w), g_{i+1}(x), \dots, g_n(x)).\end{aligned}$$

Since $g_i(w)$ is adjacent to $g_i(x)$ in $\Delta_i = \langle H_i(x) \rangle_\Delta$, it follows that $\phi(z)$ is adjacent to $\phi(y)$ in $\Delta_1 \Box \dots \Box \Delta_n$. Conversely, suppose that $\phi(z)$ is adjacent to $\phi(y)$ in $\Delta_1 \Box \dots \Box \Delta_n$. Let $g = g_1 g_2 \dots g_n$ and $h = h_1 h_2 \dots h_n$ such that $g(x) = z$ and $h(x) = y$. Then $\phi(z) = (g_1(x), \dots, g_n(x))$ and $\phi(y) = (h_1(x), \dots, h_n(x))$. Furthermore, there exists $j \in \{1, \dots, n\}$ such that $g_i(x) = h_i(x)$ for all $i \ne j$ and $h_j(x)$ is adjacent to $g_j(x)$ in $\Delta_j = \langle H_j(x) \rangle_\Delta$, and hence in Δ. Hence since the H_i commute, it follows that

$$\begin{aligned}y &= h_1 h_2 \dots h_n(x) \\ &= h_j h_1 h_2 \dots h_{j-1} h_{j+1} \dots h_n(x) \\ &= h_j g_1 g_2 \dots g_{j-1} g_{j+1} \dots g_n(x) \\ &= g_1 g_2 \dots g_{j-1} g_{j+1} \dots g_n h_j(x).\end{aligned}$$

Also $z = g_1 g_2 \dots g_{j-1} g_{j+1} \dots g_n g_j(x)$ where $g_j(x)$ is adjacent to $h_j(x)$. Then as $g_1 g_2 \dots g_{j-1} g_{j+1} \dots g_n$ is an automorphism of Δ it follows that z is adjacent to y in Δ and hence ϕ is a graph isomorphism. □

We use Proposition 11 together with Proposition 2 to obtain the following characterization of limit graphs that are Cartesian products.

Proposition 12 *Let $((\Gamma_i, x_i))_{i \ge 0}$ be a sequence in $\mathcal{G}^*$ that $(\varphi_i)_{i \ge 0}$-converges to (Γ, x), and let n be an integer, $n \ge 2$. Suppose that, for each $i \ge 0$, $G_i \le \mathrm{Aut}(\Gamma_i)$ is a vertex-transitive direct product $G_{i,1} \times \cdots \times G_{i,n}$ such that $\Gamma_i(x_i) \subseteq \cup_{1 \le k \le n} G_{i,k}(x_i)$ and $G_{i,k}(x_i) \cap \widetilde{G}_{i,k}(x_i) = \{x_i\}$ for all $k = 1, \dots, n$. For each i, k, let $\Delta_{i,k}$ be the subgraph $\langle G_{i,k}(x_i) \rangle_{\Gamma_i}$. Then there exist subgraphs $\Delta_1, \dots, \Delta_n$ of Γ with $x \in V(\Delta_1) \cap \cdots \cap V(\Delta_n)$ and an increasing sequence of non-negative integers $(i_j)_{j \ge 0}$ such that, for each $k = 1, \dots, n$, the sequence $(\Delta_{i_j,k}, x_{i_j})_{j \ge 0}$ is $(\varphi_{i_j}|_{V(\Delta_k)})_{j \ge 0}$-convergent to (Δ_k, x), and $\Gamma \cong \Delta_1 \Box \dots \Box \Delta_n$.*

Proof By Proposition 11, for each $i \ge 0$, $\Gamma_i \cong \Delta_{i,1} \Box \dots \Box \Delta_{i,n}$. Note that, for all i, k, the graphs $\Delta_{i,k}$ are connected since the Γ_i are connected.

