

Primitive decompositions of Johnson graphs*

Alice Devillers
Université Libre de Bruxelles
Département de mathématiques
Géométrie- CP 216
Boulevard du Triomphe
B-1050 Bruxelles Belgique

Michael Giudici, Cai Heng Li and Cheryl E. Praeger
School of Mathematics and Statistics
The University of Western Australia
35 Stirling Highway
Crawley WA 6009
Australia

Abstract

A *transitive decomposition* of a graph is a partition of the edge set together with a group of automorphisms which transitively permutes the parts. In this paper we determine all transitive decompositions of the Johnson graphs such that the group preserving the partition is arc-transitive and acts primitively on the parts.

1 Introduction

A *decomposition* of a graph is a partition of the edge set with at least two parts, which we interpret as subgraphs and call the *divisors* of the decomposition. If each divisor is a spanning subgraph we call the decomposition a *factorisation* and the divisors *factors*. Graph decompositions and factorisations have received much attention, see for example [2, 24]. Of particular interest [22, 23] are decompositions where the divisors are pairwise isomorphic. These are known as *isomorphic decompositions*.

A *transitive decomposition* is a decomposition \mathcal{P} of a graph Γ together with a group of automorphisms G which preserves the partition and acts transitively on the set of divisors. We refer to (Γ, \mathcal{P}) as a G -transitive decomposition. This is a special class of isomorphic decompositions and a general theory has been outlined in [21]. Sibley [35] has described all G -transitive decompositions of the complete graph K_n where G is 2-transitive on vertices. This generalised the Cameron-Korchmaros classification in [7] of the G -transitive 1-factorisations of K_n (that is, the factors have valency 1) with G acting 2-transitively on vertices. Note that a subgroup of S_n is arc-transitive on K_n if and only if it is 2-transitive. Also all G -transitive decompositions of graphs with G inducing a rank three product action on vertices have been determined in [1]. A special class of transitive decompositions called *homogeneous factorisations*, are the G -transitive decompositions (Γ, \mathcal{P}) such that the kernel M of the action of G on \mathcal{P} is vertex-transitive. This implies that each divisor is a spanning subgraph and so \mathcal{P} is indeed a factorisation. Homogeneous factorisations were first introduced in [29] for complete graphs and extended to arbitrary graphs and digraphs in [20].

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Table 1: G -primitive decompositions of $J(n, k)$ for Theorem 1.1

Γ	G	Divisor	Comments
$J(6, 3)$	A_6 or $\langle A_6, (1, 2)\tau \rangle$	Petersen graph	Example 4.3(2)
$J(12, 4)$	M_{12}	$2J(6, 4)$	Construction 2.10 and 2.1
$J(12, 4)$	M_{12}	Σ	Construction 5.6
$J(24, 4)$	M_{24}	$J(8, 4)$	Construction 2.10
$J(23, 3)$	M_{23}	$J(7, 3)$	Construction 2.10
$J(11, 3)$	M_{11}	$J(5, 2)$	Construction 2.10
$J(11, 3)$	M_{11}	2 Petersen graphs	Construction 6.11
$J(11, 3)$	M_{11}	11 Petersen graphs	Construction 6.10(2)
$J(11, 3)$	M_{11}	Π	Construction 6.10(1)
$J(9, 3)$	$\text{P}\Gamma\text{L}(2, 8)$	$\text{PSL}(2, 8)$ -orbits	Construction 6.13(1)
$J(9, 3)$	$\text{P}\Gamma\text{L}(2, 8)$	Heawood graph	Construction 6.13(4)
$J(22, 2)$	M_{22} or $\text{Aut}(M_{22})$	$J(6, 2)$	Construction 2.10
$J(2^d, 2)$, $d \geq 3$	$\text{AGL}(d, 2)$	$2^{d-2}K_{2,2,2}$	Construction 2.10 and 2.1
$J(16, 2)$	$C_2^4 \rtimes A_7$	$4K_{2,2,2}$	Construction 2.10 and 2.1
$J(q+1, 2)$	3-transitive subgroup of $\text{P}\Gamma\text{L}(2, q)$	$J(q_0+1, 2)$ $q = q_0^r$, r prime	Construction 2.10
$J(q+1, 2)$ $q \equiv 1 \pmod{4}$	3-transitive subgroup of $\text{P}\Gamma\text{L}(2, q)$	$\text{PSL}(2, q)$ -orbits	Construction 8.1

The *Johnson graph* $J(n, k)$ is the graph with vertices the k -element subsets of an n -set X , two sets being adjacent if they have $k-1$ points in common. Note that $J(n, 1) \cong K_n$ and $J(n, k) \cong J(n, n-k)$ so we always assume that $2 \leq k \leq \frac{n}{2}$. Note that $J(4, 2) \cong K_{2,2,2}$ while the complement of $J(5, 2)$ is the Petersen graph. All homogeneous factorisations of $J(n, k)$ were determined in [11, 12]. Examples only exist for $J(q+1, 2)$ for prime powers $q \equiv 1 \pmod{4}$, $J(q, 2)$ and $J(q+1, 3)$ for $q = 2^{r^f}$ with r an odd prime, and for $J(8, 3)$. However, examples of transitive decompositions exist for all values of n and k (see Construction 2.8). Constructions 2.8(1) and (2) were drawn to our attention by Michael Orrison. Both constructions were used in [27] to help determine maximal subgroups of symmetric groups while Construction 2.8(1) was used in [32] for the statistical analysis of unranked data.

In this paper we determine all G -transitive decompositions of the Johnson graphs subject to two conditions on G . The first is that G is arc-transitive while the second is that G acts primitively on the set of divisors of the decomposition. We call G -transitive decompositions for which G acts primitively on the set of divisors, *G -primitive decompositions*. We see in Lemma 2.2 that any G -transitive decomposition is the refinement of some G -primitive decomposition. By Theorem 3.4, a subgroup $G \leq S_n$ acts transitively on the set of arcs of $J(n, k)$ if and only if G is $(k+1)$ -transitive, or $(n, k) = (9, 3)$ and $G = \text{P}\Gamma\text{L}(2, 8)$. Using this, we analyse the appropriate groups to determine all primitive decompositions. In particular we prove the following theorem.

Theorem 1.1. *Let G be an arc-transitive group of automorphisms of $\Gamma = J(n, k)$ where $2 \leq k \leq n/2$. If (Γ, \mathcal{P}) is a G -primitive decomposition then one of the following holds:*

1. *the divisors are matchings or unions of cycles,*
2. *the divisors are unions of K_{n-k+1}, K_{k+1} or K_3 , or*
3. *(Γ, \mathcal{P}) is given by one of the rows of Table 1.*

The divisor graphs Σ and Π of Table 1 are investigated further in [13]. Construction 2.10 allows us to construct transitive decompositions of $J(n, k)$ with divisors isomorphic to $J(l, k)$ for any Steiner system $S(k+1, l, n)$ and this accounts for many of the examples in Table 1. Further constructions of transitive decompositions from Steiner systems are given in Section 2 and these have divisors isomorphic to unions of cliques or matchings.

2 General constructions

First we show that the study of transitive decompositions can be reduced to the study of primitive decompositions. We denote by $V\Gamma$, $E\Gamma$ and $A\Gamma$, the sets of vertices, edges and arcs respectively, of the graph Γ .

Construction 2.1. Let (Γ, \mathcal{P}) be a G -transitive decomposition and let \mathcal{B} be a system of imprimitivity for G on \mathcal{P} . For each $B \in \mathcal{B}$, let $Q_B = \cup_{P \in B} P$ and let $\mathcal{Q} = \{Q_B \mid B \in \mathcal{B}\}$. Then (Γ, \mathcal{Q}) is a G -transitive decomposition.

Lemma 2.2. Any G -transitive decomposition (Γ, \mathcal{P}) with $|\mathcal{P}|$ finite is the refinement of a G -primitive decomposition (Γ, \mathcal{Q}) .

Proof. If $G^{\mathcal{P}}$ is primitive then we are done. If not, let \mathcal{B} be a nontrivial system of imprimitivity for G on \mathcal{P} with maximal block size. Then $G^{\mathcal{B}}$ is primitive and \mathcal{P} is a refinement of the partition \mathcal{Q} yielded by Construction 2.1. Thus (Γ, \mathcal{Q}) is a G -primitive decomposition. ■

We have the following general construction of transitive decompositions.

Construction 2.3. Let Γ be a graph with an arc-transitive group G of automorphisms. Let e be an edge of Γ and suppose that there exists a subgroup H of G such that $G_e < H < G$. Let $P = e^H$ and $\mathcal{P} = \{P^g \mid g \in G\}$.

Lemma 2.4. Let (Γ, \mathcal{P}) be obtained as in Construction 2.3. Then (Γ, \mathcal{P}) is a G -transitive decomposition. Conversely, every G -transitive decomposition with G arc-transitive arises in such a manner. Moreover, if the subgroup H is maximal in G , then (Γ, \mathcal{P}) is a G -primitive decomposition.

Proof. Since G is arc-transitive and $G_e < H < G$, then \mathcal{P} is a partition of $E\Gamma$ which is preserved by G and such that $G^{\mathcal{P}}$ is transitive. Thus (Γ, \mathcal{P}) is a G -transitive decomposition. Conversely, let (Γ, \mathcal{P}) be a G -transitive decomposition such that G is arc-transitive. Let e be an edge of Γ and P the divisor containing e . Since \mathcal{P} is a system of imprimitivity for G on $E\Gamma$ it follows that for $H = G_P$ we have $G_e < H < G$ and $P = e^H$. Moreover, $\mathcal{P} = \{P^g \mid g \in G\}$ and so (Γ, \mathcal{P}) arises from Construction 2.3. The last statement follows from the fact that H is the stabiliser in G of the divisor P . ■

Remark 2.5. Lemma 2.4 implies that there are two possible ways to determine all G -transitive decompositions such that the divisor stabilisers are in a given conjugacy class H^G of subgroups of G . One is to fix an edge e and run over all subgroups conjugate to H which contain the stabiliser of e . Note that different conjugates may give different partitions. The second is to run over all edges whose stabiliser is contained in H . Again, different edges may give different partitions.

We say that two decompositions (Γ, \mathcal{P}_1) and (Γ, \mathcal{P}_2) are *isomorphic* if there exists $g \in \text{Aut}(\Gamma)$ such that $\mathcal{P}_1^g = \mathcal{P}_2$. If both are G -transitive decompositions, then they are *isomorphic G -transitive decompositions* if there is such an element $g \in N_{\text{Aut}(\Gamma)}(G)$. The following lemma gives us a condition for determining when different conjugates give the same decomposition.

Lemma 2.6. Let (Γ, \mathcal{P}_1) , (Γ, \mathcal{P}_2) be two G -transitive decompositions with G arc-transitive.

1. Let e be an edge of Γ and P_1, P_2 be the divisors of $\mathcal{P}_1, \mathcal{P}_2$ respectively that contain e . If there exists an automorphism $g \in N_{\text{Aut}(\Gamma)}(G)$ fixing e such that $G_{P_1}^g = G_{P_2}$ then (Γ, \mathcal{P}_1) and (Γ, \mathcal{P}_2) are isomorphic.
2. Let e_1, e_2 be two edges of Γ with divisors $P_1 = e_1^H$ and $P_2 = e_2^H$ of $\mathcal{P}_1, \mathcal{P}_2$ respectively. If there exists an automorphism $g \in N_{\text{Aut}(\Gamma)}(G)$ mapping e_1 onto e_2 such that $H^g = H$ then (Γ, \mathcal{P}_1) and (Γ, \mathcal{P}_2) are isomorphic.

Proof. 1. By Lemma 2.4, $P_1 = e^{G_{P_1}}$ and $P_2 = e^{G_{P_2}}$. Thus $P_2 = e^{g^{-1}G_{P_1}g} = e^{G_{P_1}g} = P_1^g$. Moreover, $\mathcal{P}_2 = P_2^G = (P_1^g)^G = (P_1^G)^g = \mathcal{P}_1^g$ and so (Γ, \mathcal{P}_1) and (Γ, \mathcal{P}_2) are isomorphic.

2. We have $P_2 = e_2^H = (e_1^g)^H = (e_1^H)^g = P_1^g$. Hence we get the same conclusion. ■

We also have the following useful lemma.

Lemma 2.7. *Let (Γ, \mathcal{P}) be a G -primitive decomposition, with H the stabiliser of a divisor P . If $L \leq G$ is such that $L \not\leq H$, L is arc-transitive on Γ and $L \cap H$ is maximal in L , then (Γ, \mathcal{P}) is a L -primitive decomposition.*

Proof. Since L is arc-transitive and contained in G , it follows that L acts transitively on \mathcal{P} . Moreover, since $H \cap L$ is the stabiliser in L of a part, it follows that L acts primitively on \mathcal{P} . ■

We now describe some general methods for constructing transitive decompositions of Johnson graphs.

Construction 2.8. Let X be an n -set.

1. For each $(k-1)$ -subset Y of X , let P_Y be the complete subgraph of $J(n, k)$ whose vertices are all the k -subsets containing Y . Then

$$\mathcal{P}_\cap = \{P_Y \mid Y \text{ is a } (k-1)\text{-subset of } X\}$$

is a decomposition of $J(n, k)$ with $\binom{n}{k-1}$ divisors, each isomorphic to K_{n-k+1} .

2. For each $(k+1)$ -subset W of X , let Q_W be the complete subgraph whose vertices are all the k -subsets contained in W . Then

$$\mathcal{P}_\cup = \{Q_W \mid W \text{ is a } (k+1)\text{-subset of } X\}$$

is a decomposition of $J(n, k)$ with $\binom{n}{k+1}$ divisors, each isomorphic to K_{k+1} .

3. For each $\{a, b\} \subseteq X$, let

$$M_{\{a,b\}} = \left\{ \{ \{a\} \cup Y, \{b\} \cup Y \} \mid Y \text{ a } (k-1)\text{-subset of } X \setminus \{a, b\} \right\}.$$

Then

$$\mathcal{P}_\ominus = \{M_{\{a,b\}} \mid \{a, b\} \subseteq X\}$$

is a decomposition of $J(n, k)$ with $\binom{n}{2}$ divisors, each of which is a matching with $\binom{n-2}{k-1}$ edges.

Given two sets A and B we denote the *symmetric difference* of A and B by $A \ominus B$.

Lemma 2.9. *Let $G \leq S_n$ such that $\Gamma = J(n, k)$ is G -arc-transitive. Let A and B be two adjacent vertices of Γ . Then $(\Gamma, \mathcal{P}_\cap)$, $(\Gamma, \mathcal{P}_\cup)$, $(\Gamma, \mathcal{P}_\ominus)$ are G -transitive decompositions. Moreover, if $G_{A \cap B}$, $G_{A \cup B}$, or $G_{A \ominus B}$ respectively is maximal in G , then the decomposition is G -primitive.*

Proof. Since $P_Y^g = P_{Y^g}$, $Q_W^g = Q_{W^g}$ and $M_{\{a,b\}}^g = M_{\{a,b\}^g}$, it follows that G preserves \mathcal{P}_\cap , \mathcal{P}_\cup and \mathcal{P}_\ominus . Since G is arc-transitive, all three decompositions are G -transitive. The divisor of \mathcal{P}_\cap , \mathcal{P}_\cup or \mathcal{P}_\ominus containing $\{A, B\}$ is $P_{A \cap B}$, $Q_{A \cup B}$ or $M_{A \ominus B}$ respectively, and the stabiliser of this divisor is $G_{A \cap B}$, $G_{A \cup B}$, or $G_{A \ominus B}$ respectively. The last assertion follows. ■

Another method for constructing transitive decompositions of $J(n, k)$ is to use Steiner systems with multiply transitive automorphism groups. A *Steiner system* $S(t, k, v) = (X, \mathcal{B})$ is a collection \mathcal{B} of k -subsets (called *blocks*) of a v -set X such that each t -subset of X is contained in a unique block.

Construction 2.10. Let $\mathcal{D} = (X, \mathcal{B})$ be an $S(k+1, l, n)$ Steiner system with automorphism group G such that G is transitive on \mathcal{B} . For each $Y \in \mathcal{B}$, let P_Y be the subgraph of $J(n, k)$ whose vertices are the k -subsets in Y and let $\mathcal{P} = \{P_Y \mid Y \in \mathcal{B}\}$.

Lemma 2.11. *The pair $(J(n, k), \mathcal{P})$ yielded by Construction 2.10 is a G -transitive decomposition with divisors isomorphic to $J(l, k)$. Moreover, the decomposition is G -primitive if and only if the stabiliser of a block of \mathcal{D} is maximal in G .*

Proof. Let $\{A, B\}$ be an edge of $J(n, k)$. Then $A \cup B$ has size $k+1$ and so is contained in a unique block Y of \mathcal{D} , and hence $\{A, B\}$ is contained in a unique part P_Y of \mathcal{P} . Thus $(J(n, k), \mathcal{P})$ is a decomposition. Since G is transitive on \mathcal{B} the pair $(J(n, k), \mathcal{P})$ is G -transitive. Moreover, each P_Y consists of all k -subsets of the l -set Y and so is isomorphic to $J(l, k)$. Since the stabiliser in G of P_Y is G_Y , the last statement follows. ■

Construction 2.12. Let $\mathcal{D} = (X, \mathcal{B})$ be an $S(k+1, l, n)$ Steiner system with automorphism group G . Let $i = l - k - 1$ and suppose that G is i -transitive on X . For each i -subset Y of X let

$$P_Y = \{\{A, B\} \mid |A| = |B| = k, |A \cap B| = k - 1 \text{ and } A \cup B \cup Y \in \mathcal{B}\}.$$

Define

$$\mathcal{P} = \{P_Y \mid Y \text{ an } i\text{-subset of } X\}.$$

Lemma 2.13. *The pair $(J(n, k), \mathcal{P})$ yielded by Construction 2.12 is a G -transitive decomposition with divisors isomorphic to mK_{k+1} , where m is the number of blocks of \mathcal{D} containing an i -set. Moreover, the decomposition is G -primitive if and only if the stabiliser of an i -set is maximal in G .*

Proof. Let $\{A, B\}$ be an edge of $J(n, k)$. Then $A \cup B$ is contained in a unique block W of \mathcal{D} and the unique part of \mathcal{P} containing $\{A, B\}$ is P_Y where $Y = W \setminus (A \cup B)$. Each block containing Y contributes a copy of $J(k+1, k) \cong K_{k+1}$ to P_Y , and since each $(k+1)$ -subset is in a unique block, no two blocks containing Y share a vertex of P_Y . Hence the m copies of K_{k+1} in P_Y , are pairwise vertex-disjoint, that is $P_Y \cong mK_{k+1}$. Since G is i -transitive, it follows that $(J(n, k), \mathcal{P})$ is a G -transitive decomposition. Since the stabiliser in G of P_Y is G_Y , the last statement follows. \blacksquare

Construction 2.14. Let $\mathcal{D} = (X, \mathcal{B})$ be an $S(k+1, k+2, n)$ Steiner system with automorphism group G such that G acts 3-transitively on X . For each 3-subset Y of X , let

$$P_Y = \left\{ \{Z \cup \{u\}, Z \cup \{v\}\} \mid |Z| = k - 1, Z \cup Y \in \mathcal{B}, u, v \in Y \right\}$$

and let $\mathcal{P} = \{P_Y \mid Y \text{ a 3-subset of } X\}$.

Lemma 2.15. *The pair $(J(n, k), \mathcal{P})$ yielded by Construction 2.14 is a G -transitive decomposition with divisors isomorphic to mK_3 , where m is the number of blocks of \mathcal{D} containing a given 3-set. Moreover, the decomposition is G -primitive if and only if the stabiliser of a 3-subset is maximal in G .*

Proof. Let $\{A, B\}$ be an edge of $J(n, k)$. Then $A \cup B$ is contained in a unique block W of \mathcal{D} and the unique part of \mathcal{P} containing $\{A, B\}$ is P_Y where $Y = W \setminus (A \cup B)$. Each block containing Y contributes a copy of K_3 to P_Y , and since each $(k+1)$ -subset is in a unique block, no two blocks containing Y share a vertex of P_Y . Hence the m copies of K_3 in P_Y are pairwise vertex-disjoint, that is, $P_Y \cong mK_3$. Since G is 3-transitive, it follows that $(J(n, k), \mathcal{P})$ is a G -transitive decomposition. Since the stabiliser in G of P_Y is G_Y , the last statement follows. \blacksquare

Construction 2.16. Let $\mathcal{D} = (X, \mathcal{B})$ be an $S(k+1, k+2, n)$ Steiner system with k -transitive automorphism group G . For each k -subset Y of X let

$$P_Y = \left\{ \{ \{u\} \cup Z, \{v\} \cup Z \} \mid Y \cup \{u, v\} \in \mathcal{B}, Z \subset Y, |Z| = k - 1 \right\}$$

and let $\mathcal{P} = \{P_Y \mid Y \text{ a } k\text{-subset of } X\}$.

Lemma 2.17. *The pair $(J(n, k), \mathcal{P})$ yielded by Construction 2.16 is a G -transitive decomposition with divisors isomorphic to mkK_2 , where m is the number of blocks of \mathcal{D} containing a given k -set. Moreover, the decomposition is G -primitive if and only if the stabiliser of a k -subset is maximal in G .*

Proof. Let $\{A, B\}$ be an edge of $J(n, k)$. Then $A \cup B$ is contained in a unique block W of \mathcal{D} and the unique part of \mathcal{P} containing $\{A, B\}$ is P_Y where $Y = W \setminus (A \cup B)$. Each block containing Y contributes a copy of kK_2 to P_Y , and since each $(k+1)$ -subset is in a unique block, no two blocks containing Y share a vertex of P_Y . Hence the m copies of kK_2 in P_Y , are pairwise vertex-disjoint, that is $P_Y \cong mkK_2$. Since G is k -transitive, it follows that $(J(n, k), \mathcal{P})$ is a G -transitive decomposition. Since the stabiliser in G of P_Y is G_Y , the last statement follows. \blacksquare

We end this section with a standard construction of arc-transitive graphs.

Let G be a group with corefree subgroup H and let $g \in G$ such that $g^2 \in H$ and $g \notin N_G(H)$. Define the graph $\Gamma = \text{Cos}(G, H, HgH)$ with vertex set the set of right cosets of H in G and Hx adjacent to Hy if and only if $xy^{-1} \in HgH$. Then G acts faithfully and arc-transitively on Γ by right multiplication. We have the following lemma, see for example [17].

Lemma 2.18. *Let Γ be a G -arc-transitive graph with adjacent vertices v and w . Let $H = G_v$, and let $g \in G$ interchange v and w . Then $\Gamma \cong \text{Cos}(G, H, HgH)$. The connected component of Γ containing v consists of all cosets of H contained in $\langle H, g \rangle$. In particular, Γ is connected if and only if $\langle H, g \rangle = G$.*

3 Groups

In this section, we determine the groups G such that $J(n, k)$ is G -vertex-transitive and G -arc-transitive.

Theorem 3.1. [4, Theorem 9.1.2] *Let n, k be positive integers and let $\Gamma = J(n, k)$. If $n > 2k$ then $\text{Aut}(\Gamma) = S_n$ with the action induced from the action of S_n on X . For $n = 2k \geq 4$, $\text{Aut}(\Gamma) = S_n \times S_2 = \langle S_n, \tau \rangle$ where τ acts on $V\Gamma$ by complementation in X .*

Given a subset A of X we denote the complement of A in X by \bar{A} . Also, if $|X| = n$ and $|A| = k$ then $\Gamma(A)$ denotes the set of neighbours of A in the graph $J(n, k)$, that is, vertices B such that $\{A, B\}$ is an edge.

Lemma 3.2. [11, Proposition 3.2] *Let $\Gamma = J(n, k)$ and $G \leq S_n$. The graph Γ is G -arc-transitive if and only if G is k -homogeneous on X and, for a k -subset A , G_A is transitive on $A \times \bar{A}$.*

Proof. Note that G is arc-transitive if and only if G is vertex-transitive and G_A is transitive on $\Gamma(A)$. By definition, Γ is G -vertex-transitive if and only if G is k -homogeneous on X . Moreover, G_A is transitive on $\Gamma(A)$ if and only if G_A is independently transitive on the set of $(k-1)$ -subsets of A and on \bar{A} , that is, if and only if G_A is transitive on $A \times \bar{A}$. ■

Corollary 3.3. *If $G \leq S_n$ is $(k+1)$ -transitive, then Γ is G -arc-transitive. If Γ is G -arc-transitive and $G \leq S_n$, then G is k - and $(k+1)$ -homogeneous.*

Theorem 3.4. *Let $n \geq 2k \geq 4$ and $G \leq S_n$. The graph $\Gamma = J(n, k)$ is G -arc-transitive if and only if G is $(k+1)$ -transitive on X or $k = 3$, $n = 9$, and $G = \text{P}\Gamma\text{L}(2, 8)$.*

Proof. If G is $(k+1)$ -transitive, then by Corollary 3.3, Γ is G -arc-transitive. If $k = 3$, $n = 9$, and $G = \text{P}\Gamma\text{L}(2, 8)$, then it is easy to check that G is arc-transitive.

