Homogeneous Factorisations of Johnson Graphs*

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Abstract

For a graph Γ, subgroups $M < G \leqslant \mathrm{Aut}(\Gamma)$, and an edge partition $\mathcal{E}$ of Γ, the pair $(\Gamma, \mathcal{E})$ is a (G, M)-homogeneous factorisation if M is vertex-transitive on Γ and fixes setwise each part of $\mathcal{E}$, while G permutes the parts of $\mathcal{E}$ transitively. A classification is given of all homogeneous factorisations of finite Johnson graphs. There are three infinite families and nine sporadic examples.

1 Introduction

Let $\mathcal{E} = \{E_1, E_2, \ldots, E_k\}$ be a partition of the edge set of a graph Γ with $k \geq 2$. Given $M \leqslant G \leqslant \mathrm{Aut}(\Gamma)$, we say that $(\Gamma, \mathcal{E})$ is a (G, M)-*homogeneous factorisation* of *index* k if M acts transitively on the vertex set $V\Gamma$ and fixes each E_i setwise while G permutes the E_i transitively. Note that $(\Gamma, \mathcal{E})$ is also a $(G, \overline{M})$-homogeneous factorisation, where $\overline{M}$ is the kernel of the action of G on $\mathcal{E}$. In this paper we determine all homogeneous factorisations of the Johnson graphs $J(n, r)$. The Johnson graph $J(n, r)$ for $2 \leq r \leq n/2$ is the graph Γ whose vertex set $V\Gamma$ consists of all the r-subsets of a set X of size n where two vertices are joined by an edge if they intersect in $r - 1$ points. The graph $J(n, r)$ has $\binom{n}{r}$ vertices and is regular with valency $r(n - r)$.

If $(\Gamma, \mathcal{E})$ is a (G, M)-homogeneous factorisation of $J(n, r)$, then M and G must be r-homogeneous groups on X, that is, they act transitively on the r-subsets of X. All finite r-transitive (and hence, also r-homogeneous) groups are known (see for example [6, p243–253]) while Kantor has shown that there are four types of r-homogeneous but not r-transitive groups (see [12]). We consider each of these cases and obtain the following results:

(1). Construction 3.2: Let $q = p^f$ be a prime power, $q \equiv 1 \pmod 4$ and let $\Gamma = J(q + 1, 2)$ where $V\Gamma$ is the set of all 2-subsets of $X = \mathrm{GF}(q) \cup \{\infty\}$. There exists a (G, M)-homogeneous factorisation $(\Gamma, \mathcal{E})$ of index 2 with $\mathrm{PSL}(2, q) \leqslant M \leqslant \mathrm{P}\Sigma\mathrm{L}(2, q)$ and G a 3-transitive subgroup

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[*] This paper forms part of an ARC Discovery grant of the last 2 authors. The second author holds an Australian Research Council Australian Research Fellowship

of $\mathrm{P\Gamma L}(2,q)$. Here $\mathrm{P\Sigma L}(2,q)$ is the group generated by $\mathrm{PSL}(2,q)$ and an element corresponding to the Frobenius automorphism of the field $\mathrm{GF}(q)$ (see the first paragraph of Section 3.1).

(2). Construction 3.4: Let $q = 2^{r^f}$ for r an odd prime and let $\Gamma = J(q,2)$ where $V\Gamma$ is the set of all 2-subsets of $X = \mathrm{GF}(q)$. There exists a (G,M)-homogeneous factorisation $(\Gamma,\mathcal{E})$ of index r with $G = \mathrm{A\Gamma L}(1,q)$ and $M = \mathrm{AGL}(1,q)$.

(3). Construction 3.7: There are 9 pairwise non-isomorphic (G,M)-homogeneous factorisations $(\Gamma,\mathcal{E})$ of index 3 for $J(8,3)$ with $M = \mathrm{AGL}(1,8)$ and $G = \mathrm{A\Gamma L}(1,8)$.

(4). Construction 3.10: Let $q = 2^{r^f}$ for r an odd prime and let $\Gamma = J(q+1,3)$ where $V\Gamma$ is the set of all 3-subsets of $X = \mathrm{GF}(q) \cup \{\infty\}$. There exists a (G,M)-homogeneous factorisation $(\Gamma,\mathcal{E})$ of index r with $G = \mathrm{P\Gamma L}(2,q)$ and $M = \mathrm{PGL}(2,q)$.

Two homogeneous factorisations $(\Gamma_1,\mathcal{E}_1)$ and $(\Gamma_2,\mathcal{E}_2)$ are said to be *isomorphic* if there exists a graph isomorphism $\phi : \Gamma_1 \to \Gamma_2$ which maps $\mathcal{E}_1$ to $\mathcal{E}_2$. Note that if $\Gamma_1 = \Gamma_2 = \Gamma$ then $\phi \in \mathrm{Aut}(\Gamma)$, and if $(\Gamma,\mathcal{E}_1)$ is a (G,M)-homogeneous factorisation then $(\Gamma,\mathcal{E}_2)$ is a (G^ϕ,M^ϕ)-homogeneous factorisation. For Constructions 3.4 and 3.10, the number of homogeneous factorisations up to isomorphism are determined in Lemmas 3.6 and 3.11 respectively.

Finally, we prove that these are the only examples.

Theorem 1.1. *If $(\Gamma,\mathcal{E})$ is a (G,M)-homogeneous factorisation of the Johnson graph $\Gamma = J(n,r)$, then $r \le 3$ and $(\Gamma,\mathcal{E})$ arises from Construction 3.2 or 3.4 (with $r = 2$), or Construction 3.7 or 3.10 (with $r = 3$).*

This classification depends on the classification of finite r-homogeneous permutation groups, for $r \ge 2$, and hence on the Finite Simple Group Classification.

One of our main methods relies on the fact (Theorem 2.1) that if $(\Gamma,\mathcal{E})$ is a (G,M)-homogeneous factorisation for any graph Γ and $\alpha \in V\Gamma$, then the actions of the groups M and G on the partition $\mathcal{E}$ of $E\Gamma$ are replicated locally by the actions of the stabiliser subgroups M_α and G_α on a partition $\mathcal{E}(\alpha)$ of the set of neighbours of α, denoted $\Gamma(\alpha)$. As a result, the existence or nonexistence of a (G,M)-homogeneous factorisation is then determined by the actions of the smaller groups M_α and G_α on the set $\Gamma(\alpha)$.

2 Techniques and Strategies

2.1 Local partitions

Let M be a group acting transitively on a set X and let $\alpha \in X$. Following Wielandt [20], given a *suborbit* U of M, that is, an orbit of M_α on $X \setminus \{\alpha\}$, we define

$$U^* = \{\alpha^{g^{-1}} \mid g \in M, \alpha^g \in U\}.$$

Then U^* is also an M_α-orbit and $U^{**} = U$ by [20, Theorems 16.1 and 16.2]. We call U and U^* *paired suborbits* and if $U^* = U$ we say that U is self-paired.

Now if $U \subseteq X \setminus \{\alpha\}$ is invariant under M_α, then $U = \cup U_i$, where each U_i is an M_α-orbit. We can then define $U^* = \cup U_i^*$. It naturally follows that U^* is also M_α-invariant and $U^{**} = U$. We call U and U^* *paired sets* and if $U^* = U$ we say that U is *self-paired.*

Now, let $(\Gamma,\mathcal{E})$ be a (G,M)-homogeneous factorisation and let $\alpha \in V\Gamma$. (As noted in the introduction we may assume that $M \lhd G$). For each E_i in $\mathcal{E}$, denote by $E_i(\alpha)$ the set of all neighbours of α in E_i, that is $E_i(\alpha) = \{\beta | \{\alpha,\beta\} \in E_i\}$. Note that $E_i(\alpha)$ is M_α-invariant, and we denote $(E_i(\alpha))^*$ by $E_i^*(\alpha)$. From this, we get the following theorem:

Theorem 2.1. *Let* $(\Gamma, \mathcal{E})$ *be a* (G, M)*-homogeneous factorisation with* $M \lhd G$, $\mathcal{E} = \{E_1, \dots, E_k\}$, $k \geq 2$, *and let* $\alpha \in V\Gamma$. *Then*

(1). $\mathcal{E}(\alpha) = \{E_1(\alpha), \dots, E_k(\alpha)\}$ *is a partition of* $\Gamma(\alpha)$*;*

(2). *For each* $i \leq k$, $E_i(\alpha)$ *is* M_α*-invariant and self-paired.*

(3). G_α *acts transitively on* $\mathcal{E}(\alpha)$*. In particular, there is an integer* $c \geq 0$ *such that each* $E_i(\alpha)$ *contains exactly* c *self-paired* M_α*-orbits.*

Proof. Part (1) was proved in [5, Theorem 3.2]. Alternatively, the fact that $\mathcal{E}(\alpha)$ is a G_α-invariant partition of $\Gamma(\alpha)$ with each part fixed setwise by M_α and $G_\alpha^{\mathcal{E}(\alpha)}$ a transitive group follows from [9, Proposition 3.3]. It remains to prove the statements concerning pairing.

Now $E_i^*(\alpha) = \{\alpha^{m^{-1}} \mid m \in M, \alpha^m \in E_i(\alpha)\}$. To show $E_i^*(\alpha) = E_i(\alpha)$, we note that $\alpha^{m^{-1}} \in E_i^*(\alpha) \iff \alpha^m \in E_i(\alpha) \iff \{\alpha, \alpha^m\} \in E_i \iff \{\alpha, \alpha^m\}^{m^{-1}} = \{\alpha^{m^{-1}}, \alpha\} \in E_i \iff \alpha^{m^{-1}} \in E_i(\alpha)$. This completes the proof of Part (2).

In order to show that each $E_i(\alpha)$ contains the same number of self-paired M_α-orbits, we first show that if O is a self-paired M_α-orbit in $\Gamma(\alpha)$ and $f \in G_\alpha$ then O^f is also a self-paired M_α-orbit. Let $O = x^{M_\alpha}$ for some $x \in \Gamma(\alpha)$. Then $O^f = x^{M_\alpha f} = x^{f M_\alpha}$ since f normalises M_α. Thus, $O^f = y^{M_\alpha}$ where $y = x^f$ and hence O^f is an M_α-orbit. Now take $y^m \in O^f$ (with $m \in M$) and $h \in M$ such that $y^{mh} = \alpha$. Let $m_1 \in M$ such that $mh = m_1$. We wish to show that $y^{m_1} = \alpha \in O^f$. Since $y = x^f$, $f \in G$ and $M \lhd G$ then $y^{m_1} = x^{f m_1} = x^{m_2 f}$ for some $m_2 \in M$. This implies $y^{m_1} \in O^f$ and so O^f is self-paired. Since G_α acts transitively on $\mathcal{E}(\alpha)$ it follows that there is an integer $c \geq 0$ such that each $E_i(\alpha)$ in $\mathcal{E}(\alpha)$ contains c self-paired M_α-orbits. ■

Theorem 2.1 shows that for a (G, M)-homogeneous factorisation $(\Gamma, \mathcal{E})$, the actions of M and G on $E\Gamma$ are replicated locally by M_α and G_α on $\Gamma(\alpha)$. Thus, when G_α fixes an M_α-orbit, then no (G, M)-homogeneous factorisation exists. This observation is useful in proving the nonexistence of homogeneous factorisations in several cases.

On the other hand we have the following construction.