Let $\Gamma(x) = \{y_1, \dots, y_d\}$. Replacing $((\Gamma_i, x_i))_{i \ge 0}$ by a subsequence if necessary, we may assume that, for all $i \ge 0$, $\Gamma_i(x_i) = \{y_{i,1}, \dots, y_{i,d}\}$, where

$y_{i,t} = \varphi_i(y_t)$ for $1 \leq t \leq d$, and, moreover, the set $\Gamma(x)$ is the union of pairwise disjoint subsets $Y_1, \ldots, Y_n$ such that, for any $i \geq 0$ and $1 \leq k \leq n$, $\varphi_i(Y_k) = \Gamma_i(x_i) \cap G_{i,k}(x_i)$. In addition, by Proposition 2 we may assume, (again considering a subsequence, if necessary) that, for each $t = 1, \ldots, d$ and $i \geq 0$, there are automorphisms $g_t \in \mathrm{Aut}(\Gamma)$ and $g_{i,t} \in G_i$ such that $g_t(x) = y_t$, $g_{i,t}(x_i) = y_{i,t}$, the sequence $(g_{i,t})_{i\geq 0}$ is $(\varphi_i)_{i\geq 0}$-convergent to g_t, and $g_{i,t} \in G_{i,k}$ where k is such that $y_t \in Y_k$. Note that, for any $i \geq 0$ and $1 \leq k \leq n$, the subgroup of $G_{i,k}$ generated by the elements $g_{i,t}$ such that $y_t \in Y_k$, for $1 \leq t \leq d$, acts vertex-transitively on $\Delta_{i,k}$ (since $\Delta_{i,k}$ is connected).

For each $k = 1, \ldots, n$, set $H_k = \langle g_t | 1 \leq t \leq d \text{ and } y_t \in Y_k\rangle$. Since $Y_k \subseteq H_k(x)$, for $1 \leq k \leq n$, and $\Gamma(x) = \cup_{1\leq k\leq n} Y_i$, we have that $\langle H_1, \ldots, H_n\rangle$ is a vertex-transitive group of automorphisms of Γ.

If $1 \leq t_1, t_2 \leq d$ and $g_{t_1}(x) \in Y_{k_1}$, $g_{t_2}(x) \in Y_{k_2}$ for distinct k_1 and K_2 such that $1 \leq k_1, k_2 \leq n$, then $[g_{t_1}, g_{t_2}] = 1$ since $[g_{i,t_1}, g_{i,t_2}] = 1$ for all $i \geq 0$ and the sequences $(g_{i,t_1})_{i\geq 0}$, $(g_{i,t_2})_{i\geq 0}$ are $(\varphi_i)_{i\geq 0}$-convergent to g_{t_1}, g_{t_2} respectively. It follows that $[H_{k_1}, H_{k_2}] = 1$ for distinct k_1, k_2 such that $1 \leq k_1, k_2 \leq n$.

If $l \geq 1$, $t_1, \ldots, t_l \in \{1, \ldots, d\}$ and $\epsilon_1, \ldots, \epsilon_l \in \{1, -1\}$, then

$$\varphi_i(g_{t_1}^{\epsilon_1} \ldots g_{t_l}^{\epsilon_l}(x)) = g_{i,t_1}^{\epsilon_1} \ldots g_{i,t_l}^{\epsilon_l}(x_i)$$

for all sufficiently large i. Since $G_{i,k}(x_i) \cap \widetilde{G}_{i,k}(x_i) = \{x_i\}$ for each $k = 1, \ldots, n$ and all $i \geq 0$, it follows that $H_k(x) \cap \widetilde{H}_k(x) = \{x\}$ for each $k = 1, \ldots, n$, where we set $\widetilde{H}_k = \langle H_{k'} | k' \in \{1, \ldots, n\} \setminus \{k\}\rangle = \langle g_t | 1 \leq t \leq d \text{ and } y_t \notin Y_k\rangle$.

Hence, for each $k = 1, \ldots, n$, $H_k \cap \widetilde{H}_k$ is a normal subgroup of the vertex-transitive group $\langle H_1, \ldots, H_n\rangle \leq \mathrm{Aut}(\Gamma)$ and $(H_k \cap \widetilde{H}_k)(x) \leq H_k(x) \cap \widetilde{H}_k(x) = \{x\}$. Hence $H_k \cap \widetilde{H}_k = 1$ for each k, and so $\langle H_1, \ldots, H_n\rangle = H_1 \times \cdots \times H_n$.