Suppose now that Γ is G -arc-transitive. By Corollary 3.3, G is k - and $(k+1)$ -homogeneous on X . If G is not $(k+1)$ -transitive, then, by [28, 31] either $2k \leq n \leq 2k+1$, or $2 \leq k \leq 3$ and G is one of a small number of groups.

Suppose first that $k = 2$. (This is an improvement on the proof of [11, Proposition 3.3].) Since G is 3-homogeneous, it is transitive on X . For $A = \{a, b\}$, Lemma 3.2 implies that G_A is transitive on $A \times \bar{A}$. Therefore using elements of G_A we can map (a, c) onto (a, d) for any $c, d \in \bar{A}$, and so $G_{a,b}$ is transitive on \bar{A} . Similarly, $G_{a,c}$ is transitive on $\bar{\{a, c\}}$ for any $c \in \{a, b\}$. Hence G_a is transitive on $\{a\}$ and so G is 3-transitive on X .

Next suppose that $k = 3$. If G is not 4-transitive then either $n = 6, 7$, or by [28], G is one of $\text{PGL}(2, 8)$, $\text{P}\Gamma\text{L}(2, 8)$ (with $n = 9$), or $\text{P}\Gamma\text{L}(2, 32)$ (with $n = 33$). Let $A = \{a, b, c\}$ and suppose that $G \neq \text{P}\Gamma\text{L}(2, 8)$.

Suppose first that $G = \text{PGL}(2, 8)$. Then $G_A \cong S_3$ and $G_{A,a} = C_2$. Hence G does not satisfy the arc-transitivity condition given in Lemma 3.2. Next suppose that $G = \text{P}\Gamma\text{L}(2, 32)$. Then $|G_{A,a}| = 10$ and so again Lemma 3.2 implies that G is not arc-transitive.

If $n = 6$, the only 3-homogeneous and 4-homogeneous group which is not 4-transitive is $\text{PGL}(2, 5)$. However, this does not satisfy the condition in Lemma 3.2 for arc-transitivity. There are no 3-homogeneous and 4-homogeneous groups of degree 7 which are not 4-transitive.

Next suppose that $k = 4$. If G is not 5-transitive, then $n = 8$ or 9. Since G is 4-homogeneous and 5-homogeneous, either G is 4-transitive, or G is one of $\text{PGL}(2, 8)$, $\text{P}\Gamma\text{L}(2, 8)$. However, these two groups are not arc-transitive as the stabiliser of a 4-subset A also stabilises a point in \bar{A} . The only 4-transitive groups of degree n are A_n and S_n and they are also 5-transitive.

If $k = 5$ and G is not 6-transitive, then $n = 10$ or 11. Since G is 5-homogeneous it is 5-transitive and so G contains A_n . Thus G is also 6-transitive. Finally, let $k \geq 6$. Since G is k -homogeneous it is k -transitive. The only k -transitive groups for $k \geq 6$ are A_n and S_n , which are also $(k+1)$ -transitive. ■

We need a couple of results for the case $n = 2k$.

Theorem 3.5. *Let $\Gamma = J(2k, k)$ and suppose that $G \leq \text{Aut}(\Gamma) = S_{2k} \times \langle \tau \rangle$ and Γ is G -arc-transitive. Then either $G \cap S_{2k}$ is arc-transitive on Γ , or $k = 2$, $G = \langle A_4, (1, 2)\tau \rangle$ and $G \cap S_4 = A_4$ has two orbits on arcs.*

Proof. Let $\hat{G} = G \cap S_{2k}$. If $\hat{G} = G$, we are done. Hence we can assume \hat{G} is an index 2 subgroup of G . The graph Γ is connected and is not bipartite, as it contains 3-cycles. It follows that \hat{G} cannot have two orbits on vertices and so \hat{G} is vertex-transitive.

Suppose that \hat{G} is not arc-transitive, and hence has two orbits of equal size on $A\Gamma$. Let $(A, B) \in A\Gamma$. Then $\hat{G}_{(A,B)} \leq G_{(A,B)}$ and $|G_A : G_{(A,B)}| = |\Gamma(A)| = k^2 = 2|\hat{G}_A : \hat{G}_{(A,B)}| = |G_A : \hat{G}_{(A,B)}|$. Hence $\hat{G}_{(A,B)} = G_{(A,B)}$ and k is even.

Suppose first that $k \geq 6$. Since \hat{G} is transitive on $V\Gamma$, \hat{G} is k -homogeneous and therefore also k -transitive. Hence $A_{2k} \leq \hat{G}$, and so \hat{G} is $(k+1)$ -transitive. It follows from Theorem 3.4 that \hat{G} is transitive on $A\Gamma$, which is a contradiction. Thus $k = 2$ or 4 .

If $k = 4$, then \hat{G} is k -homogeneous. The only 4-homogeneous groups of degree 8 contain A_8 , and so are also 5-transitive. By Theorem 3.4, \hat{G} is transitive on $A\Gamma$ in this case, and so $k = 2$.

Since \hat{G} is transitive on $V\Gamma$ and $(n, k) = (4, 2)$ we have that 6 divides $|\hat{G}|$. Since \hat{G} is 2-homogeneous it follows that $A_4 \leq \hat{G}$. Moreover, S_4 is arc-transitive and so $\hat{G} = A_4$. There are two groups $G \leq S_n \times S_2$ such that $\hat{G} = A_4$ and is of index 2 in G , namely $\langle A_4, \tau \rangle$ and $\langle A_4, (1, 2)\tau \rangle$. It is easy to check that the second group is transitive on $A\Gamma$ but not the first one. ■

We also have the following theorem about primitivity.

Theorem 3.6. *Let $\Gamma = J(2k, k)$ and $G \leq \text{Aut}(\Gamma) = S_{2k} \times \langle \tau \rangle$ such that both G and $G \cap S_{2k}$ are arc-transitive. Suppose that (Γ, \mathcal{P}) is a G -primitive decomposition. Then (Γ, \mathcal{P}) is also $(G \cap S_{2k})$ -primitive.*

Proof. Let $\hat{G} = G \cap S_{2k}$, let H be the stabiliser in G of a divisor and $\hat{H} = H \cap \hat{G} = H \cap S_{2k}$. We may suppose that $G \neq \hat{G}$. Moreover, as \hat{G} is arc-transitive it acts transitively on \mathcal{P} and so $\hat{G} \not\leq H$. Since H is maximal in G it follows that $|H : \hat{H}| = 2$.

Suppose first that $G = \hat{G} \times \langle \tau \rangle$. Now $H = \langle \hat{H}, \sigma\tau \rangle$ for some $\sigma \in \hat{G}$. Since $\hat{H} \triangleleft H$, the element $\sigma\tau$ (and hence also σ) normalises \hat{H} and \hat{H} contains $(\sigma\tau)^2 = \sigma^2$. This implies that $H \leq \langle \hat{H}, \sigma \rangle \times \langle \tau \rangle \leq G$. Since H is maximal in G , either $H = \langle \hat{H}, \sigma \rangle \times \langle \tau \rangle$ or $\langle \hat{H}, \sigma \rangle \times \langle \tau \rangle = G$. The first implies that $\sigma \in \hat{H}$ and hence $H = \hat{H} \times \langle \tau \rangle$. Thus \hat{H} is maximal in \hat{G} and so by Lemma 2.7, \mathcal{P} is \hat{G} -primitive. On the other hand, the second implies $\hat{G} = \langle \hat{H}, \sigma \rangle$. Since $\sigma^2 \in \hat{H}$, we have $|\mathcal{P}| = |\hat{G} : \hat{H}| = 2$ and so again \hat{G} is primitive on \mathcal{P} .

Suppose now that $G = \langle \hat{G}, \sigma\tau \rangle$ for some $\sigma \in S_{2k} \setminus \{1\}$ and $\tau \notin G$. Then σ normalises \hat{G} and $\sigma^2 \in \hat{G}$. Also, as $\tau \notin G$, we have $\sigma \notin \hat{G}$ and in particular $\hat{G} \neq S_{2k}$. By Theorem 3.4 and the fact that $n = 2k$, the classification of $(k+1)$ -transitive groups (see for example [6, pp194–197]) implies that $\hat{G} = A_{2k}$ and $k \geq 3$. Let $\phi : S_{2k} \times \langle \tau \rangle \rightarrow S_{2k}$ be the projection of $\text{Aut}(\Gamma)$ onto S_{2k} . Then $\phi|_G$ is an isomorphism. Moreover, for an edge $\{A, B\}$ contained in the divisor stabilised by H , $\phi(G_{A,B}) = S_{k-1} \times S_{k-1}$. Since $k \geq 3$, there is a transposition in $\phi(G_{A,B})$ and so by [34, Theorem 13.1] and since $\phi(G_{A,B}) \subseteq \phi(H)$, $\phi(H)$ is not primitive. It follows that $\phi(H)$ is a maximal intransitive subgroup of S_{2k} or a maximal imprimitive subgroup of S_{2k} preserving a partition into at most 3 parts. Thus by [30] and since $\hat{H} = \phi(H) \cap A_{2k}$, it follows that \hat{H} is a maximal subgroup of $\hat{G} = A_{2k}$. Hence again \hat{G} is primitive on \mathcal{P} . ■

4 Alternating and symmetric groups

We have already seen the S_n -transitive decompositions $\mathcal{P}_\cap, \mathcal{P}_\cup$ and \mathcal{P}_\ominus . Since $n \geq 2k$ it follows that S_n always acts primitively on \mathcal{P}_\cap . Also, S_n acts primitively on \mathcal{P}_\cup if and only if $n \neq 2k + 2$. When $n = 2k + 2$, applying Construction 2.1 to \mathcal{P}_\cup we obtain an S_n -primitive decomposition with divisors isomorphic to $2K_{k+1}$. Finally S_n acts primitively on \mathcal{P}_\ominus if and only if $(n, k) \neq (4, 2)$. This justifies the first four lines of Table 2 below. We also have the following two examples.

Table 2: S_n -primitive decompositions of $J(n, k)$

\mathcal{P}	P	G_P	(n, k)
\mathcal{P}_\cap	K_{n-k+1}	$(k-1)$ -set stabiliser	
\mathcal{P}_\cup	K_{k+1}	$(k+1)$ -set stabiliser	$n \neq 2k+2$
\mathcal{P}_\ominus	$\binom{n-2}{k-1}K_2$	2-set stabiliser	$(n, k) \neq (4, 2)$
\mathcal{P}_\cup and Construction 2.1	$2K_{k+1}$	S_{k+1} wr S_2	$n = 2k+2$
Example 4.1(1)	C_4	D_8	$(n, k) = (4, 2)$
Example 4.1(2)	$6K_2$	S_2 wr S_3	$(n, k) = (6, 3)$

Example 4.1. 1. Let $G = S_4$, $H = \langle (1, 2, 3, 4), (1, 3) \rangle \cong D_8$, $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $P = \{A, B\}^H$ is the 4-cycle

$$\left\{ \{1, 2\}, \{2, 3\}, \{2, 3\}, \{3, 4\}, \{3, 4\}, \{1, 4\}, \{1, 4\}, \{1, 2\} \right\}.$$

Since $G_{\{A, B\}} = \langle (1, 3) \rangle$ we have $G_{\{A, B\}} < H < G$ and so by Lemma 2.4 $((J(4, 2), \mathcal{P}))$ is a G -primitive decomposition with $\mathcal{P} = \{P^g \mid g \in G\}$.

2. Let $G = S_6$ and H be the stabiliser in G of the partition

$$\{1, 4\}, \{2, 3\}, \{5, 6\}$$

of $\{1, \dots, 6\}$. Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. Then $P = \{A, B\}^H$ is the matching

$$\left\{ \{1, 2, 3\}, \{2, 3, 4\}, \{2, 5, 6\}, \{3, 5, 6\}, \{1, 4, 5\}, \{1, 4, 6\}, \right. \\ \left. \{1, 5, 6\}, \{4, 5, 6\}, \{2, 3, 5\}, \{2, 3, 6\}, \{1, 4, 2\}, \{1, 4, 3\} \right\}.$$

Since $G_{\{A, B\}} < H < G$ it follows from Lemma 2.4 that $((J(6, 3), \mathcal{P}))$ is a G -primitive decomposition with $\mathcal{P} = \{P^g \mid g \in G\}$.

We have now constructed all the S_n -primitive decompositions in Table 2. It remains to prove that these are the only ones.

Theorem 4.2. *If $(J(n, k), \mathcal{P})$ is an S_n -primitive decomposition with $n \geq 2k$ then \mathcal{P} is given by one of the rows of Table 2.*

Proof. Let $\Gamma = J(n, k)$, $X = \{1, \dots, n\}$, and let $A = \{1, 2, \dots, k\}$ and $B = \{2, \dots, k+1\}$ be adjacent vertices of Γ . Then $G_{\{A, B\}} = \text{Sym}(\{1, k+1\}) \times \text{Sym}(\{2, \dots, k\}) \times \text{Sym}(\{k+2, \dots, n\})$. By Lemma 2.4, to find all G -primitive decompositions of Γ , we need to determine all maximal subgroups H of G which contain $G_{\{A, B\}}$. Since $G_{\{A, B\}}$ contains a 2-cycle, [34, Theorem 13.1] implies that there are no proper primitive subgroups of G containing $G_{\{A, B\}}$. Hence H is either imprimitive or intransitive.

Suppose first that H is intransitive. Then H is a maximal intransitive subgroup and hence it has two orbits U, W on X and $H = \text{Sym}(U) \times \text{Sym}(W)$. Since $G_{\{A, B\}} \leq H$, the only possibilities for these two orbits are:

$$\begin{array}{lll} \{1, \dots, k+1\} & \{k+2, \dots, n\} & n \neq 2k+2 \\ \{1, k+1\} & X \setminus \{1, k+1\} & (n, k) \neq (4, 2) \\ \{2, \dots, k\} & \{1, k+1, k+2, \dots, n\} & \end{array}$$

When $H = \text{Sym}(\{1, \dots, k+1\}) \times \text{Sym}(\{k+2, \dots, n\}) = G_{A \cup B}$, we obtain the decomposition $(\Gamma, \mathcal{P}_\cup)$, while $H = \text{Sym}(\{1, k+1\}) \times \text{Sym}(X \setminus \{1, k+1\}) = G_{A \ominus B}$ yields the decomposition $(\Gamma, \mathcal{P}_\ominus)$. Finally, $H = \text{Sym}(\{2, \dots, k\}) \times \text{Sym}(\{1, k+1, k+2, \dots, n\}) = G_{A \cap B}$ gives us the decomposition $(\Gamma, \mathcal{P}_\cap)$.

If H is transitive but imprimitive, then the possible systems of imprimitivity are:

$$\begin{array}{ll} \{1, \dots, k+1\}, \{k+2, \dots, 2k+2\} & \text{when } n = 2k+2 \\ \{1, 4\}, \{2, 3\}, \{5, 6\} & \text{when } (n, k) = (6, 3) \\ \{1, 3\}, \{2, 4\} & \text{when } (n, k) = (4, 2) \end{array}$$

In the first case, $P = \{A, B\}^H$ is the union of two cliques each of size $k + 1$, and has as vertices all k -subsets of $\{1, \dots, k + 1\}$ and all k -subsets of $\{k + 2, \dots, 2k + 2\}$, that is we get the decomposition obtained from applying Construction 2.1 to \mathcal{P}_\cup . The last two cases give us the two decompositions from Example 4.1. \blacksquare

By Theorem 3.4, A_n is arc-transitive on $J(n, k)$ if and only if $n \geq 5$. Moreover, all the S_n -primitive decompositions in Table 2 are A_n -primitive decompositions. We have the following extra examples for alternating groups.

- Example 4.3.** 1. Let $(n, k) = (5, 2)$, $G = A_5$, $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $G_{\{A, B\}} = \langle (1, 3)(4, 5) \rangle$ and is contained in the maximal subgroup $H = \langle (1, 2, 3, 4, 5), (1, 3)(4, 5) \rangle \cong D_{10}$ of G . Letting $P = \{A, B\}^H$ and $\mathcal{P} = \{P^g \mid g \in G\}$, Lemma 2.4 implies that $(J(5, 2), \mathcal{P})$ is an A_5 -primitive decomposition. Since $H_A \cong C_2$ it follows that the divisors are cycles of length 5.
2. Let $(n, k) = (6, 3)$, $G = A_6$, $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. Then $G_{\{A, B\}} = \langle (2, 3)(5, 6), (1, 4)(5, 6) \rangle$ and is contained in the maximal subgroup $H = \langle (2, 3)(5, 6), (1, 4, 5)(2, 3, 6) \rangle \cong \text{PSL}(2, 5)$ of G . Letting $P = \{A, B\}^H$ and $\mathcal{P} = \{P^g \mid g \in G\}$, Lemma 2.4 implies that $(J(6, 3), \mathcal{P})$ is an A_6 -primitive decomposition. Now P is a graph on 10 vertices with valency 3 admitting an arc-transitive action of $H \cong A_5$. Hence P is the Petersen graph.

Lemma 4.4. *Let \mathcal{P} be the decomposition of $J(6, 3)$ given by Example 4.3(2). Then \mathcal{P} is G -primitive if and only if $G = A_6$ or $\langle A_6, (1, 2)\tau \rangle$ where τ is the complementation map as in Theorem 3.1.*

Proof. As in the example, we take $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $P = \{A, B\}^H$ for $H = \langle (2, 3)(5, 6), (1, 4, 5)(2, 3, 6) \rangle \cong A_5$.

If $G \leq S_6$, by Theorem 3.4, G must be 4-transitive, so $A_6 \leq G$. We have seen above that \mathcal{P} is A_6 -primitive. However, S_6 does not preserve the partition \mathcal{P} of Example 4.3(2), since $(1, 4)$ preserves $\{A, B\}$ but not P . So assume $G \not\leq S_6$. By Theorems 3.5 and 3.6, \mathcal{P} is a $(G \cap S_6)$ -primitive decomposition. Thus $G \cap S_6 = A_6$ and so $G = G_1 = \langle A_6, \tau \rangle$ or $G = G_2 = \langle A_6, (1, 2)\tau \rangle$. Thus $|G| = 2|A_6|$ and so $|G_P : H| = 2$. Then as $G_{\{A, B\}} \leq G_P$ it follows that $G_{\{A, B\}}$ normalises H . However, $(2, 5)(3, 6)\tau \in (G_1)_{\{A, B\}}$ and does not normalise H , so $G \neq G_1$. Now $(G_2)_{\{A, B\}} = \langle (1, 4)(2, 5)(3, 6)\tau, H_{\{A, B\}} \rangle$ does normalise H and so fixes P . Thus $\langle H, (1, 4)(2, 5)(3, 6)\tau \rangle = (G_2)_P \cong S_5$ which is a maximal subgroup of $G_2 \cong S_6$. Hence \mathcal{P} is a G_2 -primitive decomposition. \blacksquare

We now show that Example 4.3 yields the only A_n -primitive decompositions which are not S_n -primitive.

Theorem 4.5. *Let $(J(n, k), \mathcal{P})$ be an A_n -primitive decomposition such that A_n is arc-transitive and $n \geq 2k$. Then \mathcal{P} is either an S_n -primitive decomposition, or $(n, k) = (5, 2)$ or $(6, 3)$ and \mathcal{P} is isomorphic to a decomposition given in Example 4.3.*

Proof. Let $\Gamma = J(n, k)$. Since $G = A_n$ is arc-transitive it follows from Theorem 3.4 that $n \geq 5$. Let $X = \{1, \dots, n\}$, $A = \{1, \dots, k\}$ and $B = \{2, \dots, k + 1\}$. Then

$$G_{\{A, B\}} = (\text{Sym}(\{1, k + 1\}) \times \text{Sym}(\{2, \dots, k\}) \times \text{Sym}(\{k + 2, \dots, n\})) \cap A_n.$$

We need to consider all maximal subgroups H such that $G_{\{A, B\}} < H < G$. For each such H , $P = \{A, B\}^H$ is the edge-set of a divisor of the G -primitive decomposition.

Suppose first that H is intransitive on X . Then $G_{\{A, B\}}$ has the same orbits on X as $(S_n)_{\{A, B\}}$ and so H is the intersection with A_n of one of the maximal intransitive subgroups which we considered in the S_n case in the proof of Theorem 4.2. Moreover, we obtain the decompositions in rows 1–3 in Table 2, and so (Γ, \mathcal{P}) is S_n -primitive.

Next suppose that H is imprimitive on X . Since $G_{\{A, B\}}$ is primitive on both $A \cap B$ and $\overline{A \cup B}$, the only systems of imprimitivity preserved by $G_{\{A, B\}}$ are those discussed in the S_n case. Thus H is the intersection with A_n of one of the maximal imprimitive subgroups considered in the S_n case and we obtain the decompositions in rows 4 and 6 in Table 2. Thus (Γ, \mathcal{P}) is S_n -primitive.

Finally, suppose that H is primitive on X . If $k - 1 \geq 3$ or $n - k - 1 \geq 3$, the edge stabiliser $G_{\{A, B\}}$, and hence H , contains a 3-cycle. Hence by [34, Theorem 13.3], $H = A_n$, contradicting H being a proper subgroup. Note that if $k \geq 4$ then $k - 1 \geq 3$, and so (n, k) is one of $(5, 2)$ or $(6, 3)$.

If $(n, k) = (5, 2)$ then $G_{\{A, B\}} = \langle (1, 3)(4, 5) \rangle$ and $H \cong D_{10}$. Since A_5 contains 15 involutions, D_{10} contains 5 involutions and there are six subgroups D_{10} in A_5 , it follows that there are 2 choices for H and these are

$$\begin{aligned} H_1 &= \langle (1, 2, 3, 4, 5), (1, 3)(4, 5) \rangle \\ H_2 &= \langle (1, 4, 5, 3, 2), (1, 3)(4, 5) \rangle. \end{aligned}$$

Note that $H_2 = H_1^{(1,3)}$ and $(1, 3) \in (S_n)_{\{A, B\}}$ and so by Lemma 2.6 the two decompositions obtained are isomorphic. Moreover, H_1 is the stabiliser of the divisor containing $\{A, B\}$ in the decomposition in Example 4.3(1).

If $(n, k) = (6, 3)$ then $G_{\{A, B\}} = \langle (2, 3)(5, 6), (1, 4)(5, 6) \rangle$ and $H \cong \text{PSL}(2, 5)$. A computation using MAGMA [3] showed that there are two choices for H containing $G_{\{A, B\}}$ and these are:

$$\begin{aligned} H_1 &= \langle (2, 3)(5, 6), (1, 4, 5)(2, 3, 6) \rangle \\ H_2 &= \langle (2, 3)(5, 6), (1, 4, 5)(3, 2, 6) \rangle. \end{aligned}$$

Note that $H_2 = H_1^{(2,3)}$ and $(2, 3) \in (S_n)_{\{A, B\}}$ and so the two decompositions obtained are isomorphic. Moreover, H_1 is the stabiliser of the divisor containing $\{A, B\}$ in the decomposition in Example 4.3(2). \blacksquare

We now look at the case where $n = 2k$ and G is not a subgroup of S_n .

Example 4.6. Let $(n, k) = (4, 2)$ and $G = \langle A_4, (1, 2)\tau \rangle$. Let $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $G_{\{A, B\}} = \langle (2, 4)\tau \rangle$.

1. Let $H_1 = \langle (1, 2, 4), (1, 2)\tau \rangle$ and

$$P = \{A, B\}^{H_1} = \left\{ \{ \{1, 2\}, \{2, 3\} \}, \{ \{2, 4\}, \{3, 4\} \}, \{ \{1, 4\}, \{1, 3\} \} \right\}.$$

Since $G_{\{A, B\}} \leq H_1$, it follows from Lemma 2.4 that $(J(4, 2), P^G)$ is a G -primitive decomposition, with divisors isomorphic to $3K_2$.