Construction 2.2. Consider a graph Γ and let $\alpha \in V\Gamma$ such that $\mathcal{E}(\alpha) = \{E_1(\alpha), \dots, E_k(\alpha)\}$ is a partition of $\Gamma(\alpha)$. Suppose M is a group of automorphisms transitive on $V\Gamma$ such that M_α fixes each $E_i(\alpha)$ setwise. We assume that each $E_i(\alpha)$ in $\mathcal{E}(\alpha)$ is self-paired, i.e., $E_i^*(\alpha) = E_i(\alpha)$ for each i, $1 \leq i \leq k$. Define $E_i := \{\{\alpha, \beta\}^g | \beta \in E_i(\alpha), g \in M\}$ and $\mathcal{E} = \{E_1, \dots, E_k\}$. Further, let G be a group of automorphisms containing M such that G_α is transitive on $\mathcal{E}(\alpha)$.

Theorem 2.3. *Let* $\Gamma, G, M, \mathcal{E}(\alpha)$ *be as yielded by Construction* 2.2*. Then* $(\Gamma, \mathcal{E})$ *is a* (G, M)*-homogeneous factorisation of* Γ *where* $\mathcal{E}$ *is as in the construction. Conversely, if* $(\Gamma, \mathcal{E})$ *is a* (G, M)*-homogeneous factorisation and* $\mathcal{E}(\alpha)$ *is the edge partition of* $\Gamma(\alpha)$ *as in Theorem* 2.1*, then* $(\Gamma, \mathcal{E})$ *arises from Construction* 2.2 *applied to* $\Gamma, G, M, \mathcal{E}(\alpha)$.

Proof. The forward statement is proved in [5, Theorem 3.3], while the whole theorem follows from [9, Proposition 3.3] on noting that since the $E_i(\alpha)$ are selfpaired we obtain a partition of $E\Gamma$ as opposed to just a partition of the arc set $A\Gamma$. ■

The above theorem is invoked in the constructions of homogeneous factorisations for $J(n, r)$. In order to obtain a (G, M)-homogeneous factorisation $(\Gamma, \mathcal{E})$, we need a partition $\mathcal{E}(\alpha)$ of $\Gamma(\alpha)$ whose elements are all self-paired and are M_α-invariant. Further, we need G_α to act transitively on $\mathcal{E}(\alpha)$. This strategy is used in constructing the homogeneous factorisations of $J(n, r)$ in Section 3.

2.2 Arc-Transitivity

A graph is said to be *arc-transitive* if it admits an automorphism group that is transitive on its arc-set. The following lemma characterises arc-transitivity on any graph. Its proof is straightforward and omitted.

Lemma 2.4. *Consider a graph* Γ*, a subgroup* $G \leqslant \mathrm{Aut}(\Gamma)$*, and let* $\alpha \in V\Gamma$*. Then* G *is transitive on* $V\Gamma$ *and* G_α *is transitive on* $\Gamma(\alpha)$ *if and only if* G *is arc-transitive on* Γ*.*

Consider a set X with n elements and let $\Gamma = J(n, r)$ with $V\Gamma = X^{\{r\}}$ for $r \geq 2$. By [2, Theorem 9.1.2], for $n > 2r$, $\mathrm{Aut}(\Gamma) = S_n$ with the action induced from the action of S_n on X. However, when $n = 2r \geq 4$ we have $\mathrm{Aut}(\Gamma) = S_n \times S_2 = \langle S_n, \tau\rangle$ where τ acts on $V\Gamma$ by complementation. The following lemma allows us to restrict to subgroups of S_n.

Lemma 2.5. *Let* M *be a vertex-transitive group of automorphisms of* $J(n, r)$*. Then* $M \cap S_n$ *is vertex-transitive.*

Proof. We only need to consider the case where $n = 2r$ and $M \nleqslant S_n$. In this case $M \cap S_n$ is an index 2 subgroup of M. If $M \cap S_n$ were intransitive on vertices, then it would have two orbits and $J(n, r)$ would be bipartite with the two orbits forming the two parts of the bipartition. However, this contradicts the fact that $J(n, r)$ contains 3-cycles and so $M \cap S_n$ is vertex-transitive. ■

We also have the following result about arc-transitivity of subgroups of S_n.

Proposition 2.6. *A group* $G \leqslant S_n$ *is arc-transitive on* $J(n, r)$ *if and only if* G *is* r*-homogeneous on* X*, and for* $A \in V\Gamma$*,* G_A *is transitive on the set*

$$A \times \overline{A} = \{(a, b) | a \in A, b \in X \setminus A\}.$$

Proof. Let $\Gamma = J(n, r)$ and suppose G is arc-transitive on Γ. By Lemma 2.4, G is transitive on $V\Gamma$ and is therefore r-homogeneous on X. Moreover, G_A is transitive on $\Gamma(A)$.

Consider any two elements $(a_1, b_1), (a_2, b_2) \in A \times \overline{A}$. Then there exist two elements $B_1, B_2 \in \Gamma(A)$ such that the symmetric difference $A\Theta B_1 = \{a_1, b_1\}$ while $A\Theta B_2 = \{a_2, b_2\}$. Since G is arc-transitive then by Lemma 2.4, there is an $h \in G_A$ such that $B_1^h = B_2$. This implies that $(A\Theta B_1)^h = (A\Theta B_2)$ and hence that $(a_1, b_1)^h = (a_2, b_2)$. So G_A is transitive on $A \times \overline{A}$.

Conversely, suppose G is r-homogeneous and G_A is transitive on $A \times \overline{A}$. Then G is transitive on $V\Gamma$. Let $B_1, B_2 \in \Gamma(A)$ and set $A\Theta B_1 = \{a_1, b_1\}$ and $A\Theta B_2 = \{a_2, b_2\}$ with $a_1, a_2 \in A$. Then $(a_1, b_1), (a_2, b_2) \in A \times \overline{A}$. Since G_A is transitive on $A \times \overline{A}$, there exists $h \in G_A$ such that $(a_1, b_1)^h = (a_2, b_2)$. Thus, $B_1^h = B_2$ and G_A is transitive on $\Gamma(A)$. By Lemma 2.4, G is arc-transitive on Γ. ■

We now link the concept of arc-transitivity on Johnson graphs $J(n, r)$ to r-transitivity on the underlying set X.

Lemma 2.7. *If a group* $G \leqslant S_n$ *is* $(r + 1)$*-transitive on* X *then* G *is arc-transitive on* $\Gamma = J(n, r)$*.*

Proof. Let G be $(r + 1)$-transitive on X. Then G is r-transitive and hence r-homogeneous on X. Further, for any $A = \{a_1, \ldots, a_r\} \in V\Gamma$, $x, y \notin A$ and any ordering $(a_{j_1}, \ldots, a_{j_r})$ of the subset A, there exists $g \in G$ such that $(a_1, \ldots, a_r, x)^g = (a_{j_1}, \ldots, a_{j_r}, y)$. Hence for any $a_i, a_j \in A$ and $x, y \notin A$, there exists $g \in G_A$ such that $(a_i, x)^g = (a_j, y)$. So G_A is transitive on $A \times \overline{A}$ and by Proposition 2.6, G is arc-transitive on Γ. ■

Based on the above lemma, we obtain the following theorem, which is used to show the nonexistence of homogeneous factorisations for $J(n, r)$ in several cases.

Lemma 2.8. *If $M \leqslant S_n$ and M is $(r+1)$-transitive on X then there is no (G,M)-homogeneous factorisation of $J(n,r)$ for any G such that $M < G \leqslant \mathrm{Aut}(J(n,r))$.*

Proof. Let $M \leqslant S_n$ be $(r+1)$-transitive on X. By Lemma 2.7, M is arc-transitive on Γ and so cannot fix the parts of a partition of $E\Gamma$. Thus no (G,M)-homogeneous factorisation of $J(n,r)$ exists for any G. ■

We are now able to show that there are no homogeneous factorisations with $G \not\leqslant S_n$ and so we can concentrate on the case where $G \leqslant S_n$.

Proposition 2.9. *There are no (G,M)-homogeneous factorisations of $J(2r,r)$ with $G \not\leqslant S_{2r}$.*

Proof. Suppose that $(J(2r,r),\mathcal{E})$ is (G,M)-homogeneous and $G \not\leqslant S_{2r}$. Replacing M by the kernel of the action of G on $\mathcal{E}$ if necessary, we may assume that M is normal in G. By Lemma 2.5, $M \cap S_{2r}$ is vertex-transitive and hence r-homogeneous on X. Thus by the Classification of 2-transitive and 2-homogeneous groups (see [6, §7.7 and Theorem 9.4B]), either $r = 3$ and $M \cap S_{2r} = \mathrm{PGL}(2,5)$ or $A_{2r} \leqslant M \cap S_{2r}$. Suppose first that $r = 3$ and $M \cap S_{2r} = \mathrm{PGL}(2,5)$. Since G normalises M and $\mathrm{PGL}(2,5)$ is selfnormalising in S_6 it follows that $M = \mathrm{PGL}(2,5)$ and $G = \mathrm{PGL}(2,5) \times \langle \tau \rangle$. Let $A = \{1,2,3\}$ and $M = \langle (3,5,4,6),(1,6,2)(3,4,5) \rangle \cong \mathrm{PGL}(2,5)$. Then a computation using Magma [1] shows that $M_A \cong S_3$ and $G_A = \langle M_A, (1,4,2,6,3,5)\tau \rangle$. Moreover, given $B = \{1,2,4\}$, a neighbour of A, we have $B^{M_A} = \{\{1,3,5\},\{2,3,6\},\{1,2,4\}\} \subset \Gamma(A)$, which is fixed setwise by G_A. Hence we do not obtain a homogeneous factorisation. Thus $A_{2r} \leqslant M \cap S_{2r}$. However, if $r \geq 3$ then A_{2r} is $(r+1)$-transitive on X and so by Lemma 2.8 there are no homogeneous factorisations. Thus $r = 2$. But an easy calculation shows that A_4 is edge-transitive contradicting the fact that M preserves each part of $\mathcal{E}$. Hence no such homogeneous factorisation exists. ■

Lemma 2.7 has this stronger version when $r = 2$ and $n \geq 5$. Note in this case that $\mathrm{Aut}(\Gamma) = S_n$.

Proposition 2.10. *If $n \geq 5$, then G is arc-transitive on $J(n,2)$ if and only if G is 3-transitive on X.*

Proof. By Lemma 2.7, if G is 3-transitive on X then it is arc-transitive on $J(n,2)$. So suppose G is arc-transitive on $\Gamma = J(n,2)$. By Proposition 2.6, G is 2-homogeneous on X, and therefore G is transitive on X. Also, if $A = \{a,b\} \in V\Gamma$ then $G_{\{a,b\}}$ is transitive on the set $A \times \overline{A} = \{(x,y) | x \in \{a,b\}, y \notin \{a,b\}\}$. This implies $G_{a,b}$ is transitive on $\{(a,y) \mid y \notin \{a,b\}\}$ and hence $G_{a,b}$ is transitive on $X \setminus \{a,b\}$. Thus also $G_{a,c}$ is transitive on $X \setminus \{a,c\}$ where $c \notin \{a,b\}$, and this implies that G_a is transitive on $X \setminus \{a\}$. Since $G_{a,b}$ is transitive on $X \setminus \{a,b\}$ it follows that G_a is 2-transitive on $X \setminus \{a\}$ and hence G is 3-transitive on X. ■

The following is another useful tool in proving the nonexistence of a homogeneous factorisation.