For each $k = 1, \ldots, n$, let $\Delta_k = \langle H_k(x)\rangle_\Gamma$. By Proposition 11, $\Gamma \cong \Delta_1 \square \ldots \square \Delta_n$. Now, for each $t = 1, \ldots, d$, the sequence $(g_{i,t})_{i\geq 0}$ is $(\varphi_i)_{i\geq 0}$-convergent to g_t, and for each $i \geq 0$ and $1 \leq k \leq n$, the subgroup of $G_{i,k}$ generated by all elements $g_{i,t}$ such that $y_t \in Y_k$ $(1 \leq t \leq d)$ is contained in $G_{i,k}$ and acts vertex-transitively on $\Delta_{i,k}$. It then follows easily that, for each $k = 1, \ldots, n$, the sequence $(\Delta_{i,k}, x_i)_{i\geq 0}$ is $(\varphi_i|_{V(\Delta_k)})_{i\geq 0}$-convergent to (Δ_k, x). This completes the proof. □

In contrast to the examples constructed in the previous section of graphs in $\lim(\mathcal{FP}_{\mathrm{AS}})$ it turns out that graphs in $\lim(\mathcal{FP}_{\mathrm{PA}})$ have small girth, as shown in the next result.

Proposition 13 *If $\Gamma \in \mathcal{FP}_{\mathrm{PA}}$, then the girth of Γ is at most 4.*

Proof Using the notation introduced at the beginning of this section, we have a vertex-primitive subgroup $G \leqslant \mathrm{Aut}(\Gamma)$ of type PA, and $G = NG_x$ where $x \in V(\Gamma)$, and $N = T_1 \times \cdots \times T_n$, $n \geq 2$. Also $V(\Gamma) = U^n$ for some U, and for each $i = 1, \ldots, n$ and $u \in U$, u is the only fixed point of $(T_i)_u$ in U. Now $x = (x_1, \ldots, x_n)$ for some $x_1, \ldots, x_n \in U$. Suppose first that the following holds.

(*) There exists a vertex $x' = (x'_1, \ldots, x'_n) \in \Gamma(x)$ such that $x_i \neq x'_i$ and $x_j \neq x'_j$ for some i, j with $1 \leq i < j \leq n$.

In this case there are $g_i \in (T_i)_x$ and $g_j \in (T_j)_{x'}$ such that $g_i(x') \neq x'$ and $g_j(x) \neq x$. Since $i \neq j$ we have $g_i g_j = g_j g_i$, and so $g_i g_j(x) = g_j g_i(x) = g_j(x)$. At the same time, $d_\Gamma(g_i g_j(x), g_i(x')) = d_\Gamma(g_j(x), x') = d_\Gamma(g_j(x), g_j(x')) = d_\Gamma(x, x') = 1$. Thus $g_i(x'), x, x', g_j(x)$ is a 4-cycle in Γ, and the result follows.

Suppose now that (*) does not hold. Then Γ satisfies the conditions of Proposition 11 and so $\Gamma \cong \Delta_1 \Box \ldots \Delta_n$ for some $\Delta_1, \ldots, \Delta_n$. In this case also, Γ contains a 4-cycle and the result follows. $\Box$

Corollary 1 *Every graph in* $\lim(\mathcal{FP}_{\mathrm{PA}})$ *has girth at most* 4.

Limit graphs of $\mathcal{FP}_{\mathrm{PA}}^{min}$

In order to study the limit graphs of $\mathcal{FP}_{\mathrm{PA}}^{min}$ we need to understand the structure of graphs in $\mathcal{FP}_{\mathrm{PA}}^{min}$. The next result shows that most of them are Cartesian products. Let H be a primitive permutation group on a finite set V and let Γ be a graphs with $V(\Gamma) = V$ such that $H \leqslant \mathrm{Aut}(\Gamma)$. Then $\Gamma \in \mathcal{FP}^{min}$ with respect to this group H if and only if $W = \{(v, v') \mid \{v, v'\} \in E(\Gamma)\}$ has minimum size among the H-invariant subsets U of $V \times V$ such that $U = U^*$ (where $U^* = \{(u', u) \mid (u, u') \in U\}$) and $U \neq \{(v, v) \mid v \in V\}$. For such a subset W, and $v \in V$, the set $W(v) = \{v' \mid (v, v') \in W\}$ is H_v-invariant and is called a *minimal symmetric* H_v-invariant subset of $V \backslash \{v\}$. The relationship between the minimal symmetric subsets for primitive permutation groups of type PA and those for their primitive components of type AS was elucidated in [6] and the classification obtained. This is used here to study $\lim(\mathcal{FP}_{\mathrm{PA}}^{min})$. First we consider $\mathcal{FP}_{\mathrm{PA}}^{min}$.