2. Let $H_2 = \langle (1, 2)(3, 4), (1, 3)(2, 4), (1, 3)\tau \rangle$ and $P = \{A, B\}^{H_2} =$

$$\left\{ \{ \{1, 2\}, \{2, 3\} \}, \{ \{2, 3\}, \{3, 4\} \} \{ \{3, 4\}, \{1, 4\} \}, \{ \{1, 4\}, \{1, 2\} \} \right\}.$$

Since $G_{\{A, B\}} \leq H_2$, it follows from Lemma 2.4 that $(J(4, 2), P^G)$ is a G -primitive decomposition, with divisors isomorphic to C_4 . Notice that this decomposition is the one in Example 4.1(1) and so is also S_4 -primitive.

Theorem 4.7. Let $\Gamma = J(n, k)$ with $n = 2k$ and let $G \leq \text{Aut}(\Gamma) = S_n \times S_2$ such that G is not contained in S_n . Further, suppose that (Γ, \mathcal{P}) is a G -primitive decomposition which is not $(G \cap S_n)$ -primitive. Then $n = 4$ and \mathcal{P} is isomorphic to a decomposition given by Example 4.6.

Proof. By Theorems 3.5 and 3.6, it follows that $k = 2$ and $G = \langle A_4, (1, 2)\tau \rangle$, where τ is complementation in X . Let $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $G_{\{A, B\}} = \langle (2, 4)\tau \rangle$. It is not hard to see that the only maximal subgroups of G containing $G_{\{A, B\}}$ are the groups H_1 and H_2 from Example 4.6, and $H_3 = \langle (2, 3, 4), (2, 3)\tau \rangle$. The first two give the two decompositions from Example 4.6. Note that $(1, 3)$ stabilizes $\{A, B\}$ and normalises G , and $H_3 = H_1^{(1,3)}$. So by Lemma 2.6, this yields a decomposition isomorphic to the one in Example 4.6(1). \blacksquare

5 The case $k \geq 4$

By Theorem 3.4, if $k \geq 4$ then $G \leq S_n$ is arc-transitive on $J(n, k)$ if and only if G is $(k + 1)$ -transitive on the n -set X . Hence by the Classification of finite 2-transitive permutation groups, other than A_n or S_n , the only possibilities for (n, G) when $k \geq 4$ are $(12, M_{12})$ and $(24, M_{24})$ with $k = 4$.

First we state the following well known lemmas.

Lemma 5.1. *Let (X, \mathcal{B}) be the Witt design $S(5, 6, 12)$. Then \mathcal{B} contains 132 elements, called hexads. Each point of X is contained in 66 hexads, each 2-subset in 30 hexads, each 3-subset in 12 hexads, each 4-subset in 4 hexads, and each 5-subset in a unique hexad.*

Proof. The number of hexads is given in [10, p 31] and then the number of hexads containing a given i -subset is calculated by counting i -subset-hexad pairs in two different ways. ■

Lemma 5.2. [26, Lemma 2.11.7] *Suppose that (X, \mathcal{B}) is a Witt design $S(5, 6, 12)$ preserved by $G = M_{12}$ and let $h \in \mathcal{B}$ be a hexad. Then $G_h \cong S_6$ and the actions of G_h on h and $X \setminus h$ are the two inequivalent actions of S_6 on six points.*

Since the stabiliser of a 3-set or a 2-set is maximal in $G = M_{12}$, it follows from Lemma 2.9 that \mathcal{P}_\cap and \mathcal{P}_\ominus are G -primitive decompositions. Moreover, as G acts primitively on the point set X of the Witt design, Construction 2.12 yields a G -primitive decomposition of $J(12, 4)$. We also obtain a G -primitive decomposition from Construction 2.14 as G acts primitively on 3-subsets and one from Construction 2.16 as G acts primitively on 4-subsets. The G -transitive decomposition obtained from Construction 2.10 is not primitive as the stabiliser of a hexad is contained in the stabiliser of a pair of complementary hexads. However, applying Construction 2.1 we obtain a G -primitive decomposition with divisors isomorphic to $2J(6, 4)$.

Before giving several more constructions arising from the Witt design, we need the following definition and lemma.

Definition 5.3. A *linked three* in $S(5, 6, 12)$ is a set of four triads (or 3-sets) such that the union of any two is a hexad.

Lemma 5.4. *Let A, B be two triads whose union is a hexad. Then there exists a unique linked three containing both A and B .*

Proof. By Lemma 5.1, there are exactly 12 hexads containing A . If such a hexad contains at least two points of B , then it is $A \cup B$. Let $b \in B$. Then there are 4 hexads containing A and b , and so exactly 3 hexads meet $A \cup B$ in $A \cup \{b\}$. Therefore there are 9 hexads containing A and meeting $A \cup B$ in a 4-set. Hence only two hexads contain A and are disjoint from B . These yield two triads, C and D , forming hexads with A . By Lemma 5.2, the stabiliser of A and B is $S_3 \times S_3$ which acts transitively on the remaining 6 points. Hence C and D must be disjoint. Since the complement of a hexad is a hexad, C and D must form hexads with B too. It follows that $\{A, B, C, D\}$ is the unique linked three containing A and B . ■

Construction 5.5. Let (X, \mathcal{B}) be the Witt design $S(5, 6, 12)$ and let $G = M_{12}$.

1. Let T be a linked three as in Definition 5.3. Let

$$P_T = \left\{ \{ \{u\} \cup Y, \{v\} \cup Y \} \mid Y \in T, \{u, v\} \text{ contained in some triad of } T \setminus Y \right\}$$

and $\mathcal{P} = \{P_T \mid T \text{ is a linked three}\}$. Then $P_T \cong 12K_3$, with each triad contributing $3K_3$. If $\{A, B\}$ is an edge of $J(12, 4)$ then $A \cup B$ is contained in a unique hexad $A \cup B \cup \{x\}$ for some $x \in X$, and by Lemma 5.4, $\{A \cap B, \{x\} \cup (A \ominus B)\}$ is contained in a unique linked three T . For this T , P_T is the unique part of \mathcal{P} containing $\{A, B\}$. Since G acts transitively on the set of linked threes and the stabiliser of a linked three is maximal, $(J(12, 4), \mathcal{P})$ is a G -primitive decomposition.

Table 3: M_{12} -primitive decompositions of $J(12, 4)$

\mathcal{P}	P	G_P
\mathcal{P}_\cap	K_9	$M_9 \times S_3$
\mathcal{P}_\ominus	$\binom{10}{3}K_2$	$M_{10.2}$
Constructions 2.10 and 2.1	$2J(6, 4)$	$M_{10.2}$
Construction 2.12	$66K_5$	M_{11}
Construction 2.14	$12K_3$	$M_9 \times S_3$
Construction 2.16	$16K_2$	$M_8 \times S_4$
Construction 5.5(1)	$12K_3$	$M_9 \times S_3$
Construction 5.5(2)	$36K_2$	$M_9 \times S_3$
Construction 5.6	Σ	M_{11}

2. Let T be a linked three. A 4-set intersecting each triad of T in a single point and such that its union with any triad is a hexad is called *special* for T . For fixed triads T_1, T_2 of T and points $x_1 \in T_1, x_2 \in T_2$, these conditions imply that there is at most one special 4-set containing $\{x_1, x_2\}$ and existence of such a 4-set was confirmed by MAGMA [3]. Thus there are nine special 4-sets for T . Let

$$P_T = \left\{ \left\{ \{u, x, y, z\}, \{v, x, y, z\} \right\} \mid \{x, y, z, t\} \text{ sp. 4-set for } T, \{u, v, t\} \in T \right\}$$

and $\mathcal{P} = \{P_T \mid T \text{ is a linked three}\}$. Then $P_T \cong 36K_2$, with each special 4-set contributing $4K_2$. If $\{A, B\}$ is an edge of $J(12, 4)$ then $A \cup B$ is contained in a unique hexad $A \cup B \cup \{x\}$ for some $x \in X$, and there is a unique linked three T such that $(A \cap B) \cup \{x\}$ is special for T and $\{x\} \cup (A \ominus B)$ is a triad of T (a MAGMA [3] calculation). Thus P_T is the only part of \mathcal{P} containing $\{A, B\}$. Since G acts transitively on the set of linked threes and the stabiliser of a linked three is maximal, $(J(12, 4), \mathcal{P})$ is a G -primitive decomposition.

Construction 5.6. Let $G = M_{12} < S_{12}$ and let $H = M_{11}$ be a 3-transitive subgroup of G . Then H has an orbit of length 165 on 4-subsets and this orbit forms a $3 - (12, 4, 3)$ design. Let Σ be the subgraph of $J(12, 4)$ induced on the orbit of length 165. The graph Σ was studied in [13], where it is seen that Σ has valency 8, is H -arc-transitive and given an edge $\{A, B\}$ we have $H_{\{A, B\}} \cong S_2 \times S_3 = G_{\{A, B\}}$. Thus Lemma 2.4 and the fact that H is maximal in G , imply that $\mathcal{P} = \Sigma^G$ is a G -primitive decomposition of $J(12, 4)$.

We have now seen all the M_{12} -primitive decompositions listed in Table 3. It remains to prove that these are the only ones.

Proposition 5.7. *If $(J(12, 4), \mathcal{P})$ is an M_{12} -primitive decomposition then \mathcal{P} is given by one of the rows of Table 3.*

Proof. Let $\Gamma = J(12, 4)$ and $G = M_{12}$ acting on the point set X of the Witt-design $S(5, 6, 12)$. Take adjacent vertices $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4, 5\}$ and suppose that $h = \{1, 2, 3, 4, 5, 6\}$ is the unique hexad containing $A \cup B$. Then $G_{\{A, B\}} = G_{\{1, 5\}, \{2, 3, 4\}, \{6\}} \cong S_2 \times S_3$, by Lemma 5.2. Since transpositions in the action of G_h on h act as a product of three transpositions on $X \setminus h$, and 3-cycles on h act as a product of two 3-cycles on $X \setminus h$ it follows that $G_{1, 5, 6, \{2, 3, 4\}} \cong S_3$ acts regularly on $X \setminus h$, and so $G_{\{A, B\}}$ acts transitively on $X \setminus h$.

Let H be a maximal subgroup of G such that $G_{\{A, B\}} \leq H < G$. The maximal subgroups of G are given in [10, p 33]. The orbit lengths of $G_{\{A, B\}}$ imply that $G_{\{A, B\}}$ does not preserve a system of imprimitivity on X with blocks of size 2 or 4 and so $H \not\cong C_4^2 \times D_{12}, A_4 \times S_3$, or $C_2 \times S_5$. Moreover, $|H_6|$ is even and so $H \not\cong \text{PSL}(2, 11)$.

If H is intransitive then H is one of $G_{\{2, 3, 4, 6\}}, G_{\{2, 3, 4\}}, G_{\{1, 5, 6\}}, G_{\{1, 5\}}$ or G_6 . (Note that G_h is not maximal.) The first is the stabiliser of the divisor containing $\{A, B\}$ in the decomposition yielded by Construction 2.16. The second gives \mathcal{P}_\cap while the third is the stabiliser of the divisor of the decomposition yielded by Construction 2.14 containing $\{A, B\}$. If $H = G_{\{1, 5\}}$ then we obtain the decomposition \mathcal{P}_\ominus while if $H = G_6$ we obtain the decomposition yielded by Construction 2.12.

Table 4: M_{24} -primitive decompositions of $J(24, 4)$

\mathcal{P}	P	G_P
\mathcal{P}_\cap	K_{21}	$\text{P}\Gamma\text{L}(3, 4)$
\mathcal{P}_\ominus	$\binom{22}{3}K_2$	$M_{22}.2$
Construction 2.10	$J(8, 4)$	$C_2^4 \rtimes A_8$
Construction 2.12	$21K_5$	$\text{P}\Gamma\text{L}(3, 4)$

The only hexad pair fixed by $G_{\{A,B\}}$ is $\{h, X \setminus h\}$. Now G_h is the stabiliser of the divisor of the decomposition yielded by Construction 2.10 containing $G_{\{A,B\}}$. Such a divisor is isomorphic to $J(6, 4)$ and so $G_{\{h, X \setminus h\}}$ yields the decomposition with divisors isomorphic to $2J(6, 4)$ obtained after applying Construction 2.1.

A calculation using MAGMA [3] shows that there is only one transitive subgroup of G isomorphic to M_{11} which contains $G_{\{A,B\}}$ and this yields Construction 5.6.

By the list of maximal subgroups of G given in [10, p 33], the only case left to consider is H being the stabiliser of a linked three. If T is a linked three preserved by $G_{\{A,B\}}$ then $\{1, 5, 6\}$ is a triad of T and either $\{2, 3, 4\}$ is also a triad or 2, 3, and 4 lie in distinct triads. Since a linked three is uniquely determined by any two of its triads (Lemma 5.4), there is a unique linked three T containing $\{1, 5, 6\}$ and $\{2, 3, 4\}$. Then G_T is the stabiliser of the divisor of the decomposition yielded by Construction 5.5(1) containing $\{A, B\}$. If 2, 3 and 4 are in distinct blocks, a calculation using MAGMA [3] shows that there is a unique H containing $G_{\{A,B\}}$ and we obtain the decomposition in Construction 5.5(2). ■

We need the following well known lemma to deal with the case where $G = M_{24}$.

Lemma 5.8. [26, Lemma 2.10.1] *Let (X, \mathcal{B}) be the Witt design $S(5, 8, 24)$. Then \mathcal{B} contains 759 elements, called octads. Each point of X is contained in 253 octads, each 2-subset in 77 octads, each 3-subset in 21 octads, each 4-subset in 5 octads, and each 5-subset in a unique octad. Moreover, the stabiliser of an octad in M_{24} is $C_2^4 \rtimes A_8$ where C_2^4 acts trivially on the octad and transitively on its complement.*

Proof. Then number of octads comes from [26, Lemma 2.10.1] and then the numbers of octads containing a given i -subset follows from basic counting. The statement about the stabiliser of an octad also comes from [26, Lemma 2.10.1]. ■

Since the stabilisers of a 3-set, of a 2-set, and of an octad are maximal in G , applying Constructions 2.8, 2.10 and 2.12, we get the list of M_{24} -primitive decompositions in Table 4.

Proposition 5.9. *If $(J(24, 4), \mathcal{P})$ is an M_{24} -primitive decomposition then \mathcal{P} is given by one of the rows in Table 4.*

Proof. Let $\Gamma = J(24, 4)$ and $G = M_{24}$ acting on the point-set X of the Witt-design $S(5, 8, 24)$. Take adjacent vertices $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4, 5\}$ and suppose that $\Delta = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is the unique octad containing $A \cup B$. Then looking at the stabiliser of an octad given in Lemma 5.8, we see that $G_{\{A,B\}} = G_{\{1,5\},\{2,3,4\},\{6,7,8\}} = C_2^4 \rtimes ((S_2 \times S_3^2) \cap A_8)$ with orbits in Δ of lengths 2, 3, 3. Since $G_{\{A,B\}}$ contains the pointwise stabiliser of the octad Δ , which by Lemma 5.8 acts regularly on $X \setminus \Delta$, it follows that $G_{\{A,B\}}$ is transitive on $X \setminus \Delta$.

Let H be a maximal subgroup of G such that $G_{\{A,B\}} \leq H < G$. The maximal subgroups of G are given in [10, p 96], and comparing orders we see that $H \not\cong \text{PSL}(2, 7)$ or $\text{PSL}(2, 23)$. Since $G_{\{A,B\}}$ has an orbit of length 16 and an orbit of length 3 in X , it cannot fix a pair of dodecads. Similarly, if H fixed a trio of disjoint octads, one of the three octads would be Δ and $G_{\{A,B\}}$ would interchange the other 2. However, all index 2 subgroups of $G_{\{A,B\}}$ are transitive on $X \setminus \Delta$ (a MAGMA calculation [3]) and so H does not fix a trio of disjoint octads. Suppose next that H fixes a sextet, that is, 6 sets of size 4 such that the union of any two is an octad. Then the $G_{\{A,B\}}$ -orbit $X \setminus \Delta$ is the union of four of these sets. However, the remaining $G_{\{A,B\}}$ -orbit lengths are incompatible with H fixing a partition of $\{1, \dots, 8\}$ into two sets of size 4. Thus the list of maximal subgroups of G in [10, p 96] implies that H is intransitive on X , and so $H = G_{\{1,5\}}, G_{\{2,3,4\}}, G_{\{6,7,8\}}$, or $G_{\{1,2,3,4,5,6,7,8\}}$. By Lemma

2.9, the first gives the decomposition \mathcal{P}_\ominus while the second gives \mathcal{P}_\cap . The third is the stabiliser of the divisor of the decomposition yielded by Construction 2.12 containing $\{A, B\}$ while the fourth yields the decomposition obtained from Construction 2.10. \blacksquare

6 The case $k = 3$

By Theorem 3.4, $G \leq S_n$ is arc-transitive on $J(n, 3)$ if and only if G is 4-transitive or $G = \text{P}\Gamma\text{L}(2, 8)$ and $n = 9$. Thus other than A_n or S_n the only possibilities for (n, G) are $(11, M_{11})$, $(12, M_{12})$, $(23, M_{23})$, $(24, M_{24})$ and $(9, \text{P}\Gamma\text{L}(2, 8))$.

Since the stabiliser of a 2-subset is maximal in M_{24} , it follows that \mathcal{P}_\cap and \mathcal{P}_\ominus are M_{24} -primitive decompositions with divisors K_{22} and $\binom{22}{2}K_2$ respectively. We also have a construction involving sextets.

Construction 6.1. Let S be a sextet, that is, a set of six 4-subsets such that the union of any two is an octad, and define $P_S = \{\{A, B\} \mid A \cup B \in S\}$ and $\mathcal{P} = \{P_S \mid S \text{ a sextet}\}$. Then $P_S \cong 6J(4, 3) \cong 6K_4$ with one copy of K_4 for each 4-set in S . Let $\{A, B\}$ be an edge of $J(24, 3)$. By [26, Lemma 2.3.3], $A \cup B$ is a member of a unique sextet S and so P_S is the only part of \mathcal{P} containing $\{A, B\}$. Since G acts primitively on the set of sextets, it follows that $(J(24, 3), \mathcal{P})$ is an M_{24} -primitive decomposition.

Proposition 6.2. *If $(J(24, 3), \mathcal{P})$ is an M_{24} -primitive decomposition then either $\mathcal{P} = \mathcal{P}_\ominus$ or \mathcal{P}_\cap , or \mathcal{P} arises from Construction 6.1.*

Proof. Let $\Gamma = J(24, 3)$ and $G = M_{24}$ acting on the point set X of the Witt-design $S(5, 8, 24)$. Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ be adjacent vertices in Γ . Then $G_{\{A, B\}} = G_{\{1, 4\}, \{2, 3\}}$ which is the stabiliser in $\text{Aut}(M_{22})$ of a 2-subset and so by [10, p 39], $G_{\{A, B\}} \cong 2^5 \rtimes S_5$. Since G is 5-transitive on X , $G_{\{A, B\}}$ is transitive on $X \setminus \{1, 2, 3, 4\}$.

Let H be a maximal subgroup of G such that $G_{\{A, B\}} \leq H < G$. The maximal subgroups of G can be found in [10]. Comparing orders we see that $H \not\cong \text{PSL}(2, 7)$, $\text{PSL}(2, 23)$, or the stabiliser of a trio of distinct octads. Now $G_{\{A, B\}}$ contains $G_{1,2,3,4}$ which is transitive on the remaining 20 points. Thus $G_{1,2,3,4}$ does not fix a pair of dodecads and so neither does H . Hence by the list of maximal subgroups of G in [10, p 96], either H is intransitive, or fixes a sextet. If H is intransitive, then $H = G_{\{1, 4\}}$ or $G_{\{2, 3\}}$. By Lemma 2.9, the first gives \mathcal{P}_\ominus while the second gives \mathcal{P}_\cap .

Suppose then that H fixes a sextet. The orbit lengths of $G_{\{A, B\}}$ imply that $\{1, 2, 3, 4\}$ is one of the blocks of the sextet. By [26, Lemma 2.3.3], $\{1, 2, 3, 4\}$ is contained in a unique sextet S . Thus $H = G_S$ and is the stabiliser in G of the divisor of the decomposition obtained from Construction 6.1 containing $\{A, B\}$. \blacksquare

Before dealing with $G = M_{23}$ we need the following well known result which follows from Lemma 5.8.

Lemma 6.3. *Let (X, \mathcal{B}) be the Witt design $S(4, 7, 23)$. Then \mathcal{B} contains 253 elements, called heptads. Each point of X is contained in 77 heptads, each 2-subset in 21 heptads, each 3-subset in 5 heptads, and each 4-subset in a unique heptad. Moreover, the stabiliser of a heptad is $C_2^4 \rtimes A_7$ with the pointwise stabiliser of the heptad being C_2^4 which acts regularly on the 16 points not in the heptad.*

Proof. Since (X, \mathcal{B}) is derived from the set of all blocks of the Witt design $S(5, 8, 24)$ containing a given point, this follows from Lemma 5.8. \blacksquare

Using the Witt design $S(4, 7, 23)$ and the fact that the stabiliser of a 2-set is maximal in M_{23} we get the M_{23} -primitive decompositions in Table 5. These are in fact all such decompositions.

Proposition 6.4. *If $(J(23, 3), \mathcal{P})$ is an M_{23} -primitive decomposition then \mathcal{P} is as in one of the lines of Table 5.*

Proof. Let $\Gamma = J(23, 3)$ and $G = M_{23}$ acting on the point-set X of the Witt-design $S(4, 7, 23)$. Take adjacent vertices $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. By Lemma 6.3, $\{1, 2, 3, 4\}$ is contained in a unique heptad, $h = \{1, 2, 3, 4, 5, 6, 7\}$ say, and so $G_{\{A, B\}}$ fixes h . Since the stabiliser of a heptad is isomorphic to $C_2^4 \rtimes A_7$ (Lemma 6.3), it follows that $G_{\{A, B\}}$ has order 192 and has orbits $\{1, 4\}$, $\{2, 3\}$, $\{5, 6, 7\}$ and $X \setminus h$.

Table 5: M_{23} -primitive decompositions of $J(23, 3)$

\mathcal{P}	P	G_P
\mathcal{P}_\cap	K_{21}	$\text{PSL}(3, 4)$
\mathcal{P}_\ominus	$\binom{21}{2}K_2$	$\text{PSL}(3, 4)$
Construction 2.10	$J(7, 3)$	$C_2^4 \rtimes A_7$
Construction 2.12	$5K_4$	$C_2^4 \rtimes (C_3 \times A_5) \rtimes C_2$

Let H be a maximal subgroup of G such that $G_{\{A,B\}} \leq H < G$. The maximal subgroups of G can be found in [10]. By comparing orders, $H \not\cong C_{23} \rtimes C_{11}$ and so H is intransitive. Thus $H = G_{\{1,4\}}, G_{\{2,3\}}, G_{\{5,6,7\}}$ or G_h . By Lemma 2.9, the first two give the decompositions \mathcal{P}_\ominus and \mathcal{P}_\cap respectively. Also $G_{\{5,6,7\}}$ is the stabiliser of the divisor of the decomposition obtained from Construction 2.12 containing $\{A, B\}$ while G_h is the stabiliser of the divisor of the decomposition yielded by Construction 2.10. ■

Since 4-set stabilisers and 2-set stabilisers are maximal in M_{12} , it follows from Lemma 2.9 that $\mathcal{P}_\cup, \mathcal{P}_\cap$ and \mathcal{P}_\ominus are M_{12} -primitive decompositions with divisors isomorphic to K_4, K_{10} and $\binom{10}{2}K_2$ respectively. We also have the following construction.

Construction 6.5. Let (X, \mathcal{B}) be the Witt design $S(5, 6, 12)$. Let F be a *linked four*, that is a set of three mutually disjoint tetrads (sets of size 4) admitting a refinement into six duads (called duads of F) such that the union of any three duads coming from any two tetrads is a hexad. Let

$$P_F = \left\{ \{ \{x, u, v\}, \{y, u, v\} \} \mid \{x, y, u, v\} \in F, \{u, v\}, \{x, y\} \text{ are duads of } F \right\}$$

and let $\mathcal{P} = \{P_F \mid F \text{ a linked four}\}$. Then $P_F \cong 6K_2$ with one copy of $2K_2$ for each tetrad in F . Let $\{A, B\}$ be an edge of $J(12, 3)$. It turns out (MAGMA calculation [3]) there is exactly one linked four F having $A \cup B$ as a tetrad and $A \cap B$ as a duad of F , and so P_F is the only part of \mathcal{P} containing $\{A, B\}$. Since G acts primitively on the set of linked fours, it follows that $(J(12, 3), \mathcal{P})$ is an M_{12} -primitive decomposition.