Proposition 2.11. *Let $(\Gamma,\mathcal{E})$ be a (G,M)-homogeneous factorisation of index $k \geq 2$. Then $M \neq G$ and $M \neq 1$.*

Proof. Since M is contained in the kernel of the transitive G-action on $\mathcal{E}$, it follows that $k = |\mathcal{E}|$ divides $|G : M|$, so $M \neq G$. Also, $M \neq 1$ since M is transitive on $V\Gamma$. ■

3 Constructions of Homogeneous Factorisations

We now present the (G,M)-homogeneous factorisations of $J(n,r)$ obtained in this study. For the case $r = 2$, we give two constructions.

3.1 On $J(q+1,2)$ when $q \equiv 1 \pmod 4$

Let $X = \mathrm{GF}(q) \cup \{\infty\}$. An element $x \in \mathrm{GF}(q)$ is a *square* if $x = y^2$ for some $y \in \mathrm{GF}(q)$. Let $x = \square$ mean *x is a nonzero square*, and let $x \neq \square$ mean *x is a non-square*. We now consider the group $\mathrm{PSL}(2,q) \cong \{t_{abcd} : z \mapsto \frac{az+b}{cz+d} | a,b,c,d \in \mathrm{GF}(q), ad-bc = \square\}$ which acts 2-transitively on the set X. If σ is a the Frobenius automorphism of $\mathrm{GF}(q)$, that is, $\sigma : z \to z^p$ for $z \in \mathrm{GF}(q)$, then it also acts on X by fixing ∞. We can then define the group $\mathrm{P\Sigma L}(2,q) = \langle \mathrm{PSL}(2,q), \sigma \rangle$.

Lemma 3.1. *Let $q \equiv 1 \pmod 4$, $\mathrm{PSL}(2,q) \leqslant M \leqslant \mathrm{P\Sigma L}(2,q)$ and $\Gamma = J(q+1,2)$ where $V\Gamma$ is the set of all 2-subsets of $X = \mathrm{GF}(q) \cup \{\infty\}$. For $\alpha = \{0,\infty\} \in V\Gamma$ there are two self-paired M_α-orbits in $\Gamma(\alpha)$. Further, M has two orbits in $E\Gamma$.*

Proof. Let $\alpha = \{0,\infty\} \in V\Gamma$. Then $\Gamma(\alpha)$ consists of sets of the form $\{0,x\}$ and $\{\infty,x\}$ where $x \in X \backslash \{\alpha\}$. First, we consider $L = \mathrm{PSL}(2,q)$. Then L_α is generated by the subsets $\{t_{a,0,0,a^{-1}} | a \neq 0\}$ and $\{t_{0,b,-b^{-1},0} | b \neq 0\}$. The first set fixes 0 and ∞ pointwise while the second interchanges 0 and ∞. In general, these sets take z to $a^2 z$ and $-\frac{b^2}{z}$ respectively.

If $q \equiv 1 \pmod 4$ then -1 is a square. So if $z = \square$ then $a^2 z = \square$ and $-\frac{b^2}{z} = \square$. Thus, the L_α-orbits in $\Gamma(\alpha)$ are the following sets:

$$E_1(\alpha) = \Big\{\{0,x\} | x = \square\Big\} \bigcup \Big\{\{\infty,x\} | x = \square\Big\}, \quad \text{and}$$

$$E_2(\alpha) = \Big\{\{0,y\} | y \neq \square\Big\} \bigcup \Big\{\{\infty,y\} | y \neq \square\Big\}.$$

We show that $E_i^*(\alpha) = E_i(\alpha)$ for $i = 1,2$. Let $g = t_{1,0,1,1}$. Then $g^{-1} = t_{1,0,-1,1}$ and $\alpha^g = \{0,1\} \in E_1(\alpha)$. Hence $\alpha^{g^{-1}} = \{0,-1\} \in E_1^*(\alpha)$. However, $\{0,-1\} \in E_1(\alpha)$ and so $E_1(\alpha) = E_1^*(\alpha)$. Since L_α has only 2 orbits in $\Gamma(\alpha)$, it follows that $E_2(\alpha) = E_2^*(\alpha)$ also.

Suppose $q = p^r$ for some prime p and let $\sigma \in \mathrm{Aut}(\mathrm{GF}(q))$ such that $\sigma : z \mapsto z^p$. Then for any divisor s of r, σ^s fixes $\alpha = \{0,\infty\} \in V\Gamma$. Furthermore, if $z = \square$ then $z^{\sigma^s} = \square$ so σ^s fixes the orbits of $\mathrm{PSL}(2,q)_\alpha$ in $\Gamma(\alpha)$. In general, if $M = \mathrm{PSL}(2,q) \cdot \langle \sigma^s \rangle$ then M_α has 2 self-paired orbits in $\Gamma(\alpha)$ when $q \equiv 1 \pmod 4$.

Let $E_i = \Big\{\{\alpha,\beta\}^g | \beta \in E_i(\alpha), g \in M\Big\}$ then $\mathcal{E} = \{E_1, E_2\}$ forms a partition of $E\Gamma$. Further, E_1 and E_2 are the two orbits of M in $E\Gamma$. ■

Lemma 3.1 provides us with the following construction.

Construction 3.2. Let $\Gamma = J(q+1,2)$ with $q \equiv 1 \pmod 4$, and choose M such that $\mathrm{PSL}(2,q) \leqslant M \leqslant \mathrm{P\Sigma L}(2,q)$. Let $\mathcal{E}$ be the set of two M-orbits on $E\Gamma$ and G be any 3-transitive subgroup of $\mathrm{P\Gamma L}(2,q)$ containing M. By Lemma 2.7, G is arc-transitive and since the two parts of $\mathcal{E}$ are fixed setwise by the normal subgroup $\mathrm{PSL}(2,q)$ of G, it follows that G interchanges the two parts of $\mathcal{E}$. Since M is 2-transitive it is vertex-transitive and hence $(\Gamma, \mathcal{E})$ is a (G,M)-homogeneous factorisation of index 2.

The partition $\mathcal{E}$ is equivalent to splitting the edges of the Johnson graph $J(q+1,2)$ into two classes $R_\square$ and $R_{\not\square}$ when $q \equiv 1 \pmod 4$, where $R_\square$ is the set of all pairs $\big\{\{r,s\},\{r,t\}\big\}$, for $r,s,t \in X = \mathrm{GF}(q) \cup \{\infty\}$, such that any one of the following conditions is satisfied:

(1). $r = \infty$ and $s - t = \square$

(2). $s = \infty$ and $t - r = \square$

(3). $t = \infty$ and $r - s = \square$

(4). $r,s,t \in \mathrm{GF}(q)$ and $(r-s)(s-t)(t-r) = \square$.

Otherwise, the pair is in $R_{\not\square}$.

3.2 On $J(q,2)$ with $q = 2^n$, n an odd prime power

In this section, we construct (G, M)-homogeneous factorisations $(\Gamma, \mathcal{E})$ of $J(q,2)$ with $M = \mathrm{AGL}(1,q)$ when $q = 2^n$ and n is an odd prime power. First, we look at the orbits of a vertex-stabiliser of this group.

Lemma 3.3. *Let* $X = \mathrm{GF}(q)$ *where* q *is even,* $q > 2$ *and let* $\Gamma = J(q,2)$ *with* $V\Gamma = X^{\{2\}}$. *Let* $M = \mathrm{AGL}(1,q)$ *and* $\alpha = \{0,1\} \in V\Gamma$. *Then* M_α *has* $q-2$ *orbits of length two in* $\Gamma(\alpha)$*, all of which are not self-paired.*

Proof. Take $\alpha = \{0,1\} \in V\Gamma$. Then $\Gamma(\alpha)$ consists of $2(q-2)$ sets of the form $\{0,x\}$ and $\{1,x\}$ where $x \in X \setminus \{0,1\}$.

Let $M = \mathrm{AGL}(1,q) = \{t_{ab} : z \mapsto az+b | a,b \in X, a \neq 0\}$. Then $M_\alpha = \{t_{1,0}, t_{1,1}\}$.

Consider $\{0,x\} \in \Gamma(\alpha)$. Then $\{0,x\}^{M_\alpha} = \big\{\{0,x\},\{1,x+1\}\big\}$. Further, if $m = t_{x^{-1},0}$ then $\{0,x\}^m = \{0,1\} = \alpha$ and $\alpha^m = \{0,1\}^m = \{0,x^{-1}\}$. Thus, if ω is a primitive element of X and $O_{\omega^i}(\alpha) := \{0,\omega^i\}^{M_\alpha}$ then the set $\{O_{\omega^i}(\alpha) | i = 1,\ldots,q-2\}$ is the set of all M_α-orbits in $\Gamma(\alpha)$. Furthermore, since $(\omega^i)^{-1} = \omega^{q-1-i}$ then $O^*_{\omega^i}(\alpha) = O_{\omega^{q-1-i}}(\alpha)$ for each i where $1 \leq i \leq q-2$. Note that $q-1-i \neq i$ since q is even. Hence, M_α has $q-2$ orbits of length 2 in $\Gamma(\alpha)$, all not self-paired. ■

This lemma enables us to make the following construction.

Construction 3.4. Let $X = \mathrm{GF}(q)$ where $q = 2^{r^f}$ for some odd prime r and let $\Gamma = J(q,2)$ with $V\Gamma = X^{\{2\}}$. Let $M = \mathrm{AGL}(1,q)$ and $G = \mathrm{A\Gamma L}(1,q) = \langle M, \sigma\rangle$, where $\sigma \in \mathrm{Aut}(X)$ such that $\sigma : z \mapsto z^2$.

Take $\alpha = \{0,1\} \in V\Gamma$. Then $G_\alpha = \langle M_\alpha, \sigma\rangle$. By Lemma 3.3, M_α has $q-2$ orbits $O_{\omega^i}(\alpha)$ such that $O^*_{\omega^i}(\alpha) = O_{\omega^{q-1-i}}(\alpha)$, for $1 \leq i \leq q-2$. Set $Q_i(\alpha) := O_{\omega^i}(\alpha) \cup O_{\omega^{q-1-i}}(\alpha)$. Then the set $\{Q_i(\alpha) | 1 \leq i \leq \frac{q-2}{2}\}$ is a partition of $\Gamma(\alpha)$ where each $Q_i(\alpha)$ is invariant under M_α and is self-paired.

Now, every G_α-orbit on the set $\{Q_i(\alpha) \mid 1 \leq i \leq \frac{q-2}{2}\}$ has length a power of r greater than 1. Denote by $\mathcal{Q}$ the set of all such G_α-orbits. For $Q^{G_\alpha} \in \mathcal{Q}$ where say, $Q = Q_i(\alpha)$, let $Q' := \{Q, Q^{\sigma^r}, Q^{\sigma^{2r}}, \ldots, Q^{\sigma^{r^m}}\}$ for some m, that is, the $\langle\sigma^r\rangle$-orbit containing Q. Then define

$$E_1(\alpha) := \bigcup_{Q^{G_\alpha} \in \mathcal{Q}} Q'$$

and let $E_i(\alpha) := E_{i-1}(\alpha)^\sigma$ for $2 \leq i \leq r$.