Proposition 14 *Let* $n \geq 2$ *and* $\Delta \in \mathcal{FP}_{\mathrm{AS}}^{min}$. *Then* $\Gamma \cong \Delta^{\Box n} \in \mathcal{FP}_{\mathrm{PA}}^{min}$. *Conversely, if* $\Gamma \in \mathcal{FP}_{\mathrm{PA}}^{min}$ *admitting a primitive group* G *of type PA, and if* $\mathrm{Soc}(G)$ *is not a direct power of* $\mathrm{PSL}(2,7)$ *or* $\mathrm{PSL}(2,9)$*, then* $\Gamma \cong \Delta^{\Box n}$*, for some* $n \geq 2$ *and* $\Delta \in \mathcal{FP}_{\mathrm{AS}}^{min}$.

Proof Let $\Delta \in \mathcal{FP}_{\mathrm{AS}}^{min}$, $n \geq 2$, $\Gamma = \Delta^{\Box n}$, and let H be a vertex-primitive subgroup of $\mathrm{Aut}(\Delta)$ of type AS such that, for a vertex u, $\Delta(u)$ is a minimal symmetric H_u-invariant subset of $V(\Delta)\backslash\{u\}$. Then $H \wr S_n$, in product action, is a vertex-primitive subgroup of $\mathrm{Aut}(\Gamma)$ of type PA. Hence $\Gamma \in \mathcal{FP}_{\mathrm{PA}}$, and it follows from [6, Theorem 1.4] that $\Gamma \in \mathcal{FP}_{\mathrm{PA}}^{min}$. The converse also follows from [6, Theorem 1.4] $\Box$

Propositions 12 and 14 yield the following description of limit graphs of $\mathcal{FP}_{\mathrm{PA}}^{min}$.

Theorem 8 $\Gamma \in \lim(\mathcal{FP}_{\mathrm{PA}}^{min})$ *if and only if* $\Gamma = \Delta^{\Box n}$ *for some* $\Delta \in \lim(\mathcal{FP}_{\mathrm{AS}}^{min})$ *and* $n > 1$.

Proof Suppose that $\Gamma = \Delta^{\square n}$ for some $\Delta \in \lim(\mathcal{FP}_{\mathrm{AS}}^{min})$ and $n > 1$. Then, for $u \in V(\Delta)$, we have a sequence $((\Delta_i, u_i))_{i\geq 0}$ that $(\psi_i)_{i\geq 0}$-converges to (Δ, u) with each $\Delta_i \in \mathcal{FP}_{\mathrm{AS}}^{min}$. For each i, define $\varphi : V(\Gamma) \to V(\Delta_i)^n$ by

$$\varphi : (v_1, \ldots, v_n) \mapsto (\psi_i(v_1), \ldots, \psi_i(v_n))$$

and set $x_i = (u_i, \ldots, u_i)$ and $x = (u, \ldots, u)$. Then it is easy to check that the sequence $((\Delta_i^{\square n}, x_i))_{i\geq 0}$ is $(\varphi_i)_{i\geq 0}$- convergent to (Γ, x). Since $\Delta_i \in \mathcal{FP}_{\mathrm{AS}}^{min}$, Proposition 14 implies that $\Delta_i^{\square n} \in \mathcal{FP}_{\mathrm{PA}}^{min}$, and hence $\Gamma \in \lim(\mathcal{FP}_{\mathrm{PA}}^{min})$.