Proposition 6.6. *If $(J(12, 3), \mathcal{P})$ is an M_{12} -primitive decomposition then $\mathcal{P} = \mathcal{P}_\cup, \mathcal{P}_\cap$ or \mathcal{P}_\ominus or \mathcal{P} is obtained from Construction 6.5.*

Proof. Let $\Gamma = J(12, 3)$ and $G = M_{12}$ acting on the point set X of the Witt-design $S(5, 6, 12)$. Take adjacent vertices $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. The stabiliser in G of a 4-set is $M_8 \times S_4$ such that the pointwise stabiliser M_8 of the 4-set acts regularly on the 8 remaining points. Hence $G_{\{A,B\}} = G_{\{1,4\}, \{2,3\}} = M_8 \times (S_2 \times S_2)$ which has order 32 and is transitive on the 8 points of $X \setminus \{1, 2, 3, 4\}$.

Let H be a maximal subgroup of G such that $G_{\{A,B\}} \leq H < G$. The maximal subgroups of G are given in [10], and comparing orders we see that $H \not\cong M_{11}, \text{PSL}(2, 11), M_9 \times S_3, C_2 \times S_5$ and $A_4 \times S_3$. Moreover, since $G_{\{A,B\}}$ has orbits of size 2, 2 and 8 in X it does not stabilise a hexad pair. If H is intransitive then $H = G_{\{1,2,3,4\}}, G_{\{1,4\}}$ or $G_{\{2,3\}}$. These yield $\mathcal{P}_\cup, \mathcal{P}_\ominus$ and \mathcal{P}_\cap respectively. Thus by [10, p 33] we are left to consider the case where $H \cong 4^2 \times D_{12}$. A MAGMA [3] calculation shows that there is a unique such H containing $G_{\{A,B\}}$ and we obtain the decomposition from Construction 6.5. ■

Before dealing with $G = M_{11}$ we need the following couple of lemmas, the first of which is well known.

Lemma 6.7. *Let (X, \mathcal{B}) be the Witt design $S(4, 5, 11)$. Then \mathcal{B} contains 66 elements, called pentads. Each point of X is contained in 30 pentads, each 2-subset in 12 pentads, each 3-subset in 4 pentads, and each 4-subset in a unique pentad. Moreover, the stabiliser of a pentad is isomorphic to S_5 , which acts in its natural action on the pentad and as $\text{PGL}(2, 5)$ on the complementary hexad.*

Proof. Since (X, \mathcal{B}) can be derived from the set of blocks of the Witt design $S(5, 6, 12)$ containing a given point, the first part follows from Lemma 5.1. By [10, p 18], the stabiliser of a pentad is S_5 and has two orbits on X . ■

Lemma 6.8. *Let (X, \mathcal{B}) be the Witt design $S(4, 5, 11)$ and $G = M_{11}$. Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and suppose that $p = \{1, 2, 3, 4, 5\}$ is the unique pentad containing $A \cup B$. Then $G_{\{A, B\}} \cong C_2^2$ and on $X \setminus p$ has an orbit $\{a, b\}$ of length 2 and an orbit of length 4. Moreover, $\{1, 4, 5, a, b\}$, $\{2, 3, 5, a, b\}$ and $X \setminus \{1, 2, 3, 4, a, b\}$ are pentads.*

Proof. By Lemma 6.7, G_p induces S_5 on p , and since $G_{\{A, B\}} \leq G_p$ it follows that $G_{\{A, B\}} = G_{\{2, 3\}, \{1, 4\}} \cong C_2^2$ and fixes the point 5. By [10], each involution of G fixes precisely three points of X . Two of the involutions of $G_{\{A, B\}}$ fix three points of p and so are fixed point free on $X \setminus p$. The third involution fixes the point 5 and fixes two points a, b of $X \setminus p$. It follows that $G_{\{A, B\}}$ has an orbit of length two (namely, $\{a, b\}$) and an orbit of length 4 on $X \setminus p$.

Any four points lie in a unique pentad and by Lemma 6.7, any 3-subset is contained in 4 pentads. Hence $X \setminus p$ is divided into three sets of size two by the three pentads containing $\{1, 4, 5\}$ other than $\{1, 2, 3, 4, 5\}$. Similarly, $X \setminus p$ is partitioned by the three pentads containing $\{2, 3, 5\}$. Since $G_{\{A, B\}}$ fixes $\{1, 4, 5\}$ and $\{2, 3, 5\}$, it preserves both partitions and $\{a, b\}$ must be a block of both. Hence $\{1, 4, 5, a, b\}$ and $\{2, 3, 5, a, b\}$ are pentads. Moreover, since $X \setminus (\{a, b\} \cup p)$ is an orbit of length 4 of $G_{\{A, B\}}$ and is contained in a unique pentad, the fifth point of this pentad must also be fixed by $G_{\{A, B\}}$ and hence is 5. Thus $X \setminus \{1, 2, 3, 4, a, b\}$ is a pentad. \blacksquare

Since the stabiliser of a 2-set is maximal in M_{11} , it follows from Lemma 2.9 that \mathcal{P}_\cap and \mathcal{P}_\ominus are M_{11} -primitive decompositions. We also obtain M_{11} -primitive decompositions from Constructions 2.10, 2.12, 2.14 and 2.16 by using the Witt design $S(4, 5, 11)$, since the stabilisers of a block, of a point and of a 3-subset are maximal subgroups of M_{11} .

Construction 6.9. Let (X, \mathcal{B}) be the Witt design $S(4, 5, 11)$ and $G = M_{11}$. Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ be adjacent vertices of $J(11, 3)$ and let $\{a, b\}$ be the orbit of length 2 of $G_{\{A, B\}}$ on $X \setminus \{1, 2, 3, 4, 5\}$ given by Lemma 6.8.

1. For each 3-subset Y of X let

$$P_Y = \left\{ \{ \{x, u, v\}, \{y, u, v\} \} \mid \{x, y\} \cup Y, \{u, v\} \cup Y \in \mathcal{B} \right\}$$

and let $\mathcal{P} = \{P_Y \mid Y \text{ a 3-subset}\}$. By Lemma 6.7, Y is contained in 4 pentads, and so $12K_2$. Let $Y = \{5, a, b\}$. By Lemma 6.8, $\{A, B\} \in P_Y$ and $G_{\{A, B\}} \leq G_Y = G_{P_Y}$, which is a maximal subgroup of G . Hence by Lemma 2.4, $(J(11, 3), \mathcal{P})$ is an M_{11} -primitive decomposition.

2. Since G is 4-transitive on X , Lemma 6.8 implies that the stabiliser in G of two 2-subsets of X fixes a third. For each 2-subset Y let

$$P_Y = \left\{ \{ \{x, u, v\}, \{y, u, v\} \} \mid u, v, x, y \in X \setminus Y, G_{Y, \{x, y\}} = G_{Y, \{u, v\}} \right\}$$

and let $\mathcal{P} = \{P_Y \mid Y \text{ a 2-subset}\}$. Then each $P_Y \cong \binom{9}{2}K_2$. Moreover, by Lemma 6.8 any edge of $J(11, 3)$ is contained in a unique part of \mathcal{P} ($\{A, B\} \in P_{\{a, b\}}$) and so $(J(11, 3), \mathcal{P})$ is an M_{11} -primitive decomposition.

3. For each $Y \in \mathcal{B}$ let

$$P_Y = \left\{ \{ \{x, u, v\}, \{y, u, v\} \} \mid x, y \in Y, \{u, v\} \cup (Y \setminus \{x, y\}) \in \mathcal{B} \right\}$$

and let $\mathcal{P} = \{P_Y \mid Y \in \mathcal{B}\}$. By Lemma 6.7, each 3-subset of Y is contained in three more pentads and so each part of \mathcal{P} is isomorphic to $3\binom{5}{2}K_2 = 30K_2$. By Lemma 6.8, $\{A, B\} \in P_Y$ for $Y = \{1, 4, 5, a, b\}$. Moreover, $G_{\{A, B\}}$ fixes Y and so $G_{\{A, B\}} < G_Y = G_{P_Y}$. Thus Lemma 2.4 and the fact that G acts primitively on \mathcal{B} , imply that $(J(11, 3), \mathcal{P})$ is a G -primitive decomposition.

4. For each $Y \in \mathcal{B}$ let

$$P_Y = \left\{ \{ \{x, u, v\}, \{y, u, v\} \} \mid u, v \in Y, \{x, y\} \cup (Y \setminus \{u, v\}) \in \mathcal{B} \right\}$$

and let $\mathcal{P} = \{P_Y \mid Y \in \mathcal{B}\}$. By Lemma 6.7, each 3-subset of Y is contained in three more pentads and so each part of \mathcal{P} is isomorphic to $3\binom{5}{2}K_2 = 30K_2$. By Lemma 6.8, $\{A, B\} \in P_Y$ for $Y = \{2, 3, 5, a, b\}$ and $G_{\{A, B\}} < G_Y = G_{P_Y}$. Thus Lemma 2.4 and the fact that G acts primitively on \mathcal{B} , imply that $(J(11, 3), \mathcal{P})$ is a G -primitive decomposition.

Table 6: M_{11} -primitive decompositions of $J(11, 3)$

\mathcal{P}	P	G_P
\mathcal{P}_\cap	K_9	$M_9 \rtimes C_2$
\mathcal{P}_\ominus	$\binom{9}{2}K_2$	$M_9 \rtimes C_2$
Construction 2.10	$J(5, 3) \cong J(5, 2)$	S_5
Construction 2.12	$30K_4$	M_{10}
Construction 2.14	$4K_3$	$M_8 \rtimes S_3$
Construction 2.16	$12K_2$	$M_8 \rtimes S_3$
Construction 6.9(1)	$12K_2$	$M_8 \rtimes S_3$
Construction 6.9(2)	$\binom{9}{2}K_2$	$M_9 \rtimes C_2$
Construction 6.9(3)	$30K_2$	S_5
Construction 6.9(4)	$30K_2$	S_5
Construction 6.10(1)	Π	$\text{PSL}(2, 11)$
Construction 6.10(2)	11 Petersen graphs	$\text{PSL}(2, 11)$
Construction 6.11	2 Petersen graphs	S_5

Construction 6.10. Let $H = \text{PSL}(2, 11) < M_{11} = G$. Then H has an orbit of length 55 on 3-subsets and this orbit forms a $2 - (11, 3, 3)$ design known as the Petersen design. The remaining 3-subsets form an orbit of length 110 and a $2 - (11, 3, 6)$ design [5].

1. Let Π be the subgraph of $J(11, 3)$ induced on the orbit of length 55. The graph Π was studied in [13] and is H -arc-transitive of valency 6. Given an edge $\{A, B\}$ of Π we have $H_{\{A, B\}} = C_2^2 = G_{\{A, B\}}$. Thus letting $\mathcal{P} = \{\Pi^g \mid g \in G\}$, it follows by Lemma 2.4 that $(J(11, 3), \mathcal{P})$ is a G -primitive decomposition.
2. Let Δ be the subgraph of $J(11, 3)$ induced on the orbit of length 110. Then Δ has valency 15 and given a vertex A , $H_A \cong S_3$ has orbits of length 3, 6 and 6 on the neighbours of A . Let B be a neighbour of A in the orbit of length 3 and let $P = \{A, B\}^H$. Let $g \in H$ which interchanges A and B . Then by Lemma 2.18, $P \cong \text{Cos}(H, H_A, H_A g H_A)$. Moreover, $\langle H_A, g \rangle \cong A_5$ and so P has 11 connected components, each with 10 vertices and isomorphic to the Petersen graph. Since $|H_{\{A, B\}}| = 4 = |G_{\{A, B\}}|$, it follows from Lemma 2.4 that $(J(11, 3), \mathcal{P})$ is a G -primitive decomposition with $\mathcal{P} = P^G$.

Construction 6.11. Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. By Lemma 6.8, $Y = X \setminus \{1, 2, 3, 4, a, b\}$ is a pentad fixed by $G_{\{A, B\}}$. Let $H = G_Y$ and $P = \{A, B\}^H$. Then by Lemma 6.7, H induces S_5 on Y and $\text{PGL}(2, 5)$ on $\{1, 2, 3, 4, a, b\}$. Thus $H_A \cong S_3$ and is a maximal subgroup of $A_5 \cong \text{PSL}(2, 5)$. Moreover, $g \in H_{\{A, B\}}$ which interchanges A and B induces even permutations on Y and so for such a g we have $\langle H_A, g \rangle = A_5$. By Lemma 2.18, $P \cong \text{Cos}(H, H_A, H_A g H_A)$. Since $|H : H_A| = 20$ and $\langle H_A, g \rangle \cong A_5$, it follows that P has two disconnected components with 10 vertices each. Since $|H_A : G_{A, B}| = 3$ it follows that P is a copy of two Petersen graphs. Let $\mathcal{P} = P^G$. Then as $G_{\{A, B\}} < H$, it follows from Lemma 2.4 that $(J(11, 3), \mathcal{P})$ is a G -primitive decomposition.

Proposition 6.12. *If $(J(11, 3), \mathcal{P})$ is an M_{11} -primitive symmetric decomposition then \mathcal{P} is given by Table 6.*

Proof. Let $\Gamma = J(11, 3)$ and $G = M_{11} < \text{Sym}(X)$, and consider X as the point set of the Witt-design $S(4, 5, 11)$ with automorphism group G . Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ be adjacent vertices. Suppose that $p = \{1, 2, 3, 4, 5\}$ is the unique pentad of the Witt design containing $\{1, 2, 3, 4\}$ and let H be a maximal subgroup of G containing $G_{\{A, B\}} = G_{\{2, 3, \{1, 4\}\}}$. The maximal subgroups of G are given in [10, p 18].

If H is the stabiliser of a point then $H = G_5$ and so we obtain the decomposition yielded by Construction 2.12. Next suppose that H is the stabiliser of a duad. Then H is one of $G_{\{2, 3\}}, G_{\{1, 4\}}$ or $G_{\{a, b\}}$ where $\{a, b\}$ is the orbit of length two of $G_{\{A, B\}}$ on $\{6, 7, \dots, 11\}$. The first gives \mathcal{P}_\cap the second

gives \mathcal{P}_\ominus . Finally, if $H = G_{\{a,b\}}$ then H is the stabiliser of the divisor of the decomposition obtained from Construction 6.9(2) containing $\{A, B\}$.

Next suppose that H is the stabiliser of a triad. Then H stabilises $\{1, 4, 5\}$, $\{2, 3, 5\}$ or $\{5, a, b\}$. If $H = G_{\{1,4,5\}}$ then H is the stabiliser of the divisor of the decomposition from Construction 2.14 containing $\{A, B\}$. Also $H = G_{\{2,3,5\}}$ is the stabiliser of the divisor of the decomposition yielded by Construction 2.16 containing $\{A, B\}$. Finally, $H = G_{\{5,a,b\}}$ is the stabiliser of the divisor of the decomposition obtained from Construction 6.9(1) containing $\{A, B\}$.

Next suppose that H is the stabiliser of a pentad. Since $G_{\{A,B\}}$ has only one orbit of odd length, it follows that 5 is in the pentad. Combining 5 with two orbits of $G_{\{A,B\}}$ of length two we get that $G_{\{A,B\}}$ fixes the pentads $\{1, 2, 3, 4, 5\}$, $\{1, 4, 5, a, b\}$, $\{2, 3, 5, a, b\}$ and $X \setminus \{1, 2, 3, 4, a, b\}$ (by Lemma 6.8, these 5-sets are actually pentads). Thus there are four choices for H . If $H = G_{\{1,2,3,4,5\}}$ then we obtain the decomposition from Construction 2.10. If $H = G_{\{1,4,5,a,b\}}$, then H is the stabiliser of the divisor of the decomposition from Construction 6.9(3) containing $\{A, B\}$ while $H = G_{\{2,3,5,a,b\}}$ is the stabiliser of the divisor of the decomposition yielded by Construction 6.9(4). Finally, if $H = G_{X \setminus \{1,2,3,4,a,b\}}$ then H is the stabiliser of the divisor of the decomposition produced by Construction 6.11 containing $\{A, B\}$.

We are left to consider $H \cong \text{PSL}(2, 11)$. By a calculation using MAGMA [3], there are two such H containing $G_{\{A,B\}}$. These give us the two decompositions in Construction 6.10. ■

We now give constructions for $\text{P}\Gamma\text{L}(2, 8)$ -primitive decompositions of $J(9, 3)$.

Construction 6.13. Let $G = \text{P}\Gamma\text{L}(2, 8)$ and $X = \text{GF}(8) \cup \{\infty\}$, where $\text{GF}(8)$ is defined by the relation $i^3 = i + 1$.

1. By Theorem 3.4, $T = \text{PSL}(2, 8)$ is not arc-transitive on $J(9, 3)$ and so as $T \triangleleft G$ and has index three, T has three equal sized orbits on edges. Thus the partition $\mathcal{P} = \{P_1, P_2, P_3\}$ given by these three orbits is a G -primitive decomposition. Since T is vertex-transitive, this is in fact a homogeneous factorisation and appears in [11].
2. Let $x \in X$. Then $G_x = \text{A}\Gamma\text{L}(1, 8)$ and preserves the structure of an affine space $\text{AG}(3, 2)$ (with plane-set \mathcal{B}) on $X \setminus \{x\}$. Let

$$P_x = \left\{ \{A, B\} \mid A \cup B \in \mathcal{B} \right\}$$

and $\mathcal{P} = \{P_x \mid x \in X\}$. Then since each 3-subset lies in a unique plane, $P_x \cong 14K_4$. Moreover, G_x acts transitively on the set \mathcal{B} of affine planes and for $Y \in \mathcal{B}$ we have $G_{x,Y}$ induces A_4 on Y . Thus G_x acts transitively on the set of edges in P_x and so given $\{A, B\} \in P_x$ we have $|G_{x,\{A,B\}}| = 2 = |G_{\{A,B\}}|$. Thus $G_{\{A,B\}} \leq H$ and so by Lemma 2.4, $\mathcal{P} = P_x^G$ is a G -primitive decomposition of $J(9, 3)$.

3. Let $A = \{\infty, 0, 1\}$ and $B = \{\infty, 0, i\}$. Then $G_{\{A,B\}} = \langle g \rangle \cong C_2$ where $x^g = ix^{-1}$ and has orbits $\{0, \infty\}$, $\{1, i\}$, $\{i^2, i^6\}$, $\{i^3, i^5\}$ and $\{i^4\}$. Thus $G_{\{A,B\}} \leq G_{\{i^2, i^6\}} = H$ (H has order 42) and so by Lemma 2.4, letting $P = \{A, B\}^H$ and $\mathcal{P} = P^G$ we obtain a G -primitive decomposition of $J(9, 3)$. Now $H_A = \langle h \rangle$ where $x^h = x + 1$, which has order two and so P has 21 vertices and valency 2. Moreover, $\langle H_A, g \rangle = D_{14}$ and so by Lemma 2.18, P has three connected components. Thus $P \cong 3C_7$.
4. Let $A = \{\infty, 0, 1\}$ and $B = \{\infty, 0, i\}$. Then $G_{\{A,B\}} \leq G_{\{i^3, i^5\}} = H$ and so by Lemma 2.4, letting $P = \{A, B\}^H$ and $\mathcal{P} = P^G$ we obtain a G -primitive decomposition of $J(9, 3)$. Then $H_A = \langle h \rangle$ where $x^h = (x^4 + 1)^{-1}$, which has order three. Thus P has 14 vertices and valency 3. Since g and h do not commute, $\langle H_A, g \rangle = H$ and so P is a connected graph. Moreover, P is H -arc-transitive and so by [33, p167], P is the Heawood graph.

Construction 6.14. Let $K = \text{GF}(64)$, with ξ a primitive element of K , and let $F = \{0\} \cup \{(\xi^9)^l \mid l = 0, 1, \dots, 6\} \cong \text{GF}(8)$. One can consider the projective line X on which G acts as the elements of K modulo F . Then $H = \langle \hat{\xi}, \sigma, \tau \rangle \cong D_{18} \rtimes C_3$ where $\hat{\xi} : x \rightarrow \xi x \pmod{F}$, $\sigma : x \rightarrow x^8 = x^{-1} \pmod{F}$, and $\tau : x \rightarrow x^4 \pmod{F}$.

1. Let $A = \{1, \xi, \xi^2\}$ and $B = \{\xi, \xi^2, \xi^3\}$. Then $\{A, B\}$ is an edge of $J(9, 3)$ whose ends are interchanged by $\hat{\xi}^6 \sigma \in H$. Thus letting $P = \{A, B\}^H$ and $\mathcal{P} = P^G$, Lemma 2.4 implies that

Table 7: PTL(2, 8)-primitive decompositions of $J(9, 3)$

\mathcal{P}	P	G_P
\mathcal{P}_\cap	K_7	$D_{14} \rtimes C_3$
\mathcal{P}_\ominus	$\binom{7}{2}K_2$	$D_{14} \rtimes C_3$
Construction 6.13(1)	PSL(2, 8)-orbits	PSL(2, 8)
Construction 6.13(2)	$14K_4$	AGL(1, 8)
Construction 6.13(3)	$3C_7$	$D_{14} \rtimes C_3$
Construction 6.13(4)	Heawood graph	$D_{14} \rtimes C_3$
Construction 6.14(1)	$3C_9$	$D_{18} \rtimes C_3$
Construction 6.14(2)	$27K_2$	$D_{18} \rtimes C_3$
Construction 6.14(3)	$27K_2$	$D_{18} \rtimes C_3$
Construction 6.14(4)	$27K_2$	$D_{18} \rtimes C_3$

$(J(9, 3), \mathcal{P})$ is a G -primitive decomposition. Now $H_A = \langle \hat{\xi}^7 \sigma \rangle$ and so P has 27 vertices. Moreover, $H_{A,B} = 1$ and so P has valency 2. Since $\langle \hat{\xi}^6 \sigma, \hat{x}i^7 \sigma \rangle = D_{18}$ it follows from Lemma 2.18 that P has 3 connected components and so $P \cong 3C_9$.

2. Let $A = \{1, \xi, \xi^3\}$ and $B = \{1, \xi, \xi^7\}$. Then $\{A, B\}$ is an edge of $J(9, 3)$ whose ends are interchanged by $\hat{x}i^8 \sigma \in H$. Thus letting $P = \{A, B\}^H$ and $\mathcal{P} = P^G$, Lemma 2.4 implies that $(J(9, 3), \mathcal{P})$ is a G -primitive decomposition. Now $|H_A| = 1$ and so P is a matching of 27 edges.
3. Let $A = \{1, \xi, \xi^3\}$ and $B = \{\xi, \xi^3, \xi^4\}$. Then $\{A, B\}$ is an edge of $J(9, 3)$ whose ends are interchanged by $\hat{x}i^5 \sigma \in H$. Thus letting $P = \{A, B\}^H$ and $\mathcal{P} = P^G$, Lemma 2.4 implies that $(J(9, 3), \mathcal{P})$ is a G -primitive decomposition. Now $|H_A| = 1$ and so P is a matching of 27 edges.
4. Let $A = \{1, \xi, \xi^3\}$ and $B = \{1, \xi^2, \xi^3\}$. Then $\{A, B\}$ is an edge of $J(9, 3)$ whose ends are interchanged by $\hat{x}i^6 \sigma \in H$. Thus letting $P = \{A, B\}^H$ and $\mathcal{P} = P^G$, Lemma 2.4 implies that $(J(9, 3), \mathcal{P})$ is a G -primitive decomposition. Now $|H_A| = 1$ and so P is a matching of 27 edges.

Proposition 6.15. *If $(J(9, 3), \mathcal{P})$ is a PTL(2, 8)-primitive decomposition then \mathcal{P} is as in Table 7.*

Proof. Let $G = \text{PTL}(2, 8)$ act on $\{\infty\} \cup \text{GF}(8)$ and suppose that $\text{GF}(8)$ has primitive element i such that $i^3 = i + 1$. Let $A = \{\infty, 0, 1\}$ and $B = \{\infty, 0, i\}$ be adjacent vertices in $\Gamma = J(9, 3)$. Then $G_{\{A,B\}} = G_{\{\infty, 0\}, \{1, i\}} = \langle g \rangle \cong C_2$, where $x^g = ix^{-1}$, which fixes the point i^4 and has 4 orbits of size 2. Let H be a maximal subgroup of G containing $G_{\{A,B\}}$. The maximal subgroups of G are given in [10, p 6].