Then the set $\mathcal{E}(\alpha) = \{E_1(\alpha), E_2(\alpha), \ldots, E_r(\alpha)\}$ is a partition of $\Gamma(\alpha)$ such that each $E_i(\alpha)$ is M_α-invariant and $E^*_i(\alpha) = E_i(\alpha)$ for each i. Further, G_α is transitive on $\mathcal{E}(\alpha)$. For each i define E_i as in the discussion preceding Theorem 2.3 and set $\mathcal{E} = \{E_1, \ldots, E_r\}$. Then by Theorem 2.3, $(\Gamma, \mathcal{E})$ is a (G, M)-homogeneous factorisation of index r for Γ. Note that for any 2-transitive subgroup M' of G which acts trivially on $\mathcal{E}$, $(\Gamma, \mathcal{E})$ is also a (G, M')-homogeneous factorisation.

To illustrate, consider the graph $\Gamma = J(512, 2)$ with $M = \mathrm{AGL}(1,512)$ and $G = \mathrm{A\Gamma L}(1,512)$. In this example, we have $q = 2^{r^f} = 512 = 2^9$ so $r = 3$. Let $X = \mathbb{F}_{512} = \{0, 1, \omega, \ldots, \omega^{510}\}$ and let $\alpha = \{0,1\} \in V\Gamma$.

Then there are 510 M_α-orbits in $\Gamma(\alpha)$, each of length two and these are of the form $O_{\omega^i}(\alpha) = \{0,\omega^i\}^{M_\alpha} = \big\{\{0,\omega^i\},\ \{1,\omega^i+1\}\big\}$ for each i. These M_α-orbits come in pairs where $O^*_{\omega^i}(\alpha) = O_{\omega^{511-i}}(\alpha)$. Thus, we have the following paired M_α-orbits:

$$\begin{gathered} Q_1(\alpha) = O_\omega(\alpha) \cup O_{\omega^{510}}(\alpha) \\ Q_2(\alpha) = O_{\omega^2}(\alpha) \cup O_{\omega^{509}}(\alpha) \\ \vdots \\ Q_{255}(\alpha) = O_{\omega^{255}}(\alpha) \cup O_{\omega^{256}}(\alpha). \end{gathered}$$

Since $G = M \cdot \langle \sigma \rangle$ where $\sigma : z \mapsto z^2$, the following are examples of G_α-orbits:

$$T_1 = \big\{Q_i(\alpha) \mid i \in \{1,2,4,8,16,32,64,128,255\}\big\}$$

and

$$T_2 = \big\{O_{73}(\alpha), O_{144}(\alpha), O_{219}(\alpha)\big\}.$$

From these, we include in $E_1(\alpha)$ the vertices in $\Gamma(\alpha)$ that are in every third element of T_1 and T_2 starting from the first element, that is, we include in $E_1(\alpha)$ the elements of the following:

$$Q_1(\alpha), Q_8(\alpha), Q_{64}(\alpha), \text{ and } Q_{73}(\alpha).$$

Then $E_2(\alpha)$ will include elements of

$$Q_2(\alpha), Q_{16}(\alpha), Q_{128}(\alpha), \text{ and } Q_{144}(\alpha),$$

and $E_3(\alpha)$ will include elements of

$$Q_4(\alpha), Q_{32}(\alpha), Q_{255}(\alpha), \text{ and } Q_{219}(\alpha).$$

Thus $\mathcal{E}(\alpha) = \{E_1(\alpha), E_2(\alpha), E_3(\alpha)\}$ is a partition of $\Gamma(\alpha)$ such that G_α acts transitively on this partition and we thus have a homogeneous factorisation.

Before showing that the homogeneous factorisations given by Construction 3.4 are pairwise non-isomorphic we need the following lemma.

Lemma 3.5. *There is no (G,M)-homogeneous factorisation of $\Gamma = J(p^d, 2)$ where $M = C_p^d \rtimes M_0$ and $G = C_p^d \rtimes G_0$, for $M_0, G_0 \leqslant \mathrm{GL}(d,p)$ with $\mathrm{SL}(a,q)$ normal in both M_0 and G_0 such that $p^d = q^a$ and $a \geq 2$.*

Proof. Let $\mathrm{SL}(a,q) \leqslant M_0 \leqslant \mathrm{GL}(d,p)$ where $a \geq 2$ and $p^d = q^a$, and let $\alpha = \{0, v\} \in V\Gamma$. Then $\Gamma(\alpha)$ consists of sets of the form $\{0,u\}$ and $\{v,u\}$ where $u \in X \setminus \{0,v\}$.

Let ℓ be the line in the affine geometry $\mathrm{AG}(a,q)$ containing 0 and v. Then $M_{0,v}$ fixes the line ℓ and is transitive on the points not on ℓ. Thus, the set $O(\alpha) = \{\{0,u\} | u \notin l\} \cup \{\{v,u\} \mid u \notin \ell\}$ is an M_α-orbit in $\Gamma(\alpha)$.

But G_α also fixes ℓ and is transitive on the points not on ℓ, so G_α leaves $O(\alpha)$ invariant. By Theorem 2.1, no (G,M)-homogeneous factorisation exists. ■

Lemma 3.6. *Let $G = \mathrm{A\Gamma L}(1,q)$ with $q = 2^{r^f}$ for r an odd prime, $M = \mathrm{AGL}(1,q)$ and let $\Gamma = J(q,2)$. Then there exist r^{t-1} pairwise non-isomorphic (G,M)-homogeneous factorisations $(\Gamma, \mathcal{E})$ arising from Construction 3.4, with*

$$t = \sum_{i=0}^{f-1} \frac{2^{r^i}(2^{r^i(r-1)} - 1)}{2r^{i+1}}.$$

Proof. Looking at Construction 3.4 we see that given Q^{G_α} there are r choices for Q' and so there are r^t choices for $E_1(\alpha)$, where t is the number of G_α-orbits in Q, and this is equal to the number of $\langle\sigma\rangle$-orbits on pairs $\{x, x^{-1}\}$ of elements of $\mathrm{GF}(2^{r^f}) \setminus \{0,1\}$. Since for each i, $0 \leq i < f$, σ permutes the elements of $\mathrm{GF}(2^{r^{i+1}}) \setminus \mathrm{GF}(2^{r^i})$ in cycles of length r^{i+1}, t is as given in the lemma. The partition $\mathcal{E}$ is uniquely determined by $E_1(\alpha)$, however, given $\mathcal{E}$ there are r choices for E_1. Hence the number of (G,M)-homogeneous factorisations $(\Gamma, \mathcal{E})$ obtained in this manner is r^{t-1}.

Suppose now that $(\Gamma, \mathcal{E}_1)$ and $(\Gamma, \mathcal{E}_2)$ are isomorphic (G,M)-homogeneous factorisations obtained from Construction 3.4. Then there exists $g \in \mathrm{Aut}(\Gamma) = S_q$ such that $\mathcal{E}_1^g = \mathcal{E}_2$. Moreover, $(\Gamma, \mathcal{E}_2)$

is a (G^g, M^g)-homogeneous factorisation. Let $\overline{G} = \langle G, G^g\rangle$ and $\overline{M}$ be the the normal closure in $\overline{G}$ of $\langle M, M^g\rangle$. Since both M and M^g act trivially on $\mathcal{E}_2$ and $\overline{G}$ preserves $\mathcal{E}_2$ it follows that $\overline{M}$ acts trivially on $\mathcal{E}_2$. Thus $(\Gamma, \mathcal{E}_2)$ is a $(\overline{G}, \overline{M})$-homogeneous factorisation. Now $\mathrm{AGL}(1, q) \leqslant \overline{M} \lhd \overline{G} \leqslant S_q$. By Proposition 2.10, A_q is arc-transitive on Γ and so $A_q \not\leqslant \overline{M}$. Also since $\mathrm{AGL}(1, q)$ is 2-transitive, by Burnside's Theorem, the groups $\overline{M}$, $\overline{G}$ are either of affine or almost simple type, and it follows from [19, Proposition 5.1], that both $\overline{M}$ and $\overline{G}$ are of affine type. Thus $\overline{M} = C_2^{r^f} \rtimes M_0$ for some irreducible subgroup M_0 of $\mathrm{GL}(r^f, 2)$. Moreover, $\overline{G} = C_2^{r^f} \rtimes G_0$ with $M_0 \lhd G_0$. Since M_0 and G_0 contain $\mathrm{GL}(1, q)$, [14] implies that $\mathrm{SL}(r^e, 2^{r^{f-e}})$ is normal in both M_0 and G_0 for some integer $e \geq 0$ and $M_0 \leqslant G_0 \leqslant \Gamma\mathrm{L}(r^e, 2^{r^{f-e}})$. It follows from Lemma 3.5 that $e = 0$ and so $M_0 \leqslant \mathrm{A}\Gamma\mathrm{L}(1, q)$. Hence $\overline{M} = M$ and so $g \in \mathrm{A}\Gamma\mathrm{L}(1, q) = G$. Thus $\mathcal{E}_1 = \mathcal{E}_2$. Hence the r^{t-1} (G, M)-homogeneous factorisations yielded by Construction 3.4 are pairwise non-isomorphic. ■

3.3 On $J(8, 3)$

Consider the group $\mathrm{AGL}(1, 8)$, which (see [12]) is 3-homogeneous but not 3-transitive. Let $X = \mathbb{F}_8$ and let ω be a primitive element of X such that $\omega^3 = \omega + 1$. We construct the following homogeneous factorisations for $J(8, 3)$.

Construction 3.7. Let X and ω be as given above. Let $\Gamma = J(8, 3)$ with $V\Gamma = X^{\{3\}}$ and let $\alpha = \{\omega, \omega^2, \omega^4\} \in V\Gamma$. Then $|\Gamma(\alpha)| = 15$ and $\Gamma(\alpha)$ consists of sets of the form $\{\omega, \omega^2, x\}$, $\{\omega, \omega^4, x\}$, $\{\omega^2, \omega^4, x\}$ where $x \in X \setminus \alpha$.