Conversely suppose that $\Gamma \in \lim(\mathcal{FP}_{\mathrm{PA}}^{min})$, and let $x \in V(\Gamma)$. So there exist $\Gamma_i \in \mathcal{FP}_{\mathrm{PA}}^{min}$ and $x_i \in V(\Gamma_i)$ such that $((\Gamma_i, x_i))_{i\geq 0}$ is $(\varphi_i)_{i\geq 0}$ convergent to (Γ, x). Moreover there exist primitive subgroups $G_i \leq \mathrm{Aut}(\Gamma_i)$ of type PA such that $\Gamma_i(x_i)$ is a minimal symmetric $(G_i)_{x_i}$-invariant subset of $V(\Gamma_i)\setminus\{x_i\}$. Since G_i is primitive of type PA, $G_i \leq H_i \wr S_{n_i}$ in product action on $V(\Gamma_i) = U_i^{n_i}$ for some $n_i > 1$, where H_i is a primitive permutation group on U_i of type AS. By Theorem 7, restricting to a subsequence if necessary, we may assume that there exists a positive integer $n \geq 2$ such that $n_i = n$ for all i and the H_i are all finite almost simple groups of the same Lie type and rank. Since the $|V(\Gamma_i)|$ are unbounded, we may assume that $|\mathrm{Soc}(H_i)| > 360$, and so, by Proposition 14, $\Gamma_i = \Delta_i^{\square n}$ for some $\Delta_i \in \mathcal{FP}_{\mathrm{AS}}^{min}$. Hence there exists a symmetric $(H_i)_{u_i}$-invariant subset W_i of length d_i such that $\Gamma_i(x_i)$ consists of all points $(v_1, \ldots, v_n) \in U_i^n$ having all but one of the entries equal to u_i and the remaining entry in W_i. Thus the hypotheses of Proposition 12 hold for the subgroups $\mathrm{Soc}(G_i) = T_{i,1} \times \cdots \times T_{i,n} \cong T_i^n$ of $\mathrm{Aut}(\Gamma_i)$, so (taking into account the fact that G_i acts transitively on the entries of points of U_i^n) there exists a subgraph Δ of Γ with $x \in V(\Delta)$, and a subsequence $(i_j)_{j\geq 0}$, such that $\Gamma = \Delta^{\square n}$ and, if $\Delta_i = \langle T_{i,1}(x_i)\rangle_{\Gamma_i}$, then $((\Delta_{i_j}, x_{i_j}))_{j\geq 0}$ is $(\varphi_{i_j}|_{V(\Delta)})_{j\geq 0}$ convergent to (Δ, x). Since each $\Delta_{i_j} \in \mathcal{FP}_{\mathrm{AS}}^{min}$, we have $\Delta \in \lim(\mathcal{FP}_{\mathrm{AS}}^{min})$. $\square$

7 Concluding remarks

Remark 1. It follows from Theorem 4 and Example 3, that both $\lim(\mathcal{FP}_{\mathrm{HA}})$ and $\lim(\mathcal{FP}_{\mathrm{HA}}^{min})$ are countably infinite. In addition, see Example 2, for all integers $d > 2$ there are infinitely many graphs in $\lim(\mathcal{FP}_{\mathrm{HA}})$ of valency d. However, the graphs given in Example 2 that illustrate this fact are not edge-transitive. Infinite families of edge-transitive examples have been constructed by Kostousov ([10] and private communication).

Remark 2. In the light of the results and examples of Section 5, it seems unlikely that an explicit description of the graphs in $\lim(\mathcal{FP}_{\mathrm{AS}}^{min})$ can be obtained. However the constructions given in Section 5 are sufficient to demonstrate that $\lim(\mathcal{FP}_{\mathrm{AS}}^{min})$ is infinite. A related question is the following.

Question 1 *Is* $\lim(\mathcal{FP}_{\mathrm{AS}}^{min})$ *countable?*

Independently of this question, we believe that the sets $\lim(\mathcal{FP}_{\mathrm{AS}}^{e-trans})$ and $\lim(\mathcal{FP}_{\mathrm{AS}}^{min})$ will contain some 'new' graphs with interesting properties.

Remark 3. In Theorem 8, we saw that every graph in $\lim(\mathcal{FP}_{\mathrm{PA}}^{min})$ is a cartesian power of some graph in $\lim(\mathcal{FP}_{\mathrm{AS}}^{min})$. In Problem 3 we asked for a useful description of the graphs in $\lim(\mathcal{FP}_{\mathrm{PA}})$?

Here we outline an approach for investigating the graphs in $\lim(\mathcal{FP}_{\mathrm{PA}}^{e-trans})$. We construct an edge-transitive subgroup N of limit automorphisms of Γ that has a normal subgroup Q of the form $Q_1 \times \cdots \times Q_n$, where n is as in Theorem 7, and N permutes $\{Q_1, \ldots, Q_n\}$ transitively.