If $H = \text{PGL}(2, 8)$ then we obtain the decomposition in Construction 6.13(1) while if H is a point stabiliser then $H = G_{i^4}$ and we obtain the decomposition in Construction 6.13(2).

Suppose now that $H \cong D_{14} \rtimes C_3$ is the stabiliser of a 2-subset. Then $H = G_{\{\infty, 0\}}$, $H = G_{\{1, i\}}$, $H = G_{\{i^2, i^6\}}$, or $H = G_{\{i^3, i^5\}}$. In the first case we get the decomposition \mathcal{P}_\cap , while the second yields \mathcal{P}_\ominus . The third case gives Construction 6.13(3) and the fourth gives the decomposition in Construction 6.13(4).

Let $H = \langle \hat{\xi}, \sigma, \tau \rangle \cong D_{18} \rtimes C_3$ as given in Construction 6.14. Instead of finding all conjugates of H containing $G_{\{A,B\}}$, we (equivalently) find all edge orbits $\{C, D\}^H$ such that H contains $G_{\{C,D\}}$. Note that, for such an edge, C and D lie in the same H -orbit on vertices. One sees easily that H has three orbits on vertices of $J(9, 3)$, of sizes 3 ($\{1, \xi^3, \xi^6\}^{\langle \hat{x}i \rangle}$), 27 ($\{1, \xi, \xi^2\}^{\langle \hat{x}i \rangle} \cup \{1, \xi^2, \xi^4\}^{\langle \hat{x}i \rangle} \cup \{1, \xi^4, \xi^8\}^{\langle \hat{x}i \rangle}$), and 54 (all the other vertices). The orbit of size 3 contains no edges. In the orbit of size 27, if we fix the vertex $C = \{1, \xi, \xi^2\}$, we find two vertices D , namely $\{1, \xi, \xi^8\}$ and $\{\xi, \xi^2, \xi^3\}$, such that the unique involution switching C and D is in H . Moreover, these two vertices are interchanged by H_C . Hence this vertex orbit yields one orbit of edges whose stabilisers are contained in H and we get the decomposition in Construction 6.14(1).

In the orbit of size 54, if we fix the vertex $C = \{1, \xi, \xi^3\}$, we find three vertices D , namely $\{1, \xi, \xi^7\}$, $\{\xi, \xi^3, \xi^4\}$ and $\{1, \xi^2, \xi^3\}$, such that the unique involution switching C and D is in H . Since H acts

regularly on this orbit, each choice of D gives a different H -orbit on edges and we get the three decompositions of Constructions 6.14(2,3,4). ■

7 The case $k = 2$

By Theorem 3.4, a subgroup G of S_n is arc-transitive on $J(n, 2)$ if and only if G is 3-transitive. Hence other than A_n or S_n , the possibilities for (n, G) are $(11, M_{11})$, $(12, M_{11})$, $(12, M_{12})$, $(22, M_{22})$, $(22, \text{Aut}(M_{22}))$, $(23, M_{23})$, $(24, M_{24})$, $(2^d, \text{AGL}(d, 2))$ for $d > 2$, $(16, C_2^4 \rtimes A_7)$, and $(q + 1, G)$ where G is a 3-transitive subgroup of $\text{P}\Gamma\text{L}(2, q)$ with $q \geq 4$. We treat all but the last case in this section and deal with the subgroups of $\Gamma\text{L}(2, q)$ in Section 8.

Proposition 7.1. *If $(J(11, 2), \mathcal{P})$ is an M_{11} -primitive decomposition then \mathcal{P} is \mathcal{P}_\cap , \mathcal{P}_\cup , or \mathcal{P}_\ominus .*

Proof. Let $G = M_{11}$ act on the point set X of the Witt design $S(4, 5, 11)$, and let $A = \{1, 2\}$, $B = \{2, 3\}$ be adjacent vertices. Then $G_{\{A, B\}} = G_{2, \{1, 3\}}$ and since G is strictly 4-transitive it follows that $|G_{\{A, B\}}| = 16$ and has one orbit on the 8 remaining points. Let H be a maximal subgroup of G containing $G_{\{A, B\}}$. Comparing orders and the maximal subgroups of G given in [10, p 18] we see that $H \not\cong \text{PSL}(2, 11)$ or S_5 . It follows that H stabilises either a point, a pair or a 3-subset. In the first case $H = G_2$ and so $\mathcal{P} = \mathcal{P}_\cap$. In the second case, $H = G_{\{1, 3\}}$ and we obtain the decomposition \mathcal{P}_\ominus , while in the last case $H = G_{\{1, 2, 3\}}$ and so we get the decomposition \mathcal{P}_\cup . ■

Since the stabilisers of a point and a 2-subset are maximal in M_{11} it follows from Lemma 2.9 that \mathcal{P}_\cap and \mathcal{P}_\ominus are M_{11} -primitive decompositions of $J(12, 2)$. In order to give more constructions for M_{11} -primitive decompositions of $J(12, 2)$, we will need the following lemma.

Lemma 7.2. *Let $G = M_{11}$ act 3-transitively on the point set X of the Witt design $S(5, 6, 12)$. As seen in Construction 5.6, G has an orbit of length 165 on 4-subsets, forming a $3 - (12, 4, 3)$ design with block set \mathcal{D} . In this design, each 3-set S determines uniquely another 3-set $S_{\mathcal{D}}$, namely the set of fourth points of the 3 blocks of \mathcal{D} containing S . We have $(S_{\mathcal{D}})_{\mathcal{D}} = S$ and $S \cup S_{\mathcal{D}}$ is a hexad of $S(5, 6, 12)$. Moreover if $\{S, S_{\mathcal{D}}, U, V\}$ is the unique linked three containing S and $S_{\mathcal{D}}$ as triads (see Lemma 5.4), then $U_{\mathcal{D}} = V$.*

Proof. For any 3-set S , the set $S_{\mathcal{D}}$ is obviously well defined by the properties of the $3 - (12, 4, 3)$ design. Now, an element of G stabilising S must also stabilise $S_{\mathcal{D}}$. Therefore $G_S \leq G_{S_{\mathcal{D}}}$. Since $S_{\mathcal{D}}$ is also a 3-set and G is 3-transitive, we must have $|G_S| = |G_{S_{\mathcal{D}}}|$. Therefore $G_S = G_{S_{\mathcal{D}}}$. By a computation using MAGMA [3] we find that $G_S \cong S_3 \times S_3$ has orbits of lengths 3, 3 and 6 on X . Hence $(S_{\mathcal{D}})_{\mathcal{D}} = S$.

Let u, v be two points of $S_{\mathcal{D}}$. Then $S \cup \{u, v\}$ is contained in a unique hexad h . Since G_S preserves the set of hexads containing S , and acts transitively on the 3 points of $S_{\mathcal{D}}$ and on the 6 points of $X \setminus (S \cup S_{\mathcal{D}})$, it follows that the sixth point of h must also lie in $S_{\mathcal{D}}$. Hence $S \cup S_{\mathcal{D}}$ is a hexad. Let $T = \{S, S_{\mathcal{D}}, U, V\}$ be the unique linked three containing S and $S_{\mathcal{D}}$ as triads (Lemma 5.4). Since G_S preserves T and is transitive on $U \cup V$, it follows that G_S has an index 2 subgroup $G_{S, U}$ with orbits $S, S_{\mathcal{D}}, U$ and V . Since the orbits of $G_{S, U}$ are a refinement of the orbits of G_U , $U_{\mathcal{D}}$ must be one of these orbits of size 3. Since $U_{\mathcal{D}}$ cannot be S nor $S_{\mathcal{D}}$, it follows that $U_{\mathcal{D}} = V$. ■

Construction 7.3. Let $G = M_{11}$ act 3-transitively on the point set X of the Witt design $S(5, 6, 12)$. We use the notation of Lemma 7.2.

1. Let $Y \in \mathcal{D}$. Let

$$P_Y = \left\{ \{ \{u, x\}, \{x, v\} \} \mid \{x, u, v\}_{\mathcal{D}} = Y \setminus \{x\} \right\}$$

and $\mathcal{P} = \{P_Y \mid Y \in \mathcal{D}\}$. Then $P_Y \cong 4K_2$. Let $\{\{u, x\}, \{x, v\}\}$ be an edge of $J(12, 2)$. Then it is in a unique P_Y , with $Y = \{x\} \cup \{x, u, v\}_{\mathcal{D}}$. Since G_Y is maximal in G , it follows that $(J(12, 2), \mathcal{P})$ is a G -primitive decomposition.

Table 8: M_{11} -primitive decompositions of $J(12, 2)$

\mathcal{P}	P	G_P
\mathcal{P}_\cap	K_{11}	$\text{PSL}(2, 11)$
\mathcal{P}_\ominus	$10K_2$	S_5
Construction 7.3(1)	$4K_2$	$M_8 \times S_3$
Construction 7.3(2)	$4K_3$	$M_9 \times C_2$

2. Let T be a \mathcal{D} -linked three, that is, a linked three for the $S(5, 6, 12)$ such that, for any $X \in T$, $X_{\mathcal{D}}$ is a triad of T . Let

$$P_T = \left\{ \{ \{u, x\}, \{x, v\} \} \mid \{x, u, v\} \in T \right\}$$

and $\mathcal{P} = \{P_T \mid T \text{ is a } \mathcal{D}\text{-linked three}\}$. Then $P_T \cong 4K_3$, with each triad contributing K_3 . Let $\{ \{u, x\}, \{x, v\} \}$ be an edge of $J(12, 2)$. Then $\{u, v, x\}$ and $\{u, v, x\}_{\mathcal{D}}$ must be triads of T . By Lemma 7.2, the unique linked three containing these two triads is a \mathcal{D} -linked three. It follows that there is exactly one \mathcal{D} -linked three T such that P_T contains a given edge. Since the stabiliser in G of a \mathcal{D} -linked three is maximal in G , it follows that $(J(12, 2), \mathcal{P})$ is a G -primitive decomposition.

Thus we have the M_{11} -primitive decompositions listed in Table 8.

Proposition 7.4. *If $(J(12, 2), \mathcal{P})$ is an M_{11} -primitive decomposition then \mathcal{P} is given by Table 8.*

Proof. Let $G = M_{11}$ act transitively on the point set X of the Witt design $S(5, 6, 12)$ and let \mathcal{D} be the block set of the $3 - (12, 4, 3)$ design described in Construction 5.6 (see above). Take adjacent vertices $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $G_{\{A, B\}} = G_{2, \{1, 3\}} \cong D_{12}$ which has an orbit of length 3 (namely, $\{1, 2, 3\}_{\mathcal{D}}$) and an orbit of length 6 on the remaining 9 points of X . Let H be a maximal subgroup of G containing $G_{\{A, B\}}$. Since M_{10} contains no elements of order 6, it follows that $H \not\cong M_{10}$. If H is a point stabiliser, then $H = G_2$ and we get the decomposition \mathcal{P}_\cap . If H is a pair stabiliser then $H = G_{\{1, 3\}}$, and we get the decomposition \mathcal{P}_\ominus . If $H \cong M_8 \times S_3$ then H is the stabiliser of a block in \mathcal{D} . There is a unique such block, namely the union of $\{2\}$ with $\{1, 2, 3\}_{\mathcal{D}}$. Hence H is the stabiliser of the divisor of the decomposition obtained from Construction 7.3(1) containing $\{A, B\}$.

Now let $H \cong M_9 \times S_3$. Then H is a \mathcal{D} -linked three stabiliser, namely the only one containing $\{1, 2, 3\}$ as a triad (see the construction). Hence H is the stabiliser of the divisor of the decomposition obtained from Construction 7.3(2) containing $\{A, B\}$. ■

Proposition 7.5. *If $(J(12, 2), \mathcal{P})$ is an M_{12} -primitive decomposition, then \mathcal{P} is \mathcal{P}_\cup , \mathcal{P}_\cap or \mathcal{P}_\ominus .*

Proof. Let $G = M_{12}$ act on the point set X of the Witt-design $S(5, 6, 12)$ and take adjacent vertices $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $G_{\{A, B\}} = G_{2, \{1, 3\}}$ which has order 144 and is 2-transitive on the 9 remaining points since G is 5-transitive on X . Let H be a maximal subgroup of G containing $G_{\{A, B\}}$. The maximal subgroups of G are given in [10], and comparing orders we see that $H \not\cong \text{PSL}(2, 11)$, $2 \times S_5$, $4^2 : D_{12}$, $M_8.S_4$ or $A_4 \times S_3$. Since $G_{\{A, B\}}$ fixes a point but not a hexad it follows that H is not the stabiliser of a hexad pair, and since $G_{\{A, B\}}$ is 2-transitive on $X \setminus \{1, 2, 3\}$ we also have that H is not the stabiliser of a linked three. In the action of M_{11} on 12 points, $\text{PSL}(2, 11)$ is the stabiliser of a point. Since 144 does not divide $|\text{PSL}(2, 11)|$ and $G_{\{A, B\}}$ fixes the point 2, it follows that H is not a transitive copy of M_{11} . Thus $H = G_2, G_{\{1, 3\}}$ or $G_{\{1, 2, 3\}}$. In the first case we get the decomposition \mathcal{P}_\cap , the second case yields \mathcal{P}_\ominus while the third gives \mathcal{P}_\cup . ■

Before dealing with $G = M_{22}$ we need the following well known result which follows from Lemma 6.3.

Lemma 7.6. *Let (X, \mathcal{B}) be the Witt design $S(3, 6, 22)$. Then \mathcal{B} contains 77 elements, called hexads. Each point of X is contained in 21 hexads, each 2-subset in 5 hexads, and each 3-subset in a unique hexad. Moreover, the stabiliser of a hexad is $C_2^4 \times A_6$ with the pointwise stabiliser of the hexad being C_2^4 which acts regularly on the 16 points not in the hexad.*

Proof. Since (X, \mathcal{B}) can be derived from the set of blocks of the Witt design $S(4, 5, 23)$ containing a given point, this follows from Lemma 6.3. \blacksquare

Proposition 7.7. *If $(J(22, 2), \mathcal{P})$ is an M_{22} -primitive decompositions then $\mathcal{P} = \mathcal{P}_\cap$ or \mathcal{P}_\ominus , or \mathcal{P} is obtained from Construction 2.10 and has divisors isomorphic to $J(6, 2)$.*

Proof. Let $G = M_{22}$ act on the point-set X of the Witt design $S(3, 6, 22)$ and take adjacent vertices $A = \{1, 2\}$ and $B = \{2, 3\}$. Moreover, suppose that $h = \{1, 2, 3, 4, 5, 6\}$ is the unique hexad of the Witt design containing $\{1, 2, 3\}$. By Lemma 7.6, $G_h = C_2^4 \rtimes A_6$, where C_2^4 acts trivially on h and transitively on $X \setminus h$. It follows that $G_{\{A, B\}} = G_{2, \{1, 3\}, \{4, 5, 6\}}$ had order 96 and acts transitively on $X \setminus h$.

Let H be a maximal subgroup of G containing $G_{\{A, B\}}$. Comparing orders and the maximal subgroups of G given in [10] we see that $H \cong \text{PSL}(2, 11)$, A_7 or M_{10} . Since $G_{\{A, B\}}$ does not stabilise an octad, it follows that H is either G_2 , $G_{\{1, 3\}}$ or G_h . The first gives the decomposition \mathcal{P}_\cap , while the second yields \mathcal{P}_\ominus . Finally G_h is the stabiliser of the part of the decomposition obtained from Construction 2.10 containing $\{A, B\}$ and has divisors isomorphic to $J(6, 2)$. \blacksquare

Proposition 7.8. *All $\text{Aut}(M_{22})$ -primitive decompositions of $J(22, 2)$ are M_{22} -primitive decompositions.*

Proof. By [10], a maximal subgroup of $\text{Aut}(M_{22})$ is either M_{22} or arises from a maximal subgroup of M_{22} . Since M_{22} is arc-transitive it does not give a decomposition. In all other cases, Lemma 2.7 implies that we get M_{22} -primitive decompositions. \blacksquare

Proposition 7.9. *If $(J(23, 2), \mathcal{P})$ is an M_{23} -primitive decomposition then \mathcal{P} is \mathcal{P}_\cap , \mathcal{P}_\ominus or \mathcal{P}_\cup .*

Proof. Let $G = M_{23}$ act on the point-set X of the Witt design $S(4, 7, 23)$ and take adjacent vertices $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $G_{\{A, B\}} = G_{2, \{1, 3\}} \cong 2^4 \rtimes S_5$ (see [10, p 71]) and since G is 4-transitive, $G_{\{A, B\}}$ is transitive on $X \setminus \{1, 2, 3\}$. Let H be a maximal subgroup of G containing $G_{\{A, B\}}$. Since $|G_{\{A, B\}}|$ does not divide 23.11, it follows from [10, p 71] that H is intransitive. Hence H is G_2 , $G_{\{1, 3\}}$ or $G_{\{1, 2, 3\}}$. These give us the decompositions \mathcal{P}_\cap , \mathcal{P}_\ominus and \mathcal{P}_\cup respectively. \blacksquare

Proposition 7.10. *If $(J(24, 2), \mathcal{P})$ is an M_{24} -primitive symmetric decompositions then \mathcal{P} is \mathcal{P}_\cap , \mathcal{P}_\ominus or \mathcal{P}_\cup .*

Proof. Let $G = M_{24}$ acting on the point-set X of the Witt design $S(5, 8, 24)$ and take adjacent vertices $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $G_{\{A, B\}} = G_{2, \{1, 3\}} \cong \text{P}\Sigma\text{L}(3, 4)$ (see [10, p 96]). Note that $G_{\{A, B\}}$ is transitive on $X \setminus \{1, 2, 3\}$ since G is 5-transitive on X . Let H be a maximal subgroup of G containing $G_{\{A, B\}}$. Looking at the maximal subgroups of G in [10], it follows that H is either G_2 , $G_{\{1, 3\}}$ or $G_{\{1, 2, 3\}}$. Thus we obtain the decompositions \mathcal{P}_\cap , \mathcal{P}_\ominus and \mathcal{P}_\cup respectively. \blacksquare

Let $G = \text{AGL}(d, 2)$ acting on the set X of vectors of a d -dimensional vector space over $\text{GF}(2)$. Since the stabiliser of a vector is maximal in G , Lemma 2.9 implies that \mathcal{P}_\cap is a G -primitive decomposition. The set of affine planes in the affine space $\text{AG}(d, 2)$ yields an $S(3, 4, 2^d)$ Steiner system with each point contained in $\frac{(2^d-1)(2^{d-1}-1)}{3}$ planes. In both cases, G acts transitively on planes hence we can use Construction 2.10. However, G is not primitive on planes as it preserves parallelness. Applying now Construction 2.1 yields line 2 of Table 9. As G is transitive on points and the stabiliser of a point is maximal in G , applying Construction 2.12 yields line 3 of Table 9. As G is 2-transitive, we can use Construction 2.16. However, G acts imprimitively on 2-subsets as 2-subsets correspond to lines and again G preserves parallelness. Thus we also apply Construction 2.1 and obtain line 4 of Table 9. Indeed the divisors are indexed by lines of the affine plane and are isomorphic to $2^{d-2}K_2$. Each pair Y_1, Y_2 of parallel lines yields a C_4 in the $J(4, 2)$ induced on $Y_1 \cup Y_2$. As a parallel class of lines contains 2^{d-1} lines, we have $\frac{2^{d-1}(2^{d-1}-1)}{2}$ pairs of parallel lines in the imprimitivity class.

When $d = 4$ the group $\overline{G} = C_2^4 \rtimes A_7 < \text{AGL}(4, 2)$ is 3-transitive on X and hence, by Corollary 3.3, is arc-transitive on $J(2^4, 2)$. Thus the four G -primitive decompositions in Table 9 are also \overline{G} -transitive. The stabiliser in \overline{G} of a point is A_7 which is maximal in \overline{G} . Hence the partitions in Rows 1 and 3 are \overline{G} -primitive. The stabilisers of 2-spaces and 1-spaces in A_7 are maximal in A_7 and so the remaining two partitions are also \overline{G} -primitive.

Table 9: G -primitive decompositions of $J(2^d, 2)$ for $G = \text{AGL}(d, 2)$ with $d \geq 3$, or $G = C_2^4 \rtimes A_7$ with $d = 4$

\mathcal{P}	P	G_P
\mathcal{P}_\cap	$K_{2^{d-1}}$	G_0
Constructions 2.10 and 2.1	$2^{d-2}J(4, 2) \cong 2^{d-2}K_{2,2,2}$	$C_2^d \rtimes (G_0)_{\langle v, w \rangle}$
Construction 2.12	$\frac{(2^d-1)(2^{d-1}-1)}{3}K_3$	G_{v+w}
Construction 2.16 and 2.1	$2^{d-2}(2^{d-1}-1)C_4$	$C_2^d \rtimes (G_0)_{\langle v+w \rangle}$

Before showing that these are the only G -primitive decompositions with $G \leq \text{AGL}(d, 2)$ we need a lemma.

Lemma 7.11. *Let $G = N \rtimes G_0$ where $N \cong C_p^d$ for some prime p and G_0 acts irreducibly on N . Suppose that H is a maximal subgroup of G . Then either H is a complement of N , or $M = N \rtimes H_0$ for some maximal subgroup H_0 of H .*

Proof. Since H normalises N we have $H \leq NH \leq G$. Thus as H is maximal, either $NH = H$ or $NH = G$. The first case implies that $N \leq H$ and so $H = N \rtimes H_0$ for some maximal subgroup H_0 of G_0 . Suppose now that $NH = G$. Then $H/(H \cap N) \cong G_0$, and so for each $g \in G_0$, there exists $n \in N$ such that $ng \in H$. Since N is abelian, it follows that H induces G_0 in its action on N by conjugation. Since G_0 acts irreducibly on N and H normalises $H \cap N$, it follows that $H \cap N = 1$ or N . However, $H \cap N = N$ implies that $H = G$ which is not the case. Hence $H \cap N = 1$ and $H \cong G_0$, that is H is a complement of N . \blacksquare

Proposition 7.12. *Let $d \geq 3$ and $G = \text{AGL}(d, 2)$, or $d = 4$ and $G = C_2^4 \rtimes A_7$. If $(J(2^d, 2), \mathcal{P})$ is a G -primitive decomposition then \mathcal{P} is given by Table 9.*

Proof. We can identify X with the vectors of a d -dimensional vector space over $\text{GF}(2)$. Let $A = \{0, v\}$ and $B = \{0, w\}$ where v, w are distinct non-zero elements of X . Thus $G_{\{A, B\}} = (G_0)_{\langle v, w \rangle}$ which is an index 3 subgroup of $(G_0)_{\langle v, w \rangle}$. Moreover, $G_{\{A, B\}}$ fixes the vector $v + w$ and is transitive on $X \setminus \langle v, w \rangle$.

Let H be a maximal subgroup of G containing $G_{\{A, B\}}$. By Lemma 7.11, either H is a complement of $N = \text{soc}(G)$ or $H = N \rtimes H_0$ for some maximal subgroup H_0 of G_0 .

Suppose first that H is a complement. By a MAGMA [3] calculation, $C_2^4 \rtimes A_7$ has a unique conjugacy class of complements. If $d \geq 4$ then there is a unique class of complements of N in $\text{AGL}(d, 2)$, while in $\text{AGL}(3, 2)$ there are two classes (see for example [14]). Hence either H is the stabiliser of a vector or $d = 3$ and H is transitive. In the second case $H = \text{PSL}(2, 7)$ acting transitively on V . However, a Sylow 2-subgroup of H is then regular on V , and hence H cannot contain $G_{\{A, B\}} \cong D_8$ (fixing the point 0). Thus H is the stabiliser of a vector and so $H = G_0$ or G_{v+w} . The first case yields the decomposition \mathcal{P}_\cap , while the second is the stabiliser of the divisor of the decomposition obtained from Construction 2.12 containing $\{A, B\}$.