Let $M = \mathrm{AGL}(1, 8)$. Then $M_\alpha = 1$. Hence, there are 15 M_α-orbits in $\Gamma(\alpha)$. We show that 3 of these are self-paired. These are the elements of $\Gamma(\alpha)$ containing 0, proved as follows.

$$\begin{array}{lll}
\{\omega, \omega^2, 0\}^{t_{1,\omega^4}} = \alpha, & \alpha^{t_{1,\omega^4}} = \{\omega, \omega^2, 0\} & \Rightarrow \ \big\{\{\omega, \omega^2, 0\}\big\}^* = \big\{\{\omega, \omega^2, 0\}\big\} \\
\{\omega^2, \omega^4, 0\}^{t_{1,\omega}} = \alpha, & \alpha^{t_{1,\omega}} = \{\omega^2, \omega^4, 0\} & \Rightarrow \ \big\{\{\omega^2, \omega^4, 0\}\big\}^* = \big\{\{\omega^2, \omega^4, 0\}\big\} \\
\{\omega, \omega^4, 0\}^{t_{1,\omega^2}} = \alpha, & \alpha^{t_{1,\omega^2}} = \{\omega, \omega^4, 0\} & \Rightarrow \ \big\{\{\omega, \omega^4, 0\}\big\}^* = \big\{\{\omega, \omega^4, 0\}\big\}
\end{array}$$

The following computation shows that the remaining 12 orbits come in pairs:

$$\begin{array}{lll}
\{\omega, \omega^2, 1\}^{t_{\omega^5,\omega^3}} = \alpha, & \alpha^{t_{\omega^5,\omega^3}} = \{\omega, \omega^4, \omega^5\} & \Rightarrow \ \big\{\{\omega, \omega^2, 1\}\big\}^* = \big\{\{\omega, \omega^4, \omega^5\}\big\} \\
\{\omega, \omega^2, \omega^5\}^{t_{\omega^5,\omega^5}} = \alpha, & \alpha^{t_{\omega^5,\omega^5}} = \{\omega, \omega^4, \omega^3\} & \Rightarrow \ \big\{\{\omega, \omega^2, \omega^5\}\big\}^* = \big\{\{\omega, \omega^4, \omega^3\}\big\} \\
\{\omega^2, \omega^4, 1\}^{t_{\omega^3,\omega^6}} = \alpha, & \alpha^{t_{\omega^3,\omega^6}} = \{\omega, \omega^2, \omega^3\} & \Rightarrow \ \big\{\{\omega^2, \omega^4, 1\}\big\}^* = \big\{\{\omega, \omega^2, \omega^3\}\big\} \\
\{\omega^2, \omega^4, \omega^3\}^{t_{\omega^3,\omega^3}} = \alpha, & \alpha^{t_{\omega^3,\omega^3}} = \{\omega, \omega^2, \omega^6\} & \Rightarrow \ \big\{\{\omega^2, \omega^4, \omega^3\}\big\}^* = \big\{\{\omega, \omega^2, \omega^6\}\big\} \\
\{\omega, \omega^4, 1\}^{t_{\omega^6,\omega^5}} = \alpha, & \alpha^{t_{\omega^6,\omega^5}} = \{\omega^2, \omega^4, \omega^6\} & \Rightarrow \ \big\{\{\omega, \omega^4, 1\}\big\}^* = \big\{\{\omega^2, \omega^4, \omega^6\}\big\} \\
\{\omega, \omega^4, \omega^6\}^{t_{\omega^6,\omega^6}} = \alpha, & \alpha^{t_{\omega^6,\omega^6}} = \{\omega^2, \omega^4, \omega^5\} & \Rightarrow \ \big\{\{\omega, \omega^4, \omega^6\}\big\}^* = \big\{\{\omega^2, \omega^4, \omega^5\}\big\}
\end{array}$$

Define the following sets:

$$\begin{array}{ll}
E_1(\alpha) & := \big\{\{\omega, \omega^2, 0\}, \{\omega, \omega^2, 1\}, \{\omega, \omega^4, \omega^5\}, \{\omega, \omega^2, \omega^5\}, \{\omega, \omega^4, \omega^3\}\big\} \\
E_2(\alpha) & := \big\{\{\omega^2, \omega^4, 0\}, \{\omega^2, \omega^4, 1\}, \{\omega, \omega^2, \omega^3\}, \{\omega^2, \omega^4, \omega^3\}, \{\omega, \omega^2, \omega^6\}\big\} \\
E_3(\alpha) & := \big\{\{\omega, \omega^4, 0\}, \{\omega, \omega^4, 1\}, \{\omega^2, \omega^4, \omega^6\}, \{\omega, \omega^4, \omega^6\}, \{\omega^2, \omega^4, \omega^5\}\big\}
\end{array}$$

Then $\mathcal{E}(\alpha) = \{E_1(\alpha), E_2(\alpha), E_3(\alpha)\}$ is a partition of $\Gamma(\alpha)$ such that $E_i(\alpha)$ is M_α-invariant and $E_i^*(\alpha) = E_i(\alpha)$ for each i.

Let $\sigma \in \mathrm{Aut}(X)$ such that $\sigma : z \mapsto z^2$ and take $G = M \cdot \langle\sigma\rangle = \mathrm{A}\Gamma\mathrm{L}(1, 8)$. Then $\sigma \in G_\alpha$ and G_α is transitive on $\mathcal{E}(\alpha)$ with σ permuting the $E_i(\alpha)$ cyclically. By Theorem 2.3, $(\Gamma, \mathcal{E})$ is a (G, M)-homogeneous factorisation of index 3.

We may vary this construction as follows to give nine (G, M)-homogeneous factorisations. In forming $E_1(\alpha)$, after choosing the 3-set $\{\omega, \omega^2, 0\}$, the second 3-set β may be any one of the three 3-sets in

$$\{\omega, \omega^2, 1\}^{\langle\sigma\rangle} = \{\{\omega, \omega^2, 1\}, \{\omega^2, \omega^4, 1\}, \{\omega, \omega^4, 1\}\}.$$

We must then add the pair $\{\beta\}^*$ (as given above) to $E_1(\alpha)$. Then the fourth 3-set γ may be any one of the 3-sets in

$$\{\omega,\omega^2,\omega^5\}^{\langle\sigma\rangle} = \{\{\omega,\omega^2,\omega^5\},\{\omega^2,\omega^4,\omega^3\},\{\omega,\omega^4,\omega^6\}\}$$

and the fifth 3-set in $E_1(\alpha)$ must then be $\{\gamma\}^*$.

Note that for the same values of M and G above, there are other choices for the $E_i(\alpha)$ which would lead to different homogeneous factorisations for Γ.

Lemma 3.8. *There are nine pairwise non-isomorphic (G,M)-homogeneous factorisations $(\Gamma,\mathcal{E})$ of index three for $J(8,3)$, where $M = \mathrm{AGL}(1,8)$ and $G = \mathrm{A\Gamma L}(1,8)$.*

Proof. To form the partition $\mathcal{E}(\alpha)$ we may assume that $E_1(\alpha)$ contains $\{\omega,\omega^2,0\}$. By Theorem 2.3, the remaining elements of $E_1(\alpha)$ must form M_α-self-paired subsets of $\Gamma(\alpha)$. There are exactly nine possibilities, namely those given in Construction 3.7. By Theorem 2.3, these are the only possibilities. Thus, there are 9 homogeneous factorisations corresponding to 9 different choices for $\mathcal{E}$ for $\Gamma = J(8,3)$ when $M = \mathrm{AGL}(1,8)$ and $G = \mathrm{A\Gamma L}(1,8)$.

Suppose now that $(\Gamma,\mathcal{E}_1)$ and $(\Gamma,\mathcal{E}_2)$ produced by Construction 3.7 are isomorphic homogeneous factorisations. Then there exists $g \in \mathrm{Aut}(\Gamma) = S_8$ such that $\mathcal{E}_1^g = \mathcal{E}_2$. Moreover, letting $\overline{M} = \langle M, M^g\rangle$ and $\overline{G} = \langle G, G^g\rangle$ we have that $(\Gamma,\mathcal{E}_2)$ is a $(\overline{G},\overline{M})$-homogeneous factorisation. Now $\mathrm{AGL}(1,8) < \overline{M} \lhd \overline{G} \leqslant S_8$. By Proposition 2.10, A_8 is arc-transitive and so $A_8 \not\leqslant \overline{M}$. Hence by [19, Proposition 5.2], $\overline{M} \leqslant \mathrm{AGL}(3,2)$. However, the only subgroups of $\mathrm{AGL}(3,2)$ containing $\mathrm{AGL}(1,8)$ are $\mathrm{AGL}(1,8), \mathrm{A\Gamma L}(1,8)$ and $\mathrm{AGL}(3,2)$. Since $\mathrm{A\Gamma L}(1,8)$ transitively permutes the three parts of $\mathcal{E}_2$, it follows that $\overline{M} = M = \mathrm{AGL}(1,8)$. Hence $g \in \mathrm{A\Gamma L}(1,8)$, and so $\mathcal{E}_1 = \mathcal{E}_2$. Hence the 9 homogeneous factorisations obtained from Construction 3.7 are pairwise non-isomorphic. ∎

3.4 On $J(q+1,3)$ with $q = 2^n$, n an odd prime power

We now consider the group $M = \mathrm{PGL}(2,q)$ when q is even and construct a family of (G,M)-homogeneous factorisations where $q = 2^{r^f}$ and r is an odd prime. First we consider the orbits of M_α in $\Gamma(\alpha)$.

Lemma 3.9. *Let $X = \mathrm{GF}(q) \cup \{\infty\}$ where q is even, $q > 2$. Let $\Gamma = J(q+1,3)$ with $V\Gamma = X^{\{3\}}$. Let $M = \mathrm{PGL}(2,q)$, and $\alpha = \{0,1,\infty\} \in V\Gamma$. Then M_α has $\frac{q-2}{2}$ orbits in $\Gamma(\alpha)$. Furthermore, each of these M_α-orbits has length 6, is self-paired and contains two elements of the form $\{0,1,x\}$ with $x \notin \alpha$, namely $\{0,1,x\}$ and $\{0,1,x+1\}$ for some $x \in \mathrm{GF}(q)$.*

Proof. Now

$$|M_\alpha| = \frac{|M|}{\binom{|X|}{3}} = \frac{q(q^2-1)}{\frac{(q+1)q(q-1)}{6}} = 6$$

and $M_\alpha = \{t_{1,0,0,1}, t_{1,1,1,0}, t_{0,1,1,1}, t_{0,1,1,0}, t_{1,0,1,1}, t_{1,1,0,1}\}$. Then set $\Gamma(\alpha)$ consists of sets of the form $\{0,1,x\},\{0,\infty,x\},\{1,\infty,x\}$ where $x \in X \setminus \alpha$ and $|\Gamma(\alpha)| = 3(q-2)$. First we show that every M_α-orbit in $\Gamma(\alpha)$ contains an element of the form $\{0,1,x\}$: Suppose $O = \{0,\infty,x\}^{M_\alpha}$. Then $\{0,\infty,x\}^{t_{1,0,1,1}} = \{0,1,x(x+1)^{-1}\} \in O$. On the other hand, if $O = \{1,\infty,x\}^{M_\alpha}$, then $\{1,\infty,x\}^{t_{0,1,1,0}} = \{1,0,x^{-1}\} \in O$.

Let $x,y \in X\setminus\alpha$ such that $\{0,1,y\} \in \{0,1,x\}^{M_\alpha}$. Then there exists $g \in M_\alpha$ such that $\{0,1,x\}^g = \{0,1,y\}$. This implies that either g fixes 0 and 1 pointwise or g interchanges 0 and 1. In the first case, $g = t_{1,0,0,1}$ which is the identity permutation so $x = y$. In the second case, $g = t_{1,1,0,1}$ which implies that $y = x+1$. Hence, whenever any two elements of the form $\{0,1,x\},\{0,1,y\}$ are in the same M_α-orbit then $y = x+1$. Conversely, $\{0,1,x\}$ and $\{0,1,x+1\}$ lie in the same M_α-orbit.

Since $\Gamma(\alpha)$ has $q-2$ elements of the form $\{0,1,x\}$ and every M_α-orbit contains 2 such elements then M_α has $\frac{q-2}{2}$ orbits in $\Gamma(\alpha)$. Also, since $|\Gamma(\alpha)| = 3(q-2)$, each of the M_α-orbits is of length 6.

Further, for each $\beta = \{0,1,x\}$ with $x \in X \setminus \alpha$, the element $m = t_{x,x,1,x} \in M$ interchanges 0 and 1 and also x and ∞. Thus m interchanges β and α. Hence every M_α-orbit in $\Gamma(\alpha)$ is self-paired. ■

We now construct the homogeneous factorisations. The construction is similar to that in Construction 3.4. For $x \in \mathrm{GF}(q) \setminus \{0,1\}$, let $O_x(\alpha)$ denote the M_α-orbit on $X^{\{3\}}$ containing $\{0,1,x\}$. By Lemma 3.9, $O_x(\alpha) = O_y(\alpha)$ if and only if $x + y = 1$.