Suppose that $((\Gamma_i, x_i))_{i\geq 0}$ is a sequence of finite graphs with distinguished vertices that $(\varphi_i)_{i\geq 0}$-converges to (Γ, x) and that, for each $i \geq 0$, there is a subgroup $G_i \leq \mathrm{Aut}(\Gamma_i)$ that is edge-transitive, and vertex-primitive of type PA. Replacing this sequence by a proper subsequence if necessary, we may assume (see Proposition 5) that, for each $i \geq 0$, $G_i = (T_{i,1} \times \cdots \times T_{i,n})(G_i)_{x_i}$ for a fixed $n > 1$ (independent of i), where $T_{i,1}, \ldots, T_{i,n}$ are isomorphic nonabelian simple groups, $T_{i,1} \times \cdots \times T_{i,n} \trianglelefteq G_i$, the stabiliser $(G_i)_{x_i}$ acts transitively on $\{T_{i,1}, \ldots, T_{i,n}\}$ by conjugation, and is independent of i, up to isomorphism. Next, applying Proposition 2, we may again replace this sequence of graphs by a proper subsequence if necessary, and assume further that there exists a finite subset $M = M' \cup M''$ of $\mathrm{Aut}(\Gamma)$ for which the following properties (a)-(e) all hold.

(a) The set M' is a subgroup of $\mathrm{Aut}(\Gamma)_x$ isomorphic to $(G_i)_{x_i}$ for each $i \geq 0$. In particular, M' contains a normal subgroup $R = R_1 \times \cdots \times R_n$ such that M' is transitive on $\{R_1, \ldots, R_n\}$ acting by conjugation and, for each $i \geq 0$ and $k = 1, \ldots, n$, the subgroup R_k is isomorphic to $(T_{i,k})_{x_i}$.

(b) For $1 \leq k \leq n$ and $g \in R_k$, where R_k is as defined in (a), and for each $i \geq 0$, there exists $g_i \in (T_{i,k})_{x_i}$ such that the sequence $(g_i)_{i\geq 0}$ is $(\varphi_i)_{i\geq 0}$-convergent to g.

(c) For each $g \in M'$ and for each $i \geq 0$, there exists $g_i \in G_i$ such that the sequence $(g_i)_{i\geq 0}$ is $(\varphi_i)_{i\geq 0}$-convergent to g.

(d) The set M'' consists of all the elements $g \in \mathrm{Aut}(\Gamma)$ for which $d_\Gamma(x, g(x))$ is at most 1 and, for each $i \geq 0$, there exists $g_i \in T_{i,1} \times \ldots \times T_{i,n}$ such that the sequence $(g_i)_{i\geq 0}$ is $(\varphi_i)_{i\geq 0}$-convergent to g.

(e) For each $y \in \Gamma(x)$ there exists $g \in M''$ such that $g(x) = y$.

Set $N := \langle M \rangle$ and $L := \langle R \cup M'' \rangle$. We claim:

N is a vertex-transitive and edge-transitive group of automorphisms of Γ and L is a vertex-transitive normal subgroup of N.

Note that $R \subseteq M''$ and hence $L = \langle M'' \rangle$. Moreover, by conditions (c) and (d) it follows that M' normalises M'', and hence N normalizes L. It follows from condition (e) and the connectivity of Γ that L is vertex-transitive, so also N is vertex-transitive. If N_x acts transitively on $\Gamma(x)$, then N is edge-transitive. On the other hand, if N_x acts intransitively on $\Gamma(x)$, then conditions (a) and (c), and the edge-transitivity of each G_i on Γ_i, imply that M' has two orbits on $\Gamma(x)$ which are paired orbits of N_x, and again N is edge-transitive. Thus the claim is proved, and in particular, $N = LN_x$.

Next we define Q to be the normal closure in L of R and, for $1 \leq k \leq n$, we define Q_k as the normal closure in L of R_k. We claim:

$Q = Q_1 \times \cdots \times Q_n$, Q is normal in N, and N acts transitively by conjugation on $\{Q_1, \ldots, Q_n\}$.