Suppose now that $H = N \rtimes H_0$ for some maximal subgroup H_0 of G_0 . First let $G = \text{AGL}(d, 2)$. Since $G_{\{A, B\}}$ is an index 3 subgroup of the stabiliser in $\text{GL}(d, 2)$ of the 2-space $\langle v, w \rangle$, it contains a Sylow 2-subgroup of $\text{GL}(d, 2)$. Thus H_0 contains a Sylow 2-subgroup of $\text{GL}(d, 2)$ and it follows from a Lemma of Tits (see for example [36, (1.6)]) that H_0 is a parabolic subgroup and hence is a subspace stabiliser. Now let $G = C_2^4 \rtimes A_7$. Since $G_{\{A, B\}} \cong S_4$ fixes a nonzero vector it is contained in a subgroup $\text{PSL}(2, 7)$ of A_7 and hence by [10, p 10], the elements of order 3 in $G_{\{A, B\}}$ are from the conjugacy class $3B$, that is, in the representation of A_7 on 7 points they are products of two 3-cycles. By [10, p 10], A_7 has 5 conjugacy classes of maximal subgroups. The elements of order 3 in a maximal S_5 subgroup are from the conjugacy class $3A$ ([10, p 10]), instead of $3B$ and so $H_0 \not\cong S_5$. If $H_0 \cong A_6$ then $A_6 \cong \text{PSp}(4, 2)'$ and contains two conjugacy classes of S_4 subgroups. One is the stabiliser of a vector and has orbit lengths 1, 6 and 8 on nonzero vectors and the other is the stabiliser of a totally isotropic 2-space with orbit sizes 3 and 12. Hence none of them stabilises the pair $\{v, w\}$ and so $H_0 \not\cong A_6$. The remaining three conjugacy classes of maximal subgroups of A_7 are stabilisers of subspaces. Thus for both groups G , H_0 is a subspace stabiliser. The only proper, nontrivial subspaces fixed by $G_{\{A, B\}}$ are $\langle v + w \rangle$ and $\langle v, w \rangle$. If $H_0 = (G_0)_{\langle v, w \rangle}$ then H is the stabiliser of the class of planes parallel to $\langle v, w \rangle$

and so H is the stabiliser of the divisor containing $\{A, B\}$ of the decomposition in Row 2 of Table 9. Similarly, if $H_0 = (G_0)_{\langle v+w \rangle}$ then H is the stabiliser of the class of lines parallel to $\langle v+w \rangle$ and so is the stabiliser of the divisor containing $\{A, B\}$ of the decomposition in Row 4 of Table 9. ■

8 Completing the case $k = 2$: $G \leq \text{P}\Gamma\text{L}(2, q)$

In this section we determine all G -primitive decompositions of $J(q+1, 2)$ where G is a 3-transitive subgroup of $\text{P}\Gamma\text{L}(2, q)$ for $q = p^f \geq 4$ with p a prime. The group $\text{PGL}(2, q)$ is the group of all fractional linear transformations

$$t_{a,b,c,d} : z \mapsto \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

of the projective line $X = \{\infty\} \cup \text{GF}(q)$ with the conventions $1/0 = \infty$ and $(a\infty + b)/(c\infty + d) = a/c$. Note that $t_{a,b,c,d} = t_{a',b',c',d'}$ if and only if $(a, b, c, d) = \lambda(a', b', c', d')$ for some $\lambda \neq 0$. The group $\text{PSL}(2, q)$ is then the set of all $t_{a,b,c,d}$ such that $ad - bc$ is a square in $\text{GF}(q)$. The Frobenius map $\phi : z \mapsto z^p$ also acts on X and $\phi^{-1}t_{a,b,c,d}\phi = t_{a^p,b^p,c^p,d^p}$. Then $\text{P}\Gamma\text{L}(2, q) = \langle \text{PGL}(2, q), \phi \rangle$. Another interesting family of subgroups of $\text{P}\Gamma\text{L}(2, q)$ occurs when p is odd and f is even. In this case we can define for each divisor s of $f/2$, the group $M(s, q) = \langle \text{PSL}(2, q), \phi^s t_{\xi,0,0,1} \rangle$, where ξ is a primitive element of $\text{GF}(q)$. Each $g \in \text{PGL}(2, q) \setminus \text{PSL}(2, q)$ can be written as $t_{\xi,0,0,1}h$ for some $h \in \text{PSL}(2, q)$, and so $\phi^s g \in M(s, q)$. It was shown in [18, Theorem 2.1] that G is a 3-transitive subgroup of $\text{P}\Gamma\text{L}(2, q)$ if and only if either G contains $\text{PGL}(2, q)$, or $G = M(s, q)$ for some s .

We begin with the following construction.

Construction 8.1. [11] Let $X = \{\infty\} \cup \text{GF}(q)$ be the projective line, $H = \text{PSL}(2, q)$ and $q \equiv 1 \pmod{4}$. Then H has two equal sized orbits on edges, namely $P_{\square} = \{\{\infty, 0\}, \{\infty, 1\}\}^H$, and $P_{\square} = \{\{\infty, 0\}, \{\infty, t\}\}^H$, with t not a square in $\text{GF}(q)$. Thus the partition $\mathcal{P} = \{P_{\square}, P_{\square}\}$ is a G -primitive decomposition of $J(q+1, 2)$ for any 3-transitive subgroup G of $\text{P}\Gamma\text{L}(2, q)$. The divisors are complementary spanning graphs Θ of valency $q-1$.

Proposition 8.2. *Let G be a 3-transitive subgroup of $\text{P}\Gamma\text{L}(2, q)$ and let \mathcal{P} be a G -primitive decomposition of $J(q+1, 2)$ such that $\text{PSL}(2, q)$ fixes a part. Then $q \equiv 1 \pmod{4}$ and \mathcal{P} is obtained from Construction 8.1.*

Proof. The graph $J(q+1, 2)$ contains $\frac{q(q^2-1)}{2}$ edges. For q even, $|\text{PSL}(2, q)| = q(q^2-1)$ and an edge stabiliser has order 2, so $\text{PSL}(2, q)$ is transitive on edges. Thus q is odd and so $|\text{PSL}(2, q)| = \frac{q(q^2-1)}{2}$. Whenever $(q-1)/2$ is odd, the stabiliser in $\text{PSL}(2, q)$ of a point of X has odd order. Since the stabiliser of the edge $\{\{x, y\}, \{x, z\}\}$ fixes x and interchanges y and z , it follows that no nontrivial element of $\text{PSL}(2, q)$ fixes an edge and so $\text{PSL}(2, q)$ is edge-transitive. Hence $(q-1)/2$ is even and $\text{PSL}(2, q)$ has two equal length orbits on edges, giving the G -primitive decomposition of Construction 8.1 for any 3-transitive subgroup G of $\text{P}\Gamma\text{L}(2, q)$. ■

To classify all G -primitive decompositions with G a 3-transitive subgroup of $\text{P}\Gamma\text{L}(2, q)$ we require knowledge of the maximal subgroups of all such G . First we note the following theorem.

Theorem 8.3. [19, Corollary 1.2] *Let $\text{PGL}(2, q) \leq G \leq \text{P}\Gamma\text{L}(2, q)$ and suppose that H is a maximal subgroup of G not containing $\text{PSL}(2, q)$. Then $H \cap \text{PGL}(2, q)$ is maximal in $\text{PGL}(2, q)$.*

Theorem 8.3 and Lemma 2.7 imply that we only need to find all $\text{PGL}(2, q)$ -primitive and all $M(s, q)$ -primitive decompositions. We now state all maximal subgroups of these two groups. The first is well known and follows from Dickson's classification [15] of subgroups of $\text{PSL}(2, q)$, see also [19].

Theorem 8.4. *Let $G = \text{PGL}(2, q)$ with $q \geq 4$ a power of the prime p . Then the maximal subgroups of G are:*

1. $[q] \rtimes C_{q-1}$.
2. $D_{2(q-1)}$, $q \neq 5$.
3. $D_{2(q+1)}$.

4. S_4 if $q = p \equiv \pm 3 \pmod{8}$.
5. $\text{PGL}(2, q_0)$ where $q = q_0^r$ with $q_0 > 2$, r is a prime and r is odd if q odd.
6. $\text{PSL}(2, q)$, q odd.

Theorem 8.5. [19, Theorem 1.5] *Let $G = M(s, q)$ with $q = p^f \geq 3$ for p odd and f even, and s a divisor of $f/2$. Then the maximal subgroups of G which do not contain $\text{PSL}(2, q)$ are:*

1. stabiliser of a point of the projective line,
2. $N_G(D_{q-1})$,
3. $N_G(D_{q+1})$,
4. $N_G(\text{PSL}(2, q_0))$ where $q = q_0^r$ with r an odd prime.

We require the following knowledge about the stabiliser of an edge.

Lemma 8.6. *Let $e = \{\{\infty, 0\}, \{\infty, 1\}\}$. Then*

1. $\text{PGL}(2, q)_e = \langle t_{-1,1,0,1} \rangle$,
2. $\text{PGL}(2, q)_e = \langle t_{-1,1,0,1}, \phi \rangle$ of order $2f$, and
3. $M(s, q)_e = \langle t_{-1,1,0,1}, \phi^{2s} \rangle$ of order f/s .

Proof. Since $\text{PGL}(2, q)$ is sharply 3-transitive, $\text{PGL}(2, q)_e = \langle g \rangle$ where g fixes ∞ and interchanges 0 and 1. Thus $\text{PGL}(2, q)_e$ is as in the lemma. Since ϕ fixes $\infty, 0$ and 1, the second claim follows. By [18, Corollary 2.2], $M(s, q)_{\infty,0,1} = \langle \phi^{2s} \rangle$ and since q is an even power of a prime we have $q \equiv 1 \pmod{4}$. Thus $t_{-1,1,0,1} \in \text{PSL}(2, q)$ and so $M(s, q)_e$ is as given by the lemma. \blacksquare

Instead of finding all maximal subgroups H containing the stabiliser of a fixed edge $\{A, B\}$ we solve the equivalent problem of choosing a representative H from each conjugacy class of maximal subgroups and finding all edges whose edge stabiliser is contained in H . See Remark 2.5.

Construction 8.7. Let $X = \{\infty\} \cup \text{GF}(q)$ be the projective line with q odd and let $H = \text{PGL}(2, q)_\infty = \text{AFL}(1, q)$. Let $e = \{\{0, 1\}, \{0, -1\}\}$. The stabiliser in $\text{PGL}(2, q)$ of e is $\langle \phi, t_{-1,0,0,1} \rangle$, which is contained in H . Moreover H is a maximal subgroup of $\text{PGL}(2, q)$. Thus by Lemma 2.4, letting

$$P = e^H = \left\{ \left\{ \{i, i+j\}, \{i, i-j\} \right\} \mid i, j \in \text{GF}(q), i \neq j \right\}$$

and $\mathcal{P} = P^{\text{PGL}(2, q)}$, we obtain a $\text{PGL}(2, q)$ -primitive decomposition of $J(q+1, 2)$. The divisors have valency 2 and hence are a union of cycles. Since $\text{GF}(q)$ has characteristic p it follows that each cycle has length p and so the divisors are isomorphic to $\frac{q(q-1)}{2p}C_p$. For any 3-transitive group G with socle $\text{PSL}(2, q)$, $H \cap G$ is maximal in G and so \mathcal{P} is G -primitive by Lemma 2.7.

Proposition 8.8. *Let $(J(q+1, 2), \mathcal{P})$ be a G -primitive decomposition with G a 3-transitive subgroup of $\text{PGL}(2, q)$ such that, for $P \in \mathcal{P}$, G_P is the stabiliser of a point of the projective line. Then either $\mathcal{P} = \mathcal{P}_\cap$ with divisors K_q or q is a power of an odd prime p and \mathcal{P} is obtained by Construction 8.7.*

Proof. Let $P \in \mathcal{P}$ and $\Gamma = J(q+1, 2)$. Then without loss of generality we may suppose that $H = G_P$ is the stabiliser of the point ∞ of $X = \{\infty\} \cup \text{GF}(q)$. We recall that G either contains $\text{PGL}(2, q)$ or is $M(s, q)$ for some s . Thus H acts 2-transitively on $\text{GF}(q)$ and so the orbits of H on $V\Gamma$ are $O_1 = \{\{\infty, x\} \mid x \in \text{GF}(q)\}$ and $O_2 = \{\{x, y\} \mid x, y \in \text{GF}(q)\}$. If $\{A, B\} \in P$ then H contains the stabiliser in G of $\{A, B\}$ and so either $\{A, B\} \subseteq O_1$ or $\{A, B\} \subseteq O_2$. Note that $P = \{A, B\}^H$.

Since H is 2-transitive on $\text{GF}(q)$ it follows that H acts transitively on the set of arcs between vertices of O_1 and so H contains the stabiliser in G of every edge between vertices of O_1 . Thus if $\{A, B\} \subseteq O_1$ then

$$\{A, B\}^H = \left\{ \left\{ \{\infty, x\}, \{\infty, y\} \right\} \mid x, y \in \text{GF}(q) \right\} \cong K_q.$$

Hence $\mathcal{P} = \mathcal{P}_\cap$.

Suppose now that $\{A, B\} \subseteq O_2$. We may suppose that $A = \{0, 1\}$ and $B = \{0, b\}$ for some $b \in \text{GF}(q) \setminus \{0, 1\}$. Let $g = t_{0,b,1-b,b} \in \text{PGL}(2, q)$. Then g maps $\infty \rightarrow 0 \rightarrow 1 \rightarrow b$ and so $G_{\{A, B\}} = G_{\{\infty, 0\}, \{\infty, 1\}}^g$ (this is obvious if G contains $\text{PGL}(2, q)$ and follows from the fact that $M(s, q) \triangleleft \text{PFL}(2, q)$ for $G = M(s, q)$). By Lemma 8.6, $t_{-1,1,0,1}^g \in G_{\{A, B\}} \leq H = G_\infty$, and since g does not fix ∞ and the only fixed points of $t_{-1,1,0,1}$ are ∞ and 2^{-1} (only if q is odd), it follows that q is odd and $g : 2^{-1} \rightarrow \infty$. This implies that $b = -1$. Hence ϕ^g fixes ∞ and so by Lemma 8.6, $G_{\{\{0,1\}, \{0,-1\}\}} \leq H$ in all cases. Hence \mathcal{P} is the decomposition of Construction 8.7. \blacksquare

8.1 D_{q-1} subgroups

Construction 8.9. Let $X = \{\infty\} \cup \text{GF}(q)$ be the projective line where $q = p^f$ for some odd prime p and let ξ be a primitive element of $\text{GF}(q)$. Then $\text{PFL}(2, q)_{\{0, \infty\}} = \langle t_{\xi, 0, 0, 1}, t_{0, 1, 1, 0}, \phi \rangle \cong D_{2(q-1)} \rtimes C_f$.

1. Let $H = \text{PFL}(2, q)_{\{0, \infty\}}$ and $e = \{\{0, 1\}, \{0, -1\}\}$. Then $t_{-1, 0, 0, 1} \in H$ interchanges the two vertices of e while ϕ fixes each of the vertices of e . Hence H contains the stabiliser in $\text{PFL}(2, q)$ of e and H is a maximal subgroup of $\text{PFL}(2, q)$ for $q \neq 5$. Thus by Lemma 2.4, letting

$$P = e^H = \left\{ \{ \{x, y\}, \{x, -y\} \} \mid x \in \{0, \infty\}, y \in \text{GF}(q) \setminus \{0\} \right\}$$

and $\mathcal{P} = P^{\text{PFL}(2, q)}$, we obtain a $\text{PFL}(2, q)$ -primitive decomposition of $J(q+1, 2)$. The divisors are isomorphic to $(q-1)K_2$ since the stabiliser of the vertex $\{0, 1\}$ in H is $\langle \phi \rangle$, which fixes $\{0, -1\}$. For any 3-transitive subgroup G of $\text{PFL}(2, q)$, we have $H \cap G$ is maximal in G and so \mathcal{P} is a G -primitive decomposition by Lemma 2.7.

2. Let $i < \frac{q-1}{2}$ and l be an integer such that ϕ^l fixes the set $\{\xi^i, \xi^{-i}\}$. Let $G = \langle \text{PGL}(2, q), \phi^l \rangle$ and $H = G_{\{\infty, 0\}} = \langle t_{\xi, 0, 0, 1}, t_{0, 1, 1, 0}, \phi^l \rangle$. The automorphism of $\text{PGL}(2, q)$ switching the vertices of the edge $e = \{\{1, \xi^i\}, \{1, \xi^{-i}\}\}$ is $t_{0, 1, 1, 0}$, while either ϕ^l or $t_{0, 1, 1, 0}\phi^l$ fixes both vertices of e . Hence $G_e < H$ and H is a maximal subgroup of G for $q \neq 5$. Hence by Lemma 2.4, letting

$$P = e^H = \left\{ \{ \{x, \xi^i x\}, \{x, \xi^{-i} x\} \} \mid x \in \text{GF}(q) \setminus \{0\} \right\}$$

and $\mathcal{P} = P^G$, we obtain a G -primitive decomposition of $J(q+1, 2)$. The divisors have valency 2 and hence are a union of cycles. These cycles have length the order of ξ^i , which is $\frac{q-1}{(q-1, i)}$. Thus each divisor is isomorphic to $(q-1, i)C_{\frac{q-1}{(q-1, i)}}$. In fact for any 3-transitive subgroup \overline{G} of G , $H \cap \overline{G}$ is maximal in \overline{G} and so \mathcal{P} is a \overline{G} -primitive decomposition.

Proposition 8.10. *Let $(J(q+1, 2), \mathcal{P})$ be a G -primitive decomposition such that $\text{PGL}(2, q) \leq G \leq \text{PFL}(2, q)$ and for $P \in \mathcal{P}$ we have $G_P = N_G(D_{2(q-1)})$. Then either $\mathcal{P} = \mathcal{P}_\ominus$, or q is odd and \mathcal{P} is obtained by Construction 8.9(1), or \mathcal{P} is obtained by Construction 8.9(2).*

Proof. Let $P \in \mathcal{P}$. Since $G_P \cap \text{PGL}(2, q)$ is a maximal subgroup of $\text{PGL}(2, q)$, by Lemma 2.7, \mathcal{P} is a $\text{PGL}(2, q)$ -primitive decomposition. Thus we may suppose that $G = \text{PGL}(2, q)$ and $H = G_P = \langle t_{\xi, 0, 0, 1}, t_{0, 1, 1, 0} \rangle \cong D_{2(q-1)}$. The orbits of H on vertices are $\{\{0, \infty\}\}$,

$$O_0 = \{ \{x, y\} \mid x \in \{0, \infty\}, y \in \text{GF}(q) \setminus \{0\} \}$$

and

$$O_i = \{ \{x, \xi^i x\} \mid x \in \text{GF}(q) \setminus \{0\} \}$$

for each $i \leq \frac{q-1}{2}$. Note that $|O_0| = 2(q-1)$. When q is even there are $q/2 - 1$ orbits O_i , each having length $q-1$. When q is odd there are $\frac{q-3}{2}$ of length $q-1$ and one, $O_{\frac{q-1}{2}}$, of length $\frac{q-1}{2}$.

If $\{A, B\} \in P$ then H contains the stabiliser in G of $\{A, B\}$ and so $\{A, B\}$ is contained in one of the orbits of H on vertices. Note that $P = \{A, B\}^H$.

Suppose first that $\{A, B\} \subseteq O_0$. Without loss, let $A = \{0, 1\}$. Then the neighbours of A in O_0 are $\{\infty, 1\}$ and $\{0, y\}$ such that $y \in \text{GF}(q) \setminus \{0\}$. The only ones which can be interchanged with A by an element of H are $\{\infty, 1\}$, by $t_{0, 1, 1, 0}$ and $\{0, -1\}$, by $t_{-1, 0, 0, 1}$, when q is odd. Thus the only edges

between vertices of O_0 whose stabiliser in G is contained in H are those in the orbits $\{A, \{\infty, 1\}\}^H$ and $\{A, \{0, -1\}\}^H$. The first gives the matching $\{\{\{0, y\}, \{\infty, y\}\} \mid y \in \text{GF}(q) \setminus \{0\}\}$ and hence the decomposition \mathcal{P}_\ominus while the second gives the matching $\{\{\{x, y\}, \{x, -y\}\} \mid x \in \{0, \infty\}, y \in \text{GF}(q) \setminus \{0\}\}$ and hence Construction 8.9(1). Both matchings have $q - 1$ edges and the second only occurs for q odd. Note also that both orbits are preserved by $\text{P}\Gamma\text{L}(2, q)_{\{0, \infty\}}$ and so both decompositions are also $\text{P}\Gamma\text{L}(2, q)$ -decompositions.

Note that when q is odd the orbit $O_{\frac{q-1}{2}}$ contains no edges. Thus suppose next that $\{A, B\} \subseteq O_i$ for $i < \frac{q-1}{2}$. Without loss of generality, let $A = \{1, \xi^i\}$. Then the neighbours of A in O_i are $\{1, \xi^{-i}\}$ and $\{\xi^i, \xi^{2i}\}$ and these are interchanged by $H_A = \langle t_{0, \xi^i, 1, 0} \rangle \cong C_2$. Hence H acts transitively on the set of edges between vertices of O_i . Moreover, $\langle t_{0, 1, 1, 0} \rangle$ is the stabiliser H of the edge $\{\{1, \xi^i\}, \{1, \xi^{-i}\}\}$ and so H contains the stabiliser in G of an edge between two vertices of O_i . Thus \mathcal{P} is obtained by Construction 8.9(2). Moreover, an overgroup $\overline{G} = \langle \text{PGL}(2, q), \phi^l \rangle$ of $\text{PGL}(2, q)$ in $\text{P}\Gamma\text{L}(2, q)$ preserves \mathcal{P} if and only if $\overline{G}_{\{0, \infty\}} = \langle H, \phi^l \rangle$ fixes O_i . Since ϕ^l fixes 1, it follows that ϕ^l fixes O_i if and only if ϕ^l fixes $\{\xi^i, \xi^{-i}\}$ and so \overline{G} is as stated in Construction 8.9(2). \blacksquare

Construction 8.11. Let $G = M(s, q)$ and ξ be a primitive element of $\text{GF}(q)$ with $q = p^f$ for some odd prime p and even integer f . Let i be an integer and assume that either

- $s = f/2$ and $(\xi^i)^{\langle \phi^s \rangle}$ has length 2 and does not contain ξ^{-i} , or
- $s = f/4$ and $(\xi^i)^{\langle \phi^s \rangle}$ has length 4 and does contain ξ^{-i} .

Let $H = G_{\{0, \infty\}} = \langle \text{PSL}(2, q)_{\{0, \infty\}}, \phi^s t_{\xi, 0, 0, 1} \rangle$ and note that $\text{PSL}(2, q)_{\{0, \infty\}} = \langle t_{\xi^2, 0, 0, 1}, t_{0, 1, 1, 0} \rangle$.

1. Suppose that i is even and let $e = \{\{1, \xi^i\}, \{1, \xi^{-i}\}\}$ and $P = e^H$. Then

$$P = \left\{ \left\{ \{x^2, x^2 \xi^i\}, \{x^2, x^2 \xi^{-i}\} \right\} \mid x \in \text{GF}(q) \setminus \{0\} \right\} \\ \cup \left\{ \left\{ \{y, y \xi^{ip^s}\}, \{y, y \xi^{-ip^s}\} \right\} \mid y = \not\equiv \right\}$$

Then P has valency 2 (as the two neighbours of $\{1, \xi^i\}$ are $\{1, \xi^{-i}\}$ and $\{\xi^i, \xi^{2i}\}$) and so is a union of cycles. Each cycle has length the order of ξ^i and so $P \cong (q - 1, i)C_{\frac{q-1}{(q-1, i)}}$.

Now $|\{1, \xi^i\}^H| = q - 1$ and by Lemma 8.6, $|G_e| = f/s$. Since $|H| = (q - 1)f/s$ it follows that $|H_e| = f/s$ and so $H_e = G_e$. Hence by Lemma 2.4 and the fact that H is maximal in G , letting $\mathcal{P} = P^G$ we get that \mathcal{P} is a G -primitive decomposition.