Construction 3.10. Let $q = 2^{r^f}$ for some odd prime r and let $\Gamma = J(q+1,3)$ where $V\Gamma$ is the set of all 3-subsets of $X = \mathrm{GF}(q) \cup \{\infty\}$. Let $M = \mathrm{PGL}(2,q)$, $G = \mathrm{P\Gamma L}(2,q)$ and $\alpha = \{0,1,\infty\}$. Note that $G = \langle M, \sigma\rangle$, where $\sigma : z \mapsto z^2$ for each $z \in X$.

We claim that σ does not fix setwise any of the $O_x(\alpha)$. Suppose to the contrary that σ fixes setwise the M_α-orbit containing the vertices $\{0,1,x\}$ and $\{0,1,x+1\}$, where $x \in X \setminus \alpha$. Then σ fixes the set $\{x, x+1\}$. Now $\{x,x+1\}^\sigma = \{x^2, x^2+1\}$, so $x^2 = x+1$ (since $x \neq x^2$ for $x \neq 0,1$). Now σ maps x^2 to $x^4 = (x^2)^2 = (x+1)^2 = x^2+1 = (x+1)+1 = x$. This implies that σ has even order, which contradicts the fact that $|\sigma| = n = r^f$ is odd. This proves our claim.

Denote by $\mathcal{Q}$ the set of G_α-orbits on the set $\{O_x(\alpha) \mid x \in \mathrm{GF}(q)\setminus\{0,1\}\}$. For each $Q^{G_\alpha} \in \mathcal{Q}$, where $Q = O_x(\alpha)$ for some x, let $Q' := \{Q, Q^{\sigma^r}, Q^{\sigma^{2r}}, \ldots, Q^{\sigma^{rm}}\}$, the $\langle\sigma^r\rangle$-orbit containing Q. Then define

$$E_1(\alpha) := \bigcup_{Q \in \mathcal{Q}} Q'$$

and let $E_i(\alpha) := E_{i-1}(\alpha)^\sigma$ for $2 \leq i \leq r$.

Then the set $\mathcal{E}(\alpha) = \{E_1(\alpha), E_2(\alpha), \ldots, E_r(\alpha)\}$ is a partition of $\Gamma(\alpha)$ such that $E_i(\alpha)$ is M_α-invariant and $E_i^*(\alpha) = E_i(\alpha)$ for each i. Further, G_α is transitive on $\mathcal{E}(\alpha)$. Hence by Theorem 2.3 we can use Construction 2.2 to construct a (G,M)-homogeneous factorisation $(\Gamma, \mathcal{E})$. Note that $(\Gamma,\mathcal{E})$ is also a (G,M')-homogeneous factorisation with $M' = \langle \mathrm{PGL}(2,q), \sigma^s\rangle$ for any divisor $s > 1$ of r^f.

Lemma 3.11. *Let $G = \mathrm{P\Gamma L}(2,q)$ with $q = 2^{r^f}$ for r an odd prime and let $\Gamma = J(q+1,3)$. Then there exist r^{t-1} pairwise non-isomorphic (G,M)-homogeneous factorisations $(\Gamma,\mathcal{E})$ where*

$$t = \sum_{i=0}^{f-1} \frac{2^{r^i}(2^{r^i(r-1)} - 1)}{2r^{i+1}}.$$

Proof. Looking at Construction 3.10 we see that given Q^{G_α} there are r choices for Q' and so there are r^t choices for $E_1(\alpha)$, where t is the number of orbits of $\langle\sigma\rangle$ on pairs $\{x, x+1\}$ of elements of $\mathrm{GF}(2^{r^f}) \setminus \{0,1\}$. Since for each i, $0 \leq i \leq f-1$, σ permutes the elements of $\mathrm{GF}(2^{r^{i+1}}) \setminus \mathrm{GF}(2^{r^i})$ in cycles of length r^{i+1}, t is as given in the lemma. The partition $\mathcal{E}$ is uniquely determined by $E_1(\alpha)$. However, given $\mathcal{E}$ there are r choices for E_1. Hence the number of (G,M)-homogeneous factorisations obtained is r^{t-1}.

Suppose now that $(\Gamma,\mathcal{E}_1)$ and $(\Gamma,\mathcal{E}_2)$ produced by Construction 3.10 are isomorphic homogeneous factorisations. Then there exists $g \in \mathrm{Aut}(\Gamma) = S_{q+1}$ such that $\mathcal{E}_1^g = \mathcal{E}_2$. Let $\overline{G} = \langle G, G^g\rangle$ and let $\overline{M}$ be the normal closure of $\langle M, M^g\rangle$ in $\overline{G}$. Since M and M^g act trivially on $\mathcal{E}_2$ and $\overline{G}$ preserves $\mathcal{E}_2$ it follows that $\overline{M}$ acts trivially on $\mathcal{E}_2$. Thus $(\Gamma,\mathcal{E}_2)$ is a $(\overline{G},\overline{M})$-homogeneous factorisation. Now $\mathrm{PGL}(2,q) < \overline{M} \lhd \overline{G} \leqslant S_{q+1}$. By Proposition 2.10, A_{q+1} is arc-transitive and so $A_{q+1} \not\leqslant \overline{M}$. Thus by [18], $\overline{M} \leqslant \mathrm{P\Gamma L}(2,q)$ (recall that q is even). Thus $\overline{G} = \mathrm{P\Gamma L}(2,q) = G$ and hence $g \in G$. Hence $\mathcal{E}_1 = \mathcal{E}_2$ and so the homogeneous factorisations obtained from Construction 3.10 are pairwise non-isomorphic. ■

4 Almost Simple Groups and $J(n,2)$

In this section, we consider all 2-transitive groups which are almost simple and prove the following result:

Theorem 4.1. *There is no (G,M)-homogeneous factorisation $(\Gamma,\mathcal{E})$ for $\Gamma = J(n,2)$, where M,G are almost simple groups, except for that given in Construction* 3.2.

We may, and shall assume that $M \lhd G$, so M contains the simple socle of G. If G is a 2-transitive group of degree n which is almost simple then the simple normal subgroup L of G is one of the following: A_n, A_7 (with $n = 15$), $\mathrm{PSL}(d,q)$ (with $n = \frac{q^d-1}{q-1}$), $\mathrm{PSL}(2,11)$ (with $n = 11$), $\mathrm{PSU}(3,q)$ (with $n = q^3+1$), $Sp(2m,2)$ (with $n = 2^{2m-1} \pm 2^{m-1}$) , $Sz(q)$ (with $n = q^2+1$), $Ree(q)'$ (with $n = q^3+1$), M_{11} (with $n = 11$ or 12), M_n (with $n = 12, 22, 23$ or 24), HS (with $n = 176$) or Co_3 (with $n = 276$). We consider each of these cases and use the techniques/strategies discussed in Section 2. We obtain the following results:

4.1 Alternating, sporadic, symplectic and linear socle

Lemma 4.2. *There is no (G,M)-homogeneous factorisation of $J(n,2)$ with $n \geq 5$, $M \lhd G$, and $\mathrm{soc}(M) = A_n$, A_7 (with $n = 15$), $\mathrm{PSL}(2,11)$ (with $n = 11$), $Sp(2m,2)$ with $m \geq 3$, any of the Mathieu groups, HS or Co_3.*

Proof. Since A_n and the Mathieu groups are at least 3-transitive when $n \geq 5$ then by Proposition 2.10 no homogeneous factorisation exists for $J(n,2)$ when $\mathrm{soc}(M)$ is one of these groups. On the other hand, when M is any of the other groups in the statement then M is self-normalising in S_n; hence by Proposition 2.11, there is no possible group G that would give a (G,M)-homogeneous factorisation $(J(n,2),\mathcal{E})$. ■

Now we consider the projective groups. We saw a homogeneous factorisation of $J(q+1,2)$ when $q \equiv 1 \pmod 4$ and $\mathrm{PSL}(2,q) \leqslant M \leqslant \mathrm{P\Sigma L}(2,q)$ in Construction 3.2. Recall that $x = \square$ and $y \neq \square$ are used to denote that x is a nonzero squared element and y is a non-squared element, respectively of the field $\mathrm{GF}(q)$. We now obtain the following lemma.

Lemma 4.3. *Let $\Gamma = J(q+1,2)$ with q a prime power. If $q \not\equiv 1 \pmod 4$, then $\mathrm{PSL}(2,q)$ is edge-transitive on $J(q+1,2)$ and no (G,M)-homogeneous factorisation exists where M contains $\mathrm{PSL}(2,q)$. If $q \equiv 1 \pmod 4$ and $(\Gamma,\mathcal{E})$ is a (G,M)-homogeneous factorisation where M contains $\mathrm{PSL}(2,q)$, then $(\Gamma,\mathcal{E})$ is as in Construction* 3.2.

Proof. Let $X = \mathrm{GF}(q) \cup \{\infty\}$, $\Gamma = J(q+1,2)$ with $V\Gamma = X^{\{2\}}$, and $\alpha = \{0,\infty\} \in V\Gamma$. From the proof of Lemma 3.1, we recall that if $L = \mathrm{PSL}(2,q)$ then L_α is generated by the permutations $t_{a,0,0,a^{-1}}$ and $t_{0,b,-b^{-1},0}$ taking z to a^2z and $-\frac{b^2}{z}$, respectively.

Suppose $q \not\equiv 1 \pmod 4$. If q is even, then $L = \mathrm{PGL}(2,q)$ is 3-transitive. Thus, by Proposition 2.10, if $L \leqslant M$ then M is arc-transitive, hence edge-transitive, on Γ and so no (G,M)-homogeneous factorisations exist.

If $q \equiv 3 \pmod 4$ then -1 is not a square in $\mathrm{GF}(q)$. Hence, the M_α-orbits in $\Gamma(\alpha)$ are as follows:

$$E_1(\alpha) = \Big\{\{0,x\}\Big|x = \square\Big\} \bigcup \Big\{\{\infty,y\}\Big|y \neq \square\Big\}, \quad \text{and}$$

$$E_2(\alpha) = \Big\{\{0,y\}|y \neq \square\Big\} \bigcup \Big\{\{\infty,x\}|x = \square\Big\}.$$

Taking $\beta = \{0,1\} \in E_1(\alpha)$ and choosing $m_\beta = t_{1,0,-1,1}$ gives $\beta^{m_\beta} = \alpha$ and $\alpha^{m_\beta} = \{0,\infty\}^{t_{1,0,-1,1}} = \{0,-1\} \in E_2(\alpha)$. So $\beta \in E_2^*(\alpha) \cap E_1(\alpha)$ and hence $E_1(\alpha) = E_2^*(\alpha)$. Thus the G-orbits $E_1 = \{\alpha,\beta\}^G$ and $E_2 = \{\alpha,\{0,-1\}\}^G$ are equal and hence if $L \leqslant M$, then M is edge-transitive on Γ. Thus again no (G,M)-homogeneous factorisations exist.

Finally, suppose $q \equiv 1 \pmod 4$. By Lemma 3.1, $\mathrm{PSL}(2,q)$ has two orbits in $E\Gamma$ and so the only (G,M)-homogeneous factorisation with $\mathrm{PSL}(2,q) \leqslant M$ is that given by Construction 3.2. ■

Note that in the succeeding lemmas of the section, we conclude that no (G,M)-homogeneous factorisations exist for the remaining possibilities for M by showing that G_α fixes a particular M_α-orbit and invoking Theorem 2.1.