By condition (d), for each $g \in L$ and each $i \geq 0$, there exists $g_i \in T_{i,1} \times \cdots \times T_{i,n}$ such that the sequence $(g_i)_{i \geq 0}$ is $(\varphi_i)_{i \geq 0}$-convergent to g. Similarly, for each $k = 1, \ldots, n$ and $g \in R_k$, there exist elements $g_i \in (T_{i,k})_{x_i}$, for $i \geq 0$, such that the sequence $(g_i)_{i \geq 0}$ is $(\varphi_i)_{i \geq 0}$-convergent to g. Thus for each $k = 1, \ldots, n$ and $g \in Q_k$, there are $g_i \in (T_{i,k})_{x_i}$, for $i \geq 0$, such that the sequence $(g_i)_{i \geq 0}$ is $(\varphi_i)_{i \geq 0}$-convergent to g. From this we deduce as follows that $Q = Q_1 \times \cdots \times Q_n$. If $1 \leq k \leq n$ and if, for some elements $g_i \in T_{i,k}$ and $g'_i \in \prod_{k' \neq k} T_{i,k'}$ (for $i \geq 0$), the two sequences $(g_i)_{i \geq 0}$ and $(g'_i)_{i \geq 0}$ are $(\varphi_i)_{i \geq 0}$-convergent to the same element g, then this element $g = 1$; for if this were not the case then, for any positive integer r, there would exist an integer i such that the non-identity element $(g_i)^{-1} g'_i \in G_i$ stabilizes pointwise the ball $B_{\Gamma_i}(x_i, r)$, contradicting the boundedness of $|(G_i)_{x_i}| = |M'|$. Thus $Q_k \cap (\prod_{k' \neq k} Q_{k'}) = 1$ and so $Q = Q_1 \times \cdots \times Q_n$. For each $k = 1, \ldots, n$, since $M'' \subseteq L$ and L normalises Q_k, it follows that M'' normalises Q_k. Also, since M' normalises both L and R, and since M' acts transitively on $\{R_1, \ldots, R_n\}$ by conjugation, it follows that M' normalises the normal closure Q in L of R, and M' acts transitively on the set $\{Q_1, \ldots, Q_n\}$ of normal closures in L of the R_k. Thus $N = \langle M' \cup M'' \rangle$ normalises Q and acts transitively on $\{Q_1, \ldots, Q_n\}$. Thus the claim is proved.

Let σ be the set of Q-orbits in $V(\Gamma)$ and let Γ/σ denote the corresponding quotient graph of Γ having vertex set σ and edges $\{\sigma_1, \sigma_2\}$ whenever there exist $y_i \in \sigma_i$ such that $\{y_1, y_2\}$ is an edge of Γ. Then L acts vertex-transitively on Γ/σ with kernel containing Q. Since $R \leq Q_x$, and R acts non-trivially on $\Gamma(x)$, it follows that $deg(\Gamma/\sigma) < deg(\Gamma)$. Moreover the graph theoretic structure of the subgraphs induced on the Q-orbits can be recognized using the $(\varphi_i)_{i \geq 0}$-convergent sequence $((\Gamma_i, x_i))_{i \geq 0}$. It would be interesting to describe Γ/σ.

8 Appendix

The English translation [16] contains the following misprints:

p. 221_{14-13}, "either" and ", or $h = 1$" should be deleted;

p. 225_{15}, it should be $2(\deg \Gamma)^{\varepsilon_n \cdot q \cdot n}$ instead of $(\deg \Gamma)^{\varepsilon_n \cdot q \cdot n}$ and it should be $1 + \varepsilon_n \cdot q \cdot n \cdot \log_2 \deg \Gamma$ instead of $\varepsilon_n \cdot q \cdot n \cdot \log_2 \deg \Gamma$;

p. 226^{13}, it should be $h_2^{\lambda_0} h_2^{\lambda_1 h_1} \ldots h_2^{\lambda_k h_1^k} = h_2^{\lambda_0}(h_1^{-1} h_2^{\lambda_1}) \ldots (h_1^{-1} h_2^{\lambda_k}) h_1^k$ instead of $h_2^{\lambda_0} h_2^{\lambda_1 h_1} \ldots h_2^{\lambda_n h_1^n} = h_2^{\lambda_0}(h_1^{-1} h_2^{\lambda_1}) \ldots (h_1^{-1} h_2^{\lambda_n}) h_1^n$;

p. 226_9, after $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ it should be added "be a non-decreasing function";