2. Suppose now that i is odd and let $e = \{\{1, \xi^i\}, \{1, \xi^{-i}\}\}$ and $P = e^H$. Then

$$P = \left\{ \left\{ \{x^2, x^2 \xi^i\}, \{x^2, x^2 \xi^{-i}\} \right\} \mid x \in \text{GF}(q) \setminus \{0\} \right\} \\ \cup \left\{ \left\{ \{y, y \xi^{ip^s}\}, \{y, y \xi^{-ip^s}\} \right\} \mid y = \not\equiv \right\}$$

Then $|P| = q - 1$ and so $|H_e| = f/s = |G_e|$, by Lemma 8.6. The only neighbour of $\{1, \xi^i\}$ in P is $\{1, \xi^{-i}\}$ and so $P = (q - 1)K_2$. By Lemma 2.4 and the fact that H is maximal in G , letting $\mathcal{P} = P^G$ we get that \mathcal{P} is a G -primitive decomposition.

Proposition 8.12. Let $(J(q + 1, 2), \mathcal{P})$ be a G -primitive decomposition with $G = M(s, q)$ for some s such that for $P \in \mathcal{P}$, $G_P = N_G(D_{q-1})$. Then either $\mathcal{P} = \mathcal{P}_\ominus$, or \mathcal{P} arises from Construction 8.9(1), 8.9(2) or 8.11.

Proof. A subgroup $N_G(D_{q-1})$ of G is a pair-stabiliser in G . Without loss of generality we may suppose that $H = G_{\{0, \infty\}} = \langle \text{PSL}(2, q)_{\{0, \infty\}}, \phi^s t_{\xi, 0, 0, 1} \rangle$. Note that $q \equiv 1 \pmod{4}$ and so $\text{PSL}(2, q)_{\{0, \infty\}} = \langle t_{\xi^2, 0, 0, 1}, t_{0, 1, 1, 0} \rangle$. Since G is 3-transitive it follows that

$$O_0 = \{\{x, y\} \mid x \in \{0, \infty\}, y \in \text{GF}(q) \setminus \{0\}\}$$

is an H -orbit on vertices and as in the proof of Lemma 8.10, if $\{A, B\} \subset O_0$ is an edge whose stabiliser in G is contained in H we obtain either $\mathcal{P} = \mathcal{P}_\ominus$ or \mathcal{P} is obtained by Construction 8.9(1).

Now suppose $\{A, B\} \not\subset O_0$. Since H is transitive on $\text{GF}(q) \setminus \{0\}$, we can assume that $A = \{1, \xi^i\}$ where $1 \leq i \leq q-2$ and that $A \cap B = \{1\}$, say $B = \{1, t\}$. We need to find the neighbours B of A such that $G_{\{A, B\}} \leq H$. Let $g \in \text{PGL}(2, q)$ map $\{\{\infty, 0\}, \{\infty, 1\}\}$ onto $\{A, B\}$. Then $G_{\{A, B\}} = \langle t_{-1,1,0,1}, \phi^{2s} \rangle^g$ by Lemma 8.6. Hence $t_{-1,1,0,1}$ and ϕ^{2s} must stabilise $\{0, \infty\}^{g^{-1}}$. Note that $\infty^g \neq \infty$ (since $\infty \notin A$) and $\infty^g \neq 0$ (since $0 \notin A$).

Since $B = \{1, t\}$, we can take $g = t_{a, \xi^i, a, 1}$ where $a = \frac{\xi^i - t}{t-1}$, and then $\{0, \infty\}^{g^{-1}} = \{-\frac{\xi^i}{a}, -\frac{1}{a}\}$. Recall that $t_{-1,1,0,1}$ stabilises this set. Now $t_{-1,1,0,1}$ fixes only the points $\infty, 2^{-1}$, and if $\{0, \infty\}^{g^{-1}} = \{\infty, 2^{-1}\}$ we would have $\infty^g \in \{0, \infty\}$ which is not the case. Hence $t_{-1,1,0,1}$ interchanges $-\frac{\xi^i}{a}$ and $-\frac{1}{a}$. Thus $-\frac{\xi^i}{a} = 1 + \frac{1}{a}$, that is $a = -1 - \xi^i = \frac{\xi^i - t}{t-1}$, and so $t = \xi^{-i}$. For this value of t , $\{0, \infty\}^{g^{-1}} = \{\frac{\xi^i}{\xi^i+1}, \frac{1}{\xi^i+1}\}$ and this set is stabilised by $t_{-1,1,0,1}$ and ϕ^{2s} . The equality $\{\frac{\xi^i}{1+\xi^i}, \frac{1}{1+\xi^i}\}^{\phi^{2s}} = \{\frac{\xi^i}{1+\xi^i}, \frac{1}{1+\xi^i}\}$, is equivalent to either $\frac{\xi^{ip^{2s}}}{1+\xi^{ip^{2s}}} = \frac{\xi^i}{1+\xi^i}$ and $\frac{1}{1+\xi^{ip^{2s}}} = \frac{1}{1+\xi^i}$, or $\frac{\xi^{ip^{2s}}}{1+\xi^{ip^{2s}}} = \frac{1}{1+\xi^i}$ and $\frac{1}{1+\xi^{ip^{2s}}} = \frac{\xi^i}{1+\xi^i}$. In the first case $\xi^{ip^{2s}} = \xi^i$; in the second case $\xi^{ip^{2s}} = \xi^{-i}$. That means $O = (\xi^i)^{\langle \phi^s \rangle}$ has length 1, 2 or 4.

Set $e = \{A, \{1, \xi^{-1}\}\}$. If O has length 1, or O has length 2 and $(\xi^i)^{\phi^s} = \xi^{-i}$, then e^H yields a decomposition in Construction 8.9(2). If O has length 2 and $(\xi^i)^{\phi^s} \neq \xi^{-i}$, or O has length 4 and $\xi^{ip^{2s}} = \xi^{-i}$, then e^H yields a decomposition Construction 8.11(1) if i is even and in Construction 8.11(2) if i is odd. \blacksquare

8.2 D_{q+1} subgroups

Before dealing with the case where $H \cap \text{PSL}(2, q) = D_{q+1}$ we need a new model for the group action. Let $K = \text{GF}(q^2)$ for $q = p^f$ with primitive element ξ , and let $F = \{0\} \cup \{(\xi^{q+1})^l \mid l = 0, 1, \dots, q-2\} \cong \text{GF}(q)$. The element ξ acts on K by multiplication and induces an F -linear map. Moreover, under the induced action of F , K is a 2-dimensional vector space over F . The field automorphism φ of K of order $2f$ mapping each element of K to its p^{th} power is F -semilinear, that is, φ preserves addition and for each $x \in K$, $\lambda \in F$, we have $(\lambda x)^\varphi = \lambda^p x^\varphi$. Then $\text{GL}(2, q) = \langle \text{GL}(2, q), \varphi \rangle$. Note that φ^f is an F -linear map so $\varphi^f \in \text{GL}(2, q)$.

We can identify the projective line X on which $\text{PGL}(2, q)$ acts with the elements of K modulo F , that is, $X = \{\xi^i F \mid i = 0, 1, \dots, q\}$. Then $\text{PTL}(2, q) = \langle \text{PGL}(2, q), \varphi \rangle$. Multiplication by ξ induces the map $\hat{\xi}$ of order $q+1$ and $\langle \hat{\xi} \rangle$ is normalised by φ . Moreover, for each i , $(\xi^i F)^{\varphi^f} = \xi^{iq} F = \xi^{-i} F$ and so φ^f inverts $\hat{\xi}$. Hence $\langle \hat{\xi}, \varphi^f \rangle \cong D_{2(q+1)}$.

Construction 8.13. Let X be the projective line modelled as above. Let $1 \leq i < \frac{q+1}{2}$ and $e = \{\{1F, \xi^i F\}, \{1F, \xi^{-i} F\}\}$ and let s be a positive integer dividing f such that $\langle \varphi^s \rangle$ has $\{\xi^i F, \xi^{-i} F\}$ as an orbit on X . Let $G = \langle \text{PGL}(2, q), \varphi^s \rangle$ and $H = \langle \hat{\xi}, \varphi^s \rangle \cong C_{q+1} \rtimes C_{2f/s}$. Now $\langle \varphi^s \rangle$ fixes e and has order $2f/s$, which by Lemma 8.6 is the order of G_e . Hence $G_e < H$ and H is a maximal subgroup of G . Thus by Lemma 2.4, letting

$$P = e^H = \left\{ \{ \{x F, x \xi^i F\}, \{x F, x \xi^{-i} F\} \} \mid x \in \text{GF}(q) \setminus \{0\} \right\}$$

and $\mathcal{P} = P^G$, we obtain a G -primitive decomposition of $J(q+1, 2)$. The divisors have valency 2 and hence are unions of cycles. These cycles have length the order of $\xi^i F$, which is $\frac{q+1}{(q+1, i)}$. Thus each divisor is isomorphic to $(q+1, i) C_{\frac{q+1}{(q+1, i)}}$.

Proposition 8.14. Let $(J(q+1, 2), \mathcal{P})$ be a G -primitive decomposition such that $\text{PGL}(2, q) \leq G \leq \text{PTL}(2, q)$ and, for $P \in \mathcal{P}$, $G_P = N_G(D_{2(q+1)})$. Then \mathcal{P} is obtained by Construction 8.13.

Proof. Since $\text{PTL}(2, q) = \langle \text{PGL}(2, q), \varphi \rangle$ and $\varphi^f \in \text{PGL}(2, q)$ we have $G = \langle \text{PGL}(2, q), \varphi^s \rangle$ for some s dividing f . Let $L = \langle \hat{\xi}, \varphi^f \rangle \cong D_{2(q+1)}$. Then $N_G(L) = \langle \hat{\xi}, \varphi^s \rangle \cong C_{q+1} \rtimes C_{2f/s}$ and we may assume that $H = G_P = N_G(L)$. Let $e \in \mathcal{P}$. Since H is transitive on X we may also assume that $e = \{\{1F, \xi^i F\}, \{1F, \xi^j F\}\}$ for some integers i and j . Since $H_{1F} = \langle \varphi^s \rangle$ and by Lemma 8.6, $|G_e| = 2f/s$, it follows that $G_e \leq H$ if and only if $\langle \varphi^s \rangle$ has $\{\xi^i F, \xi^j F\}$ as an orbit on X . Since $\varphi^f \in \langle \varphi^s \rangle$ and maps $\xi^i F$ to $\xi^{-i} F$ it follows that $j = -i$. Since $\xi^{-i} F = \xi^{q+1-i} F$ we may assume that $1 \leq i \leq (q+1)/2$.

Moreover, if $i = (q+1)/2$ then q is odd and $\xi^{-(q+1)/2}F = \xi^{(q+1)/2}F$. Thus we may further assume that $1 \leq i < (q+1)/2$. Hence \mathcal{P} arises from Construction 8.13. \blacksquare

Next we need the following lemma about the normaliser in $M(s, q)$ of a subgroup D_{q+1} in $\text{PSL}(2, q)$.

Lemma 8.15. *Suppose $q = p^f$ where f is even and p is an odd prime. Let $L = \langle \hat{\xi}, \varphi^f \rangle \cap \text{PSL}(2, q)$ and $G = M(s, q)$ for some divisor s of $f/2$. Then*

1. $L = \langle \hat{\xi}^2, \varphi^f \rangle \cong D_{q+1}$.
2. If $p \equiv 1 \pmod{4}$ or s is even then $N_G(L) = \langle \hat{\xi}^2, \varphi^s \hat{\xi} \rangle$, and is transitive on the projective line.
3. If $p \equiv 3 \pmod{4}$ and s is odd then $N_G(L) = \langle \hat{\xi}^2, \varphi^s \rangle$, and has two equal sized orbits on the projective line.

Proof. Now $\{1, \xi^{(q+1)/2}\}$ is a basis for K over F . Define $\phi : K \rightarrow K$ such that, for all $\lambda_1, \lambda_2 \in F$, $(\lambda_1 + \lambda_2 \xi^{(q+1)/2})^\phi = \lambda_1^p + \lambda_2^p \xi^{(q+1)/2}$. Then $\Gamma L(2, q) = \langle \text{GL}(2, q), \phi \rangle$. Since also $\Gamma L(2, q) = \langle \text{GL}(2, q), \varphi \rangle$, we must have $\varphi = \phi g$ for some $g \in \text{GL}(2, q)$. Since φ and ϕ fix 1, so does g . Moreover, ϕ fixes $\xi^{(q+1)/2}$ while $(\xi^{(q+1)/2})^\varphi = \xi^{p(q+1)/2} = \xi^{\frac{(p-1)(q+1)}{2}} \xi^{\frac{q+1}{2}}$. Note that $\xi^{\frac{(p-1)(q+1)}{2}} \in F$ and so $\xi^{(q+1)/2}$ is an eigenvector for g . Thus with respect to the basis $\{1, \xi^{(q+1)/2}\}$, the element g is represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \xi^{\frac{(p-1)(q+1)}{2}} \end{pmatrix},$$

and $\det(g) = \xi^{\frac{(p-1)(q+1)}{2}}$ is a square in $\text{GF}(q)$ if and only if $p \equiv 1 \pmod{4}$. Furthermore, φ^f is represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that an element of $\text{GL}(2, q)$ induces an element of $\text{PSL}(2, q)$ if and only if its determinant is a $\text{GF}(q)$ -square. Since $q \equiv 1 \pmod{4}$ it follows that $\varphi^f \in \text{PSL}(2, q)$. Now $\langle \hat{\xi}^2 \rangle \cong C_{(q+1)/2}$ and $\hat{\xi}^2 \in \text{PSL}(2, q)$, and since φ^f inverts $\hat{\xi}$ it also inverts $\hat{\xi}^2$. Hence L is as in part (1) of the lemma. Moreover, L has two orbits on the projective line X , these being $\{1F, \xi^2 F, \dots, \xi^{q-1} F\}$ and $\{\xi F, \xi^3 F, \dots, \xi^q F\}$.

Now $\varphi = \phi g$ and $g \in \text{PSL}(2, q)$ if and only if $p \equiv 1 \pmod{4}$. By definition it follows that $G = M(s, q) = \langle \text{PSL}(2, q), \phi^s t \rangle$ for any $t \in \text{PGL}(2, q) \setminus \text{PSL}(2, q)$. Suppose first that $p \equiv 1 \pmod{4}$. Then $\varphi = \phi g$ with $g \in \text{PSL}(2, q)$ and so $G = \langle \text{PSL}(2, q), \varphi^s \hat{\xi} \rangle$. When $p \equiv 3 \pmod{4}$ we have $\varphi = \phi g$ with $g \in \text{PGL}(2, q) \setminus \text{PSL}(2, q)$. Thus for odd s we have $G = \langle \text{PSL}(2, q), \varphi^s \rangle$ while for even s we have $G = \langle \text{PSL}(2, q), \varphi^s \hat{\xi} \rangle$. Now $(\varphi^f)^{\varphi^s \hat{\xi}} = (\varphi^f)^{\hat{\xi}} = \varphi^f \hat{\xi}^{-p^s+1} \in L$. Hence for $p \equiv 1 \pmod{4}$ or s even we have $N_G(L) = \langle \hat{\xi}^2, \varphi^s \hat{\xi} \rangle$. Since $\varphi^s \hat{\xi}$ interchanges the two L -orbits on X , $N_G(L)$ is transitive on X and so we have proved part (2). For $p \equiv 3 \pmod{4}$ and s odd we have $N_G(L) = \langle \hat{\xi}^2, \varphi^s \rangle$. Since φ^s fixes each L -orbit it follows that $N_G(L)$ has two orbits and the proof is complete. \blacksquare

Construction 8.16. Let $q = p^f$ where p is odd and f even and let $G = M(s, q)$ for some divisor s of $f/2$. Suppose that either $p \equiv 1 \pmod{4}$ or s is even. Let $1 \leq i < (q+1)/2$ such that $\langle \varphi^{2s} \rangle$ has $\{\xi^i F, \xi^{-i} F\}$ as an orbit on X . Let $H = \langle \hat{\xi}^2, \varphi^s \hat{\xi} \rangle$ and $e = \{\{1F, \xi^i F\}, \{1F, \xi^{-i} F\}\}$. Now $\langle \varphi^{2s} \rangle$ fixes e , lies in G , and has order f/s . Since this is the same order as G_e (Lemma 8.6) it follows that $G_e < H$. Hence by Lemma 2.4, letting $P = e^H$ and $\mathcal{P} = P^G$ we obtain a G -primitive decomposition.

1. Suppose first that i is even. Then $H_{\{1F, \xi^i F\}} = \langle \varphi^f \hat{\xi}^i, \varphi^{4s} \rangle$ whose orbit containing $\{1F, \xi^{-i} F\}$ is $\{\{1F, \xi^{-i} F\}, \{\xi^i F, \xi^{2i} F\}\}$. Thus P has valency 2 and so is a union of cycles of length the order of $\hat{\xi}^i$, that is, $P \cong (q+1, i)C_{\frac{q+1}{(q+1, i)}}$.
2. Suppose now that i is odd. An element of H mapping $1F$ to $\xi^i F$ is of the form $h = \varphi^{st} \hat{\xi}^i$ with t odd. Since $\langle \varphi^{2s} \rangle$ has $\{\xi^i F, \xi^{-i} F\}$ as an orbit on X , we have that h maps $\xi^i F$ onto $\xi^{i(1+p^s)} F$ or onto $\xi^{i(1-p^s)} F$, according as $t \equiv 1$ or $3 \pmod{4}$ respectively. Hence, for h to map $\xi^i F$ onto $1F$, we need $q+1$ to divide $i(1+p^s)$ or $i(1-p^s)$ respectively. Since $p^{2s} - 1$ divides $p^f - 1 = q - 1$, it follows that $\gcd(q+1, p^s+1) = 2$ and $\gcd(q+1, p^s-1) = 2$, and so $\frac{q+1}{2}$ must divide i , which is not possible since $1 \leq i < \frac{q+1}{2}$. Thus $(\xi^i F)^h \neq 1F$. Hence $H_{\{1F, \xi^i F\}} = H_{1F, \xi^i F} = \langle \varphi^{4s} \rangle$, which also fixes $\xi^{-i} F$ and hence fixes e . Thus P is a matching with $q+1$ edges.

Construction 8.17. Let $p \equiv 3 \pmod{4}$ and let $G = M(s, q)$ for $q = p^f$ and s an odd divisor of $f/2$. Let $1 \leq i < (q+1)/2$ such that $\langle \varphi^{2s} \rangle$ has $\{\xi^i F, \xi^{-i} F\}$ as an orbit on X . Let $H = \langle \hat{\xi}^2, \varphi^s \rangle$ and $e = \{\{1F, \xi^i F\}, \{1F, \xi^{-i} F\}\}$. Now $\langle \varphi^{2s} \rangle$ fixes e , lies in G and has order f/s . Since this is the same order as G_e (Lemma 8.6) it follows that $G_e < H$ and so by Lemma 2.4, letting $P = e^H$ and $\mathcal{P} = P^G$, we obtain a G -primitive decomposition.

1. Suppose first that i is even. Then $H_{\{1F, \xi^i F\}} = \langle \varphi^f \hat{\xi}^i, \varphi^{4s} \rangle$ and the H -orbit containing $\{1F, \xi^{-i} F\}$ has length 2. Thus P is a union of cycles of length the order of $\hat{\xi}^i$, so $P \cong (q+1, i)C_{\frac{q+1}{(q+1, i)}}$.
2. If i is odd then $1F$ and $\xi^i F$ lie in different H -orbits and so $H_{\{1F, \xi^i F\}} = H_{1F, \xi^i F} = \langle \varphi^{4s} \rangle$ which also fixes $\xi^{-i} F$ and hence fixes e . Thus P is a matching with $q+1$ edges.

Construction 8.18. Let $p \equiv 3 \pmod{4}$ and let $G = M(s, q)$ for $q = p^f$ and s an odd divisor of $f/2$. Let $1 \leq i < \frac{q+1}{2}$ such that $\langle \hat{\xi}^{-1} \varphi^{2s} \hat{\xi} \rangle$ has $\{\xi^{i+1} F, \xi^{-i+1} F\}$ as an orbit on X . Let $H = \langle \hat{\xi}^2, \varphi^s \rangle$ and $e = \{\{\xi F, \xi^{i+1} F\}, \{\xi F, \xi^{-i+1} F\}\}$. Now $\langle \hat{\xi}^{-1} \varphi^{2s} \hat{\xi} \rangle \leq H$, fixes e , and has the same order as G_e . Thus $G_e < H$ and so by Lemma 2.4, letting $P = e^H$ and $\mathcal{P} = P^G$, we obtain a G -primitive decomposition.

1. Suppose first that i is odd. Then ξF and $\xi^{i+1} F$ lie in different H -orbits. Hence $H_{\{\xi F, \xi^{i+1} F\}} = H_{\xi F, \xi^{i+1} F} = \langle \hat{\xi}^{-1} \varphi^{4s} \hat{\xi} \rangle$ which also fixes $\xi^{-i+1} F$ and so P is a matching with $q+1$ edges.
2. If i is even then $\varphi^f \hat{\xi}^{i+2} \in H$ interchanges ξF and $\xi^{i+1} F$, and so $H_{\{\xi F, \xi^{i+1} F\}} = \langle \hat{\xi}^{-1} \varphi^{4s} \hat{\xi}, \varphi^f \hat{\xi}^{i+2} \rangle$, whose orbit containing $\{\xi F, \xi^{-i+1} F\}$ has size 2. Hence P is a union of cycles of length the order of $\hat{\xi}^i$. Thus $P = (q+1, i)C_{\frac{q+1}{(q+1, i)}}$.

Proposition 8.19. Let \mathcal{P} be an $M(s, q)$ -primitive decomposition of $J(q+1, 2)$ with divisor stabiliser $N_{M(s, q)}(D_{q+1})$. Then \mathcal{P} can be obtained from Construction 8.16, 8.17 or 8.18.

Proof. Let $G = M(s, q)$ and suppose first that $q = p^f$ where $p \equiv 1 \pmod{4}$ or s is even. We may assume that $H = \langle \hat{\xi}^2, \varphi^s \hat{\xi} \rangle$ by Lemma 8.15. Let $e \in \mathcal{P} \in \mathcal{P}$. By Lemma 8.15 again, H is transitive on X and so we can assume that $e = \{\{1F, \xi^i F\}, \{1F, \xi^j F\}\}$ for some i and j . Now $H_{1F} = \langle \varphi^{2s} \rangle$, which has order f/s . By Lemma 8.6, this is the same order as G_e . Hence $G_e < H$ if and only if $H_{1F} = G_e$, which holds if and only if $\{\xi^i F, \xi^j F\}$ is an orbit of $\langle \varphi^{2s} \rangle$. Since $\varphi^f \in \langle \varphi^{2s} \rangle$ and maps $\xi^i F$ to $\xi^{-i} F$ it follows that $j = -i$ and we may assume as before that $1 \leq i < (q+1)/2$. Thus \mathcal{P} comes from Construction 8.16.

Suppose now that $p \equiv 3 \pmod{4}$ and s is odd. Then by Lemma 8.15, we may assume that $H = \langle \hat{\xi}^2, \varphi^s \rangle$. Let $e \in \mathcal{P} \in \mathcal{P}$. By Lemma 8.15, H has 2 orbits on X and so we may assume that $e = \{\{1F, \xi^i F\}, \{1F, \xi^j F\}\}$ or $\{\{\xi F, \xi^{i+1} F\}, \{\xi F, \xi^{j+1} F\}\}$. Suppose that e is the first edge. Now $H_{1F} = \langle \varphi^s \rangle$ which has order $2f/s$ while G_e has order f/s by Lemma 8.6. Since H_{1F} has a unique subgroup of order f/s it follows that $G_e < H$ if and only if $G_e = \langle \varphi^{2s} \rangle$, that is, if and only if $\langle \varphi^{2s} \rangle$ has $\{\xi^i F, \xi^j F\}$ as an orbit on X . Since $\varphi^f \in \langle \varphi^{2s} \rangle$ we have $j = -i$ and may assume $1 \leq i < (q+1)/2$. It follows that \mathcal{P} is as constructed in Construction 8.17. If on the other hand $e = \{\{\xi F, \xi^{i+1} F\}, \{\xi F, \xi^{j+1} F\}\}$, then $H_{\xi F} = \langle \hat{\xi}^{-1} \varphi^s \hat{\xi} \rangle$ which has order $2f/s$. Its only index two subgroup is $\langle \hat{\xi}^{-1} \varphi^{2s} \hat{\xi} \rangle$ and so by order arguments again this must have $\{\xi^{i+1} F, \xi^{j+1} F\}$ as an orbit. Since $\hat{\xi}^{-1} \varphi^f \hat{\xi} \in \langle \hat{\xi}^{-1} \varphi^{2s} \hat{\xi} \rangle$ and maps $\xi^{i+1} F$ to $\xi^{-i+1} F$ it follows that $j = -i$. Once again we have $1 \leq i < \frac{q+1}{2}$. Hence \mathcal{P} is as given by Construction 8.18. \blacksquare

8.3 S_4 -subgroups

First we have the following lemma on the orbit lengths of a subgroup S_4 of $\text{PGL}(2, q)$ which we have adapted from [8].