We next consider the subgroups of $\mathrm{P\Gamma L}(d,q)$ for $d \geq 3$.

Lemma 4.4. *There is no (G,M)-homogeneous factorisation of $J(n,2)$ with $n = \frac{q^d-1}{q-1}$, $d \geq 3$, and $\mathrm{PSL}(d,q) \leqslant M \lhd G \leqslant \mathrm{P\Gamma L}(d,q)$.*

Proof. Let $F = \mathrm{GF}(q)$ and $U = F^d$ with $d \geq 3$. Let $\{e_1, e_2, \ldots, e_d\}$ be a basis for U and let X be the set of 1-dimensional subspaces of U. Then for $\Gamma = J(n,2)$ with $n = \frac{q^d-1}{q-1}$, we may take $V\Gamma = X^{\{2\}}$.

Let $M \geqslant \mathrm{PSL}(d,q)$ and $\alpha = \{\langle e_1\rangle, \langle e_2\rangle\} \in V\Gamma$. Then $\Gamma(\alpha)$ consists of sets of the form $\{\langle e_1\rangle, \langle x\rangle\}$ and $\{\langle e_2\rangle, \langle x\rangle\}$ where $\langle x\rangle \in X \setminus \alpha$.

We may consider X as the set of points in the projective geometry $\mathrm{PG}(d-1,q)$. Let ℓ be the line containing $\langle e_1\rangle, \langle e_2\rangle$. Then M_α fixes ℓ and is transitive both on the points not on ℓ and the points on $\ell \setminus \alpha$. Hence, the orbits of M_α on $\Gamma(\alpha)$ are the sets $O_1(\alpha)$ and $O_2(\alpha)$, where

$$O_1(\alpha) := \{\{\langle e_i\rangle, \langle x\rangle\} \mid \langle x\rangle \in \ell, i = 1,2\} \text{ and}$$

$$O_2(\alpha) := \{\{\langle e_i\rangle, \langle x\rangle\} \mid \langle x\rangle \notin \ell, i = 1,2\}.$$

However, G_α also fixes the line ℓ, thus leaves the M_α-orbit $O_1(\alpha)$ invariant. By Theorem 2.1, no (G,M)-homogeneous factorisation exists. ■

4.2 Rank 1 unitary case

We now consider the unitary groups. Let $K = \mathbb{F}_{q^2}$ and $U = K^3$. Let $\overline{\alpha} = \alpha^q$ for $\alpha \in K$, and for $u = (\alpha, \beta, \gamma) \in U$, let $\overline{u} = (\overline{\alpha}, \overline{\beta}, \overline{\gamma})$. A vector u is *isotropic* if $u.\overline{u} = \alpha\overline{\gamma} + \beta\overline{\beta} + \gamma\overline{\alpha} = 0$. Let X be the set of 1-dimensional subspaces of U generated by isotropic vectors. Then $X = \{\langle(1,0,0)\rangle\} \cup \{\langle(\lambda,\beta,1)\rangle \mid \lambda + \overline{\lambda} + \beta\overline{\beta} = 0,\ \lambda, \beta \in K\}$, $|X| = q^3 + 1$ and $\mathrm{PSU}(3,q)$ acts 2-transitively on X (see [6, p 248]).

Consider the following matrix in $\mathrm{GU}(3,q)$:

$$h_{\gamma,\delta} = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & (\overline{\gamma})^{-1} \end{bmatrix}.$$

Let $H = \{h_{\gamma,\delta} | \gamma, \delta \in K, \gamma \neq 0, \delta\overline{\delta} = 1\}$. The group H contains the scalar subgroup $Z \cap \mathrm{GU}(3,q) = \{h_{\delta,\delta} | \delta \in K, \delta\overline{\delta} = 1\}$. Define the subgroup $\overline{H} = \frac{H}{Z \cap \mathrm{GU}(3,q)}$ of $\mathrm{PGU}(3,q)$. Then $|\overline{H}| = q^2 - 1$ and $\overline{H}$ is isomorphic to the multiplicative group K^*. Also, $\mathrm{PGU}(3,q)_{\langle e_1\rangle, \langle e_3\rangle} = \overline{H}$, where $e_1 = (1,0,0)$ and $e_3 = (0,0,1)$ (see [6, p 248–249]).

Now, consider the set of matrices Q in $\mathrm{SU}(3,q)$ given by

$$Q = \left\{ u_{\lambda,\beta} = \begin{bmatrix} 1 & 0 & 0 \\ \beta & 1 & 0 \\ \lambda & -\overline{\beta} & 1 \end{bmatrix} \middle| \lambda, \beta \in K,\ \lambda + \overline{\lambda} + \beta\overline{\beta} = 0 \right\}.$$

Again, denote by $\overline{Q}$ the corresponding subgroup in $\mathrm{PSU}(3,q)$.

From [11], the group $\mathrm{PSU}(3,q)$ has a maximal subgroup $\langle \overline{Q},\ n\rangle$ where the corresponding matrix n in $\mathrm{SU}(3,q)$ is represented by:

$$n = \mathrm{diag}(\rho) = \left[\begin{array}{ccc} \rho & 0 & 0 \\ 0 & \rho^{q-1} & 0 \\ 0 & 0 & (\overline{\rho})^{-1} \end{array}\right]$$

where ρ is a generator of the multiplicative group K^*. Further, $\mathrm{PSU}(3,q)$ is generated by $\langle \overline{Q},\ n\rangle$ and one extra element r whose corresponding matrix in $\mathrm{SU}(3,q)$ may be taken as

$$r = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{array}\right].$$

We use the above discussion in the proof of the following lemma:

Lemma 4.5. *There is no (G,M)-homogeneous factorisation of $J(q^3+1,2)$ with $\mathrm{PSU}(3,q) \leqslant M \lhd G \leqslant \mathrm{P\Gamma U}(3,q)$.*

Proof. Let $\Gamma = J(q^3+1,2)$ with $V\Gamma = X^{\{2\}}$, where X is as defined above.

Let $\alpha = \{\langle e_1\rangle, \langle e_3\rangle\} \in V\Gamma$ where $e_1 = (1,0,0)$ and $e_3 = (0,0,1)$. Also, let $L = \mathrm{PSU}(3,q)$. Then $L_{\langle e_1\rangle,\langle e_3\rangle} = \langle n\rangle$ and $L_\alpha = \langle n\rangle \cdot \langle r\rangle$, where n and r are as given above.

Consider the element $\langle(\lambda,0,1)\rangle \in X \setminus \alpha$. Then $\langle(\lambda,0,1)\rangle^n = \langle(\lambda\rho, 0, (\overline{\rho})^{-1})\rangle = \langle(\lambda\rho^{q+1},0,1)\rangle$. So $\langle(\lambda,0,1)\rangle^{L_{\langle e_1\rangle,\langle e_3\rangle}} = \{\langle(\lambda,0,1)\rangle \mid \lambda \in K, \lambda \neq 0, \lambda + \overline{\lambda} = 0\}$.

Now, we have $\langle(\lambda,0,1)\rangle^r = \langle(1,0,\lambda)\rangle = \langle(\lambda^{-1},0,1)\rangle$. Thus the L_α-orbit $\{\langle e_1\rangle, \langle(\lambda,0,1)\rangle\}^{L_\alpha}$ in $\Gamma(\alpha)$ consists of all the pairs $\{\langle e_i\rangle, \langle(\lambda,0,1)\rangle\}$ for $i = 1,3$ and $\lambda \in K, \lambda \neq 0, \lambda + \overline{\lambda} = 0$.

Since $\overline{H}$ fixes $\langle e_1\rangle$ and $\langle e_3\rangle$ and fixes the set of all one-dimensional subspaces $\{\langle(\lambda,0,1)\rangle\}$ then $\overline{H}$ fixes the orbit $\{\langle e_1\rangle, \langle(\lambda,0,1)\rangle\}^{L_\alpha}$. The Frobenius automorphism also fixes $\langle e_1\rangle, \langle e_3\rangle$ and the set $\{\langle(\lambda,0,1)\rangle | \lambda \in K, \lambda \neq 0, \lambda + \overline{\lambda} = 0\}$. Hence, $\mathrm{P\Gamma U}(3,q)_\alpha$ also fixes the orbit $\{\langle e_1\rangle, \langle(\lambda,0,1)\rangle\}^{L_\alpha}$. Therefore, from Theorem 2.1, there can be no (G,M)-homogeneous factorisation with M containing $\mathrm{PSU}(3,q)$. ∎

4.3 Suzuki groups

We now consider the Suzuki groups. Let $F = \mathrm{GF}(q)$, where $q = 2^{2m+1} > 2$, and $\sigma \in \mathrm{Aut}(F)$ such that $\sigma : z \mapsto z^{2^{m+1}}$. Using the notation from [6, p 250], take $X = \{(x,y,z) \in F^3 | z = xy + x^{\sigma+2} + y^\sigma\} \cup \{\infty\}$. Then $|X| = q^2+1$ and $Sz(q)$ acts 2-transitively on X where $Sz(q) = \langle T, N, w\rangle$, with $T = \{t_{ab} | a, b \in F\}$ and $N = \{n_c | c \in F, c \neq 0\}$ such that t_{ab} and n_c fix ∞, and

$$\begin{array}{rccl} t_{ab}: & (x,y,z) & \mapsto & (x+a, y+b+a^\sigma x, z+ab+a^{\sigma+2}+b^\sigma+ay+a^{\sigma+1}x+bx), \\ n_c: & (x,y,z) & \mapsto & (cx, c^{\sigma+1}y, c^{\sigma+2}z), \mathit{and} \\ w: & (x,y,z) & \mapsto & (\frac{y}{z}, \frac{x}{z}, \frac{1}{z}), z \neq 0 \\ & \infty & \leftrightarrow & (0,0,0). \end{array}$$

(Note that if $(x,y,0) \in X$ then $x = y = 0$ and so w is well defined.) Moreover, $|Sz(q)| = (q^2+1)q^2(q-1)$.

We now have the following lemma:

Lemma 4.6. *There is no (G,M)-homogeneous factorisation of $J(q^2+1,2)$ with $Sz(q) \leq M \lhd G \leq \mathrm{Aut}(Sz(q))$.*

Proof. Let $\Gamma = J(q^2+1, 2)$ with $V\Gamma = X^{\{2\}}$ and $\alpha = \{(0,0,0), \infty\} \in V\Gamma$. Let $L = Sz(q)$. Since N is a group of order $q-1$ fixing $(0,0,0)$ and ∞, we have $L_\alpha = \langle N, w\rangle$.

Consider the L_α-orbit $O(\alpha)$ containing $\beta := \{(0,0,0), (1,0,1)\}$. We claim that $O(\alpha) = \beta^{L_\alpha}$ is self-paired. Choose $m = wt_{0,1}$ so that

$$\beta^m = \{(0,0,0), (1,0,1)\}^{wt_{0,1}} = \{\infty, (0,1,1)\}^{t_{0,1}} = \{\infty, (0,0,0)\} = \alpha.$$

Then $\alpha^m \in (O(\alpha))^*$ and is equal to

$$\{(0,0,0), \infty\}^{wt_{0,1}} = \{\infty, (0,0,0)\}^{t_{0,1}} = \{\infty, (0,1,1)\} = \beta^w \in \beta^{L_\alpha} = O(\alpha).$$

Hence $\alpha^m \in O^*(\alpha) \cap O(\alpha)$. Therefore, $O^*(\alpha) = O(\alpha)$ and $O(\alpha)$ is self-paired.

Let $\phi \in \mathrm{Sym}(X)$ such that $\phi : (x,y,z) \mapsto (x^2, y^2, z^2)$ and ϕ maps ∞ to itself. Then ϕ fixes α and β. Moreover, ϕ normalises L and $\mathrm{Aut}(L) \cong L \cdot \langle\phi\rangle$. Thus we may identify $\mathrm{Aut}(L)$ with the subgroup $L \cdot \langle\phi\rangle$ of $\mathrm{Sym}(X)$, and we have that $\phi \in (\mathrm{Aut}(L))_\alpha$ and $(\mathrm{Aut}(L))_\alpha$ fixes the L_α-orbit $O(\alpha)$ setwise. So by Theorem 2.1, no (G, M)-homogeneous factorisation exists when M contains L. ■

4.4 Ree groups

The following discussion of the Ree groups is analogous to the discussion of the Suzuki groups. Again, we use the notation from [6, p 251] (see also Errata which can be accessed via the website http://math.carleton.ca/ jdixon/).

Let $F = \mathrm{GF}(q)$ and $\sigma \in \mathrm{Aut}(F)$ such that $\sigma : z \mapsto z^{3^{m+1}}$. Let X consist of ∞ and the set of sixtuples $(x, y, z, \lambda_1, \lambda_2, \lambda_3)$ such that $x, y, z \in F$ and

$$\begin{aligned}
\lambda_1 &= x^2y - xz + y^\sigma - x^{\sigma+3} \\
\lambda_2 &= x^\sigma y^\sigma - z^\sigma + xy^2 + yz - x^{2\sigma+3} \\
\lambda_3 &= xz^\sigma - x^{\sigma+1}y^\sigma + x^{\sigma+3}y + x^2y^2 - y^{\sigma+1} - z^2 + x^{2\sigma+4}.
\end{aligned}$$

Then $|X| = q^3 + 1$ and $Ree(q)$ acts 2-transitively on X, where $Ree(q) = \langle T, N, w\rangle$, with $T = \{t_{abc} | a, b, c \in F\}$, $N = \{n_k | k \in F, k \neq 0\}$ and t_{abc}, n_k, w permutations on X given by

$$\begin{aligned}
(x, y, z, \lambda_1, \lambda_2, \lambda_3)^{t_{abc}} &= (x + a, y + b + a^\sigma x, z + c - ay + bx - a^{\sigma+1}x, *, *, *), \\
(x, y, z, \lambda_1, \lambda_2, \lambda_3)^{n_k} &= (kx, k^{\sigma+1}y, k^{\sigma+2}z, k^{\sigma+3}\lambda_1, k^{2\sigma+3}\lambda_2, k^{2\sigma+4}\lambda_3), \\
(x, y, z, \lambda_1, \lambda_2, \lambda_3)^{w} &= (\tfrac{\lambda_2}{\lambda_3}, \tfrac{\lambda_1}{\lambda_3}, \tfrac{z}{\lambda_3}, \tfrac{y}{\lambda_3}, \tfrac{x}{\lambda_3}, \tfrac{1}{\lambda_3}) \text{ for } \lambda_3 \neq 0
\end{aligned}$$

and such that t_{abc}, n_k fix ∞ while w switches ∞ and $(0,0,0,0,0,0)$. Moreover, $|Ree(q)| = (q^3 + 1)q^3(q-1)$.

Note that since the last 3 components of any element of X are determined by the values of the first 3, then the missing values in the action of t_{abc} may be calculated from the formulae given above for the elements in X.

Using the above discussion, we have the following lemma:

Lemma 4.7. *There is no (G, M)-homogeneous factorisation of $J(q^3+1, 2)$ where $Ree(q) \leq M \lhd G \leq \mathrm{Aut}(R(q))$.*

Proof. Let $\Gamma = J(q^3+1, 2)$ where $V\Gamma = X^{\{2\}}$. Denote by $\widehat{0}$ the element $(0,0,0,0,0,0) \in X$ and let $\alpha = \{\widehat{0}, \infty\} \in V\Gamma$. Let $L = Ree(q)$. Then N fixes $\widehat{0}$ and ∞, and has order $q-1$. Thus $L_\alpha = \langle N, w\rangle$.

Consider the L_α-orbit $O(\alpha)$ containing $\beta := \{\widehat{0}, (1,0,0,-1,-1,1)\} \in \Gamma(\alpha)$. We claim that $O(\alpha)$ is self-paired.

Choose $m = wt_{1,-1,0}$ so that

$$\begin{aligned}\beta^m &= \{\widehat{0}, (1,0,0,-1,-1,1)\}^{wt_{1,-1,0}}\\ &= \{\infty, (-1,-1,0,0,1,1)\}^{t_{1,-1,0}}\\ &= \{\infty, \widehat{0}\} = \alpha.\end{aligned}$$

Now, $\alpha^m = \{\widehat{0}, \infty\}^{wt_{1,-1,0}} = \{\infty, (1,-1,0,0,-1,1)\}$. To determine the orbit to which α^m belongs, we apply wn_{-1} and get

$$\begin{aligned}(\alpha^m)^{wn_{-1}} &= \{\infty, (1,-1,0,0,-1,1)\}^{wn_{-1}}\\ &= \{\widehat{0}, (-1,0,0,-1,1,1)\}^{n_{-1}}\\ &= \{\widehat{0}, (1,0,0,-1,-1,1)\} = \beta.\end{aligned}$$

Hence $\beta \in O^*(\alpha) \cap O(\alpha)$. Therefore $O^*(\alpha) = O(\alpha)$.

Define a map ϕ on X by $(x,y,z,\lambda_1,\lambda_2,\lambda_3)^\phi = (x^3,y^3,z^3,\lambda_1^3,\lambda_2^3,\lambda_3^3)$ and $\infty^\phi = \infty$. It follows from the definition of X that $\phi \in \mathrm{Sym}(X)$. Also ϕ normalises L and we may identify $\mathrm{Aut}(L)$ with $L \cdot \langle\phi\rangle$.

Then ϕ fixes α and β, so $\phi \in (\mathrm{Aut}(L))_\alpha$ and $(\mathrm{Aut}(L))_\alpha$ fixes the L_α-orbit $O(\alpha)$. Hence, by Theorem 2.1, no (G,M)-homogeneous factorisation exists when M contains $Ree(q)$. ■

We have now considered all 2-transitive groups which are almost simple. Theorem 4.1 follows from Lemmas 4.2 to 4.7.

5 Affine Groups and $J(n,2)$

The main result of this section is summarised in the following theorem:

Theorem 5.1. *If* $(\Gamma,\mathcal{E})$ *is a* (G,M)*-homogeneous factorisation of* $\Gamma = J(n,2)$ *and* M, G *are groups of affine type, then* $(\Gamma,\mathcal{E})$ *arises from Construction* 3.4.

Since G, M are transitive on $V\Gamma$, where $\Gamma = J(n,2)$ based on X, both groups are 2-homogeneous on X. If a group H is 2-homogeneous but not 2-transitive then X may be identified with $\mathrm{GF}(q)$ for some odd prime power $q \equiv 3 \pmod 4$, and $H \leqslant \mathrm{A\Gamma L}(1,q)$, with $|H|$ odd (see [12, Theorem 1]). On the other hand, if H is affine and 2-transitive then Hering's Theorem classifies all possibilities as follows (see [17, Appendix 1] or [3, p 194]):

Theorem 5.2 (Hering's Theorem)**.** *Let* H *be a* 2*-transitive affine group of degree* p^d*, with socle* $N = (\mathbb{Z}_p)^d$ *for some prime* p*, and let* H_0 *be the stabiliser of the zero vector. Then* $H_0 \leqslant \Gamma\mathrm{L}(a,q)$ *where* $q^a = p^d$ *and* H_0 *belongs to one of the following classes:*

(1). $a = 1$ *and* $H_0 \leqslant \Gamma\mathrm{L}(1,q)$*;*

(2). $a = 6,$ q *is even and* $G_2(q)' \trianglelefteq H_0$*;*

(3). $a \geq 2$ *and* $\mathrm{SL}(a,q) \trianglelefteq H_0$*;*

(4). $a \geq 4,$ a *is even and* $\mathrm{Sp}(a,q)' \trianglelefteq H_0$*;*

(5). $a = 2,$ $p = q = 5, 7, 11,$ *or* 23 *and* $\mathrm{SL}(2,3) \trianglelefteq H_0$*;*

(6). $a = 4,$ $p = 3$ *and* H_0 *has an extraspecial normal subgroup* E *of order* 2^5*,* $H_0/E \leqslant S_5$*, and* 5 *divides* $|H_0|$*;*

(7). $a = 2,$ $q = 9, 11, 19, 29$ *or* 59 *and* $\mathrm{SL}(2,5) \trianglelefteq H_0$*;*

(8). *$d = 6$, $p = 3$ and $H_0 = \mathrm{SL}(2, 13)$;*

(9). *$d = 4$, $p = 2$ and $H_0 \cong A_7$.*

As usual we will assume that $M \lhd G$. Then $G = NG_0$ and $M = NM_0$, where N is the translation group and $M_0 \lhd G_0 \leqslant \Gamma\mathrm{L}(a, q)$. In order to prove Theorem 5.1, we first make the following observation that uses the fact that $M_0 \lhd G_0$.

Remark 5.3. *Let $(\Gamma, \mathcal{E})$ be a (G, M)-homogeneous factorisation for $\Gamma = J(n, 2)$ where M, G are 2-transitive affine groups. If M_0 belongs to any one of the classes in items (1)–(9) of Hering's Theorem then G_0 belongs to the same class as M_0.*

We now consider the 2-homogeneous but not 2-transitive groups, and each of the classes in Hering's Theorem as possibilities for G. In most of these cases, we show that G_α fixes a particular M_α-orbit and use Theorem 2.1 to show that no homogeneous factorisation exists. In some instances, we invoke Proposition 2.11, namely that $M \neq G$. For an affine type subgroup $H \leqslant \mathrm{A\Gamma L}(a, q)$, we write $H = TH_0$ where T is the group of translations and $H_0 \leqslant \Gamma\mathrm{L}(a, q)$.

Proposition 5.4. *Let $(\Gamma, \mathcal{E})$ be a (G, M)-homogeneous factorisation for $\Gamma = J(q, 2)$ where $M_0 \leqslant G_0 \leqslant \Gamma\mathrm{L}(1, q)$. Then $q = 2^{r^f}$ for some odd prime r, $G = \mathrm{A\Gamma L}(1, q)$, M is a 2-transitive subgroup of $\langle \mathrm{AGL}(1, q), \sigma^s \rangle$ for some proper divisor s of r^f and $\mathcal{E}$ is a partition yielded by Construction 3.4.*

Proof. Let $\alpha = \{0, 1\}$ and $R = \mathrm{A\Gamma L}(1, q)$ where $q = p^d$ for some prime p. Given $a \in \mathrm{GF}(q)\backslash\{0\}$ and $b \in \mathrm{GF}(q)$, let $t_{a,b} : x \mapsto ax + b$ for each $x \in \mathrm{GF}(q)$. Then $R_\alpha = \langle t_{-1,1}, \sigma \rangle \cong C_2 \times C_d$ where σ is the field automorphism $\sigma : x \