p. 227_1, after $\langle B_{\Gamma_i}(x_i, r_i)\rangle$ it should be added "mapping x to x_i";
p. 228_{12}, G and g should be interchanged;
p. 228_{11}, it should be $\varphi_i(g(y))$ instead of $g(y)$;
p. 229^2 and p. 229_{12}, it should be "for some non-decreasing function $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$, $f(n) = o(n)$, some" instead of "for some";
p. 233^{17}, the second x and x' should be interchanged;
p. 233^{18}, it should be X instead of x;
p. 233^{19}, it should be x instead of X;
p. 233_{14}, it should be $\hat{K}$ instead of K;
p. 234^{14}, it should be $B_\beta > B_{\beta+1}$ instead of $B_{\beta+1} > B_\beta$;
p. 234^{14} and p. 236^3, it should be $|B_\beta : B_{\beta+1}|$ instead of $|B_{\beta+1} : B_\beta|$.

References

[1] M. Aschbacher, On the maximal subgroups of the finite classical groups, Invent. Math. 76 (1984), 469–514.

[2] P. J. Cameron, C. E. Praeger, J. Saxl, G. M. Seitz, On the Sims conjecture and distance transitive graphs, Bull. London Math. Soc. 15 (1983), 499–506.

[3] J. D. Dixon B. Mortimer, Permutation groups, Springer-Verlag, New York, 1996.

[4] The GAP Group, GAP–Groups, Algorithms, and Programming, Version 4.3 (2002). (`http://www.gap-system.org`).

[5] M. Giudici, C. H. Li, C. E. Praeger, Á. Seress V. Trofimov, Limits of vertex-transitive graphs, to appear in: Z. Arad, M. Bianchi, W. Herfort, P. Longobardi, M. Maj and C. Scoppola (Eds), Ischia Group Theory 2004: Proceedings of a Conference in honour of Marcel Herzog, Contemporary Mathematics, 402 American Mathematical Society, Providence, RI, 2006.

[6] M. Giudici, C. H. Li, C. E. Praeger, Á. Seress V. Trofimov, On minimal subdegrees of finite primitive permutation groups, in: A. Hulpke, R. Liebler, T. Penttila and À. Seress (Eds), Finite Geometries, Groups and Computation, Walter de Gruyter 2006 pp 75–94.

[7] P. B. Kleidman M. W. Liebeck, The Subgroup Structure of the Finite Classical Groups, in: London Mathematical Society Lecture Note Series, 129, Cambridge University Press, Cambridge, 1990.

[8] A. S. Kondrat'ev V. I. Trofimov, Stabilizers of graph vertices, and a strengthened version of the Sims conjecture, (Russian) Dokl. Akad. Nauk., 364 (1999), 741–743.

[9] Problem 12.89, in: Kourovka Notebook. Unsolved problems in the theory of groups. Fifteenth augmented edition. Russian Acad. Sci. Siberian Branch. Inst. of Math. Novosibirsk, 2002. 172 pp.

[10] K. Kostousov, Cayley graphs of groups $\mathbf{Z}^d$ and limits of vertex-primitive graphs of HA-type, (Russian) *in preparation.*

[11] L. G. Kovács, Primitive subgroups of wreath products in product action, Proc. London Math. Soc. (3) 68 (2) (1989), 306–322.

[12] M. W. Liebeck G. M. Seitz, Maximal subgroups of exceptional groups of Lie type, finite and algebraic, Geom Dedicata, 35 (1990), 353-387.

[13] L. L. Scott, Representations in Characteristic p, in: The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), pp. 319–331, Proc. Sympos. Pure Math., 37, Amer. Math. Soc., Providence, 1980.

[14] D. E. Taylor M.-Y. Xu, Vertex-primitive half-transitive graphs, J. Austral. Math. Soc.(Series A), 57 (1994), 113-124.

[15] V. Trofimov, Automorphisms of graphs and a characterization of lattices, Math. USSR Izv. 22 (1984), 379-392.

[16] V. I. Trofimov, On the action of primitive groups (Russian), Algebra i Logika, 28 (1989), 337–363, 367; translation in: Algebra and Logic, 28 (1989), 220–237 (1990).

[17] V. I. Trofimov, Vertex stabilizers of graphs and tracks, I, *to appear in European J. Combin.*