Lemma 8.20. [8, Lemma 10] Let $q = p \equiv \pm 3 \pmod{8}$, $q > 3$, $G = \text{PGL}(2, q)$ acting on the projective line X , and H a subgroup of G isomorphic to S_4 . Then H has the following orbits of length less than 24 on X .

1. If $q \equiv 5 \pmod{24}$, then H has one orbit of length 6.

2. If $q \equiv 11 \pmod{24}$, then H has one orbit of length 12.
3. If $q \equiv 13 \pmod{24}$, then H has one orbit of length 6 and one of length 8.
4. If $q \equiv 19 \pmod{24}$, then H has one orbit of length 8 and one of length 12.

Construction 8.21. Let $X = \{\infty\} \cup \text{GF}(q)$ be the projective line.

1. Let $q \equiv \pm 3 \pmod{8}$ be a prime ($q > 3$) and $H = S_4$. Choose $x, y_1, y_2 \in X$ such that $(|x^H|, |y_1^H|) = (6, 8), (6, 24), (12, 8)$ or $(12, 24)$, and there exists in H_x an element switching y_1 and y_2 . Let $P = \{\{x, y_1\}, \{x, y_2\}\}^H$ and $\mathcal{P} = P^{\text{PGL}(2, q)}$. Then by Lemma 2.4, $(J(q+1, 2), \mathcal{P})$ is a $\text{PGL}(2, q)$ -primitive decomposition. Since $|\{x, y_1\}^H| = 24$, the stabiliser in H of $\{x, y_1\}$ is trivial. Hence the divisors are isomorphic to $12K_2$.
2. Let $q \equiv 5 \pmod{8}$ be a prime and $H = S_4$. Let $P = \{\{x, y_1\}, \{x, y_2\}\}^H$ where x, y_1, y_2 all lie in an H -orbit of length 6 and there exists in H_x an element switching y_1 and y_2 . By Lemma 8.20, there is a unique orbit of O_6 of length 6. The group H acts imprimitively on O_6 with blocks of size 2, and $H_x \cong C_4$ contains an element interchanging y_1, y_2 if and only if $\{y_1, y_2\}$ is a block not containing x . Moreover, $P \cong 3C_4$. Let $\mathcal{P} = P^{\text{PGL}(2, q)}$. Then by Lemma 2.4 $(J(q+1, 2), \mathcal{P})$, is a $\text{PGL}(2, q)$ -primitive decomposition.
3. Let $q \equiv 3 \pmod{8}$ be a prime and $H = S_4$. Let $P = \{\{x, y_1\}, \{x, y_2\}\}^H$ where x, y_1, y_2 all lie in an H -orbit of length 12 and there exists in H_x an element switching y_1 and y_2 . By Lemma 8.20, there is a unique orbit O_{12} of length 12. We can see this action as S_4 acting on ordered pairs, denoted by $[a, b]$. Then for $x = [1, 2] \in O_{12}$, H_x is the transposition $(3, 4)$ in S_4 . It fixes one remaining point of O_{12} , namely $[2, 1]$ and interchanges the 5 pairs $\{[2, 3], [2, 4]\}, \{[3, 1], [4, 1]\}, \{[1, 3], [1, 4]\}, \{[3, 2], [4, 2]\},$ and $\{[3, 4], [4, 3]\}$. If we take $\{y_1, y_2\}$ as in the first two cases, then the stabiliser in H of $\{x, y_1\}$ is trivial and so we get a matching $12K_2$ in each case. In the last three cases, the stabiliser in H of $\{x, y_1\}$ has order 2, and we get unions of cycles. It is easy to see that in the third and fourth case, we get $4C_3$, while in the last case we get $3C_4$. Let $\mathcal{P} = P^{\text{PGL}(2, q)}$. Then by Lemma 2.4, $(J(q+1, 2), \mathcal{P})$ is a $\text{PGL}(2, q)$ -primitive decomposition.

Proposition 8.22. Let $(J(q+1, 2), \mathcal{P})$ be a G -primitive decomposition with $G = \text{PGL}(2, q)$ for $q = p \equiv \pm 3 \pmod{8}$ with $q \geq 5$ and given $P \in \mathcal{P}$ we have $G_P \cong S_4$. Then P is obtained by Construction 8.21(1), (2) or (3).

Proof. Let $P \in \mathcal{P}$ and $H = G_P \cong S_4$. If $\{x, y\} \subseteq X$ with x and y in different H -orbits of length 24 then $|\{x, y\}^H| = 24$ and that orbit contains no edges of $J(q+1, 2)$. Thus if x and y come from different H -orbits O_1 and O_2 respectively, we may assume by Lemma 8.20, that $|O_1| < |O_2|$ and so $\{x, y\}^H$ has length $\text{lcm}(|O_1|, |O_2|)$ and contains edges. Moreover, H contains the stabiliser in G of such an edge $\{\{x, y_1\}, \{x, y_2\}\}$ if and only if H_x contains an element interchanging y_1 and y_2 . If x is in an orbit of size 8 then $|H_x| = 3$ and so no such element exists, and if x is in an orbit of size 24 then $|H_x| = 1$ and so no such element exists. Thus the possibilities for $(|O_1|, |O_2|)$ are $(6, 8), (6, 24), (8, 12)$ or $(12, 24)$. In the first two cases x must be in the orbit of length 6 and in the last two cases x must be in the orbit of length 12. Thus we get the decomposition of Construction 8.21(1).

Suppose now $e = \{\{x, y_1\}, \{x, y_2\}\}$ is an edge such that x, y_1, y_2 lie in the same H -orbit O_i . Then H contains G_e if and only if H_x interchanges y_1 and y_2 . Thus $|H_x|$ is even and so $|O_i| \neq 8, 24$. If $q \equiv 5 \pmod{8}$ and O_i is the unique orbit of size 6 then we obtain the decomposition in Construction 8.21(2). If $q \equiv 3 \pmod{8}$ and O_i is the unique orbit of size 12 then we obtain the decompositions in Construction 8.21(3). ■

8.4 Subfield subgroups

Suppose now that $q = q_0^r$. Then $S = \{\infty\} \cup \text{GF}(q_0)$ is a subset of the projective line $X = \{\infty\} \cup \text{GF}(q)$ which is an orbit of the subgroup $\text{PTL}(2, q_0)$ of $\text{PTL}(2, q)$. Notice that ϕ fixes the set S . Moreover, by [9, I, Example 3.23], if $\mathcal{B} = S^{\text{PGL}(2, q)}$ then (X, \mathcal{B}) is a $S(3, q_0 + 1, q + 1)$ Steiner system. Since ϕ fixes S and $\text{PTL}(2, q) = \langle \text{PGL}(2, q), \phi \rangle$ it follows that $\mathcal{B} = S^{\text{PGL}(2, q)}$. Thus by Lemma 2.11, we can construct a $\text{PTL}(2, q)$ -transitive decomposition of $J(q+1, 2)$ with divisors isomorphic to $J(q_0 + 1, 2)$. The stabiliser

of a divisor is $\text{PGL}(2, q_0)$. Moreover, this decomposition is G -transitive for any 3-transitive subgroup G of $\text{PGL}(2, q)$. For further constructions we need the orbits of $\text{PGL}(2, q_0)$ on $\text{GF}(q) \setminus \text{GF}(q_0)$.

Lemma 8.23. [8, Lemma 14] *Let $q = q_0^r$ for some prime r and let $H = \{t_{a,b,c,d} \mid a, b, c, d \in \text{GF}(q_0), ad - bc \neq 0\}$. If r is odd then H acts semiregularly on $\text{GF}(q) \setminus \text{GF}(q_0)$, while if $r = 2$ then H is transitive on $\text{GF}(q) \setminus \text{GF}(q_0)$.*

Construction 8.24. Let $X = \{\infty\} \cup \text{GF}(q)$ be the projective line. Let $q = q_0^r$, where $q_0 > 2$, r is a prime and r is odd if q is odd. Let $e = \{\{\infty, w_1\}, \{\infty, w_2\}\}$ such that $w_1, w_2 \in \text{GF}(q) \setminus \text{GF}(q_0)$ but $w_1 + w_2 \in \text{GF}(q_0)$. Let l be a positive integer such that ϕ^l fixes $\{w_1, w_2\}$. Then let $G = \langle \text{PGL}(2, q), \phi^l \rangle$ and $H = \langle \text{PGL}(2, q_0), \phi^l \rangle$. Let $P = e^H$ and $\mathcal{P} = P^G$. Then by Lemma 8.6, $G_e = \langle t_{-1, w_1 + w_2, 0, 1}, \phi^l \rangle$ which is in H . Therefore by Lemma 2.4, $(J(q+1, 2), \mathcal{P})$ is a G -primitive decomposition. The stabiliser $H_{\{\infty, w_1\}}$ fixes ∞ and w_1 as they are in different H -orbits. We claim that $\text{PGL}(2, q_0)_{\infty, w_1} = 1$. Indeed, an element in that subgroup must be of the form $t_{a,b,0,1}$ with $a, b \in \text{GF}(q_0)$, whose only fixed point is $\frac{b}{1-a} \in \text{GF}(q_0)$ if it is not the identity. Hence there is a unique element of $\text{PGL}(2, q_0)_\infty$ interchanging w_1 and w_2 , this being $t_{-1, w_1 + w_2, 0, 1}$. Then as ϕ^l fixes $\{w_1, w_2\}$ and ∞ , it follows that H_{∞, w_1} fixes w_2 . Hence P is isomorphic to $\frac{q_0(q_0^2 - 1)}{2} K_2$.

Proposition 8.25. *Let $(J(q+1, 2), \mathcal{P})$ be a G -primitive decomposition such that $\text{PGL}(2, q) \leq G \leq \text{PGL}(2, q)$ and for $P \in \mathcal{P}$, $G_P \cong N_G(\text{PGL}(2, q_0))$ where $q = q_0^r$, $q_0 > 2$, r is a prime and r is odd if q is odd. Then \mathcal{P} is obtained by Construction 2.10 or 8.24.*

Proof. By Theorem 8.3, \mathcal{P} is also a $\text{PGL}(2, q)$ -primitive decomposition so we may suppose that $G = \text{PGL}(2, q)$ and $H = G_P = \{t_{a,b,c,d} \mid a, b, c, d \in \text{GF}(q_0), ad - bc \neq 0\}$. We have already seen that H has an orbit $\{\infty\} \cup \text{GF}(q_0)$ of length $q_0 + 1$ on X . Moreover, by Lemma 8.23, when r is odd, H has $q_0^{r-3} + q_0^{r-5} + \dots + q_0^2 + 1$ other orbits, all of length $q_0(q_0^2 - 1)$, while when $r = 2$, H is transitive on $\text{GF}(q) \setminus \text{GF}(q_0)$.

Suppose that H contains the stabiliser in G of the edge $e = \{\{v, w_1\}, \{v, w_2\}\}$. Then H_v contains the unique nontrivial element interchanging w_1 and w_2 (see Lemma 8.6). Now v must lie in the unique H -orbit of length $q_0 + 1$. For, if r is odd and v lies in an H -orbit of length $q_0(q_0^2 - 1)$ then $H_v = 1$, while if $r = 2$ and v lies in $\text{GF}(q) \setminus \text{GF}(q_0)$, then $|H_v| = q_0 + 1$ which is odd. Without loss of generality we may suppose that $v = \infty$.

Then $G_e = \langle t_{-1, w_1 + w_2, 0, 1} \rangle$, so $G_e \leq H$ if and only if $w_1 + w_2 \in \text{GF}(q_0)$. If w_1 and w_2 lie in the orbit of length $q_0 + 1$, that is, are in $\text{GF}(q_0)$ then we obtain the decomposition from Construction 2.10, which is in fact preserved by $\text{PGL}(2, q)$. If $w_1 \notin \text{GF}(q_0)$ and $w_2 = a - w_1$ with $a \in \text{GF}(q_0)$, then we get a decomposition obtained from Construction 8.24. \blacksquare

For a primitive element μ of $\text{GF}(q_0)$, $t_{\mu, 0, 0, 1} \in \text{PGL}(2, q) \setminus \text{PSL}(2, q)$. Thus $\phi^s t_{\mu, 0, 0, 1} \in M(s, q)$ and normalises $\text{PSL}(2, q_0)$. Hence $N_{M(s, q)}(\text{PSL}(2, q_0)) = \langle \text{PSL}(2, q_0), \phi^s t_{\mu, 0, 0, 1} \rangle$.

We will need the following lemma.

Lemma 8.26. *Let $G = M(s, q)$ with $q = q_0^r = p^f$ for some odd primes r and p , and even integer f , and let $H = \langle \text{PSL}(2, q_0), \phi^s t_{\mu, 0, 0, 1} \rangle$ where μ is a primitive element of $\text{GF}(q_0)$. Let $e = \{\{\infty, w_1\}, \{\infty, w_2\}\}$.*

1. *Then $G_e \leq H$ if and only if both $w_1 + w_2$ and $(w_2 - w_1)^{p^{2s} - 1}$ lie in $\text{GF}(q_0)$.*
2. *There exist $w_1, w_2 \notin \text{GF}(q_0)$ such that $w_1 + w_2$ and $(w_2 - w_1)^{p^{2s} - 1}$ lie in $\text{GF}(q_0)$ if and only if $\gcd(\frac{q-1}{q_0-1}, p^{2s} - 1) \neq 1$.*

Proof. 1. By Lemma 8.6, $G_e = \langle t_{-1, w_1 + w_2, 0, 1}, (\phi^{2s})^g \rangle$ where $g = t_{w_2 - w_1, w_1, 0, 1}$. Since f is even and $q = q_0^r$ with r odd, q_0 is an even power of p and hence -1 is a square in $\text{GF}(q_0)$. Thus $t_{-1, w_1 + w_2, 0, 1} \in H$ if and only if $w_1 + w_2 \in \text{GF}(q_0)$. Moreover,

$$\begin{aligned} g^{-1} \phi^{2s} g &= t_{1, -w_1, 0, w_2 - w_1} \phi^{2s} t_{w_2 - w_1, w_1, 0, 1} \\ &= \phi^{2s} t_{1, -w_1^{p^{2s}}, 0, (w_2 - w_1)^{p^{2s}}} t_{w_2 - w_1, w_1, 0, 1} \\ &= \phi^{2s} t_{w_2 - w_1, -(w_2 - w_1) w_1^{p^{2s}} + w_1 (w_2 - w_1)^{p^{2s}}, 0, (w_2 - w_1)^{p^{2s}}} \\ &= \phi^{2s} t_{1, w_1 (w_2 - w_1)^{p^{2s} - 1} - w_1^{p^{2s}}, 0, (w_2 - w_1)^{p^{2s} - 1}}. \end{aligned}$$

Let $h = t_{1, w_1(w_2 - w_1)^{p^{2s} - 1} - w_1^{p^{2s}}, 0, (w_2 - w_1)^{p^{2s} - 1}}$. As $\phi^{2s} \in H$, it follows that $(\phi^{2s})^g \in H$ if and only if $h \in \text{PSL}(2, q_0)$. Now if $h \in \text{PSL}(2, q_0)$ then $(w_2 - w_1)^{p^{2s} - 1} \in \text{GF}(q_0)$. Thus if $G_e \leq H$ then both $w_1 + w_2$ and $(w_2 - w_1)^{p^{2s} - 1}$ lie in $\text{GF}(q_0)$. Conversely, suppose that $w_1 + w_2 = a \in \text{GF}(q_0)$ and $w_2 - w_1 = u$ with $u^{p^{2s} - 1} = b \in \text{GF}(q_0)$. Then writing $\frac{1}{2}$ for $2^{-1} \in \text{GF}(p)$ and noting that $2^{p^{2s}} = 2$, $w_1(w_2 - w_1)^{p^{2s} - 1} - w_1^{p^{2s}} = \frac{a-u}{2}b - \frac{a^{p^{2s}} - u^{p^{2s}}}{2p^{2s}} = \frac{ab - a^{p^{2s}}}{2} \in \text{GF}(q_0)$. Thus $h \in \text{PGL}(2, q_0)$, and since $p^{2s} - 1$ is even $h \in \text{PSL}(2, q_0)$.

2. Let ξ be a primitive element of $\text{GF}(q)$. Then $\text{GF}(q_0) = \{0\} \cup \{\xi^i \frac{q-1}{q_0-1} \mid i = 1, \dots, q_0 - 1\}$ and we can choose $\mu = \xi^{\frac{q-1}{q_0-1}}$. For $w_2 - w_1 = \xi^j \in \text{GF}(q) \setminus \{0\}$, if $(w_2 - w_1)^{p^{2s} - 1}$ lies in $\text{GF}(q_0)$, that means that $\xi^{j(p^{2s} - 1)} = \xi^{i \frac{q-1}{q_0-1}}$ for some integer i . If $\gcd(\frac{q-1}{q_0-1}, p^{2s} - 1) = 1$, we must have j a multiple of $(q-1)/(q_0-1)$, and so $w_2 - w_1 \in \text{GF}(q_0)$. If we also have $w_1 + w_2 \in \text{GF}(q_0)$, then this implies that $w_1, w_2 \in \text{GF}(q_0)$. Hence if $w_1, w_2 \notin \text{GF}(q_0)$ such that $w_1 + w_2$ and $(w_2 - w_1)^{p^{2s} - 1}$ lie in $\text{GF}(q_0)$ then $\gcd(\frac{q-1}{q_0-1}, p^{2s} - 1) \neq 1$. Conversely, suppose $\gcd(\frac{q-1}{q_0-1}, p^{2s} - 1) = d \neq 1$ and choose $j = (q-1)/d(q_0-1)$. Then take $w_2 = \xi^j/2$ and $w_1 = -\xi^j/2$. We obviously have $w_1 + w_2 \in \text{GF}(q_0)$ and $w_1, w_2 \notin \text{GF}(q_0)$. Moreover $(w_2 - w_1)^{p^{2s} - 1} = \xi^{\frac{p^{2s}-1}{d} \frac{q-1}{q_0-1}} \in \text{GF}(q_0)$. \blacksquare

Construction 8.27. Let $G = M(s, q)$ and let $X = \{\infty\} \cup \text{GF}(q)$ be the projective line. Let $q = q_0^r = p^f$ for some odd primes r and p , and f an even integer, and let $H = \langle \text{PSL}(2, q_0), \phi^s t_{\mu, 0, 0, 1} \rangle$ where μ is a primitive element of $\text{GF}(q_0)$. Assume $\gcd(\frac{q-1}{q_0-1}, p^{2s} - 1) \neq 1$, so that by Lemma 8.26, there exist $w_1, w_2 \notin \text{GF}(q_0)$ such that $w_1 + w_2, (w_2 - w_1)^{p^{2s} - 1} \in \text{GF}(q_0)$. Let $e = \{\{\infty, w_1\}, \{\infty, w_2\}\}$. By Lemma 8.6, $G_e = \langle t_{-1, w_1 + w_2, 0, 1}, (\phi^{2s})^g \rangle$, where $g = t_{w_2 - w_1, w_1, 0, 1}$, and by Lemma 8.26, $G_e \leq H$. Thus letting $P = e^H$ and $\mathcal{P} = P^G$, $(J(q+1, 2), \mathcal{P})$ is a G -primitive decomposition by Lemma 2.4.

We claim that the divisors of \mathcal{P} are either matchings or unions of cycles. Since $\text{PSL}(2, q_0)_\infty \triangleleft H_\infty = \langle \text{PSL}(2, q_0)_\infty, \phi^s t_{\mu, 0, 0, 1} \rangle$, H_∞ (of order $\frac{q_0(q_0-1)}{2} \frac{f}{s}$) acts on the set of $\text{PSL}(2, q_0)_\infty$ -orbits. Now $t_{-1, w_1 + w_2, 0, 1} \in \text{PSL}(2, q_0)_\infty$ interchanges w_1 and w_2 , and hence w_1, w_2 lie in the same $\text{PSL}(2, q_0)_\infty$ -orbit, θ say. By Lemma 8.23, $\text{PGL}(2, q_0)$ acts semiregularly on $\text{GF}(q) \setminus \text{GF}(q_0)$ and hence $|\theta| = |\text{PSL}(2, q_0)| = \frac{q_0(q_0-1)}{2}$. Note that $H_{\{\infty, w_1\}} = H_{\infty, w_1}$ and $H_{\{\infty, w_2\}} = H_{\infty, w_2}$. Also $H_{\{\infty, w_1\}, \{\infty, w_2\}} = \langle (\phi^{2s})^g \rangle$ has order $\frac{f}{2s}$. Notice that $(\phi^s t_{\mu, 0, 0, 1})^2 = \phi^{2s} t_{\mu^{p^s+1}, 0, 0, 1}$. Hence $\langle \text{PSL}(2, q_0)_\infty, (\phi^{2s})^g \rangle$ has index 2 in H_∞ and fixes θ . Therefore H_∞ either fixes θ or switches it with another $\text{PSL}(2, q_0)_\infty$ -orbit θ' . In the first case, $H_{\{\infty, w_1\}}$ has order $\frac{f}{s}$, while $H_{\{\infty, w_1\}, \{\infty, w_2\}}$ has order $\frac{f}{2s}$, hence the divisor has valency 2 and is a union of cycles. In the second case, $H_{\{\infty, w_1\}}$ and $H_{\{\infty, w_1\}, \{\infty, w_2\}}$ both have order $\frac{f}{2s}$, and so the divisor is a matching $\frac{q_0(q_0-1)}{2} K_2$.

Remark 8.28. We have not determined the length of the cycles occurring in the first case of Construction 8.27. This case happens if and only if there exists $w \in \text{GF}(q) \setminus \text{GF}(q_0)$ such that $w^{\phi^s t_{\mu, 0, 0, 1}} = w^{p^s} \mu \in \{a^2 w + b \mid a, b \in \text{GF}(q_0)\}$. We have not been able to find any instances where this condition holds.

Proposition 8.29. Let $(J(q+1, 2), \mathcal{P})$ be a G -primitive decomposition with $G = M(s, q)$ and for $P \in \mathcal{P}$ we have that $G_P = N_G(\text{PSL}(2, q_0))$ where $q = q_0^r$ for some odd prime r . Then \mathcal{P} is obtained by Construction 2.10 or 8.27.

Proof. Let $q = p^f$ with p a prime and f an even integer. As seen in the discussion before Lemma 8.26, $H := G_P = \langle \text{PSL}(2, q_0), \phi^s t_{\mu, 0, 0, 1} \rangle$ where μ is a primitive element of $\text{GF}(q_0)$. Let $X = \{\infty\} \cup \text{GF}(q)$. Then one orbit of H on X is $\{\infty\} \cup \text{GF}(q_0)$. Since H is maximal in G , H is exactly the stabiliser in G of $\{\infty\} \cup \text{GF}(q_0)$.

Suppose that H contains G_e for some edge $e = \{\{v, w_1\}, \{v, w_2\}\}$.

Then by Lemma 8.6, H contains an element of $\text{PSL}(2, q)$, and hence of $\text{PSL}(2, q_0)$, which fixes v and interchanges w_1 and w_2 . Since, by Lemma 8.23, $\text{PSL}(2, q_0)$ acts semiregularly on $\text{GF}(q) \setminus \text{GF}(q_0)$, it follows that $v \in \{\infty\} \cup \text{GF}(q_0)$. Without loss of generality we may suppose that $v = \infty$. By Lemma 8.26, this means that both $w_1 + w_2$ and $(w_2 - w_1)^{p^{2s} - 1}$ lie in $\text{GF}(q_0)$. This is of course satisfied if $w_1, w_2 \in \text{GF}(q_0)$, and then we get Construction 2.10 using $\mathcal{B} = (\{\infty\} \cup \text{GF}(q_0))^{\text{PGL}(2, q)}$, as G is transitive on \mathcal{B} . Now assume $w_1, w_2 \notin \text{GF}(q_0)$. Then by Lemma 8.26, $\gcd(\frac{q-1}{q_0-1}, p^{2s} - 1) \neq 1$. Moreover, $P = e^H$ and $\mathcal{P} = P^G$ are as obtained in Construction 8.27. \blacksquare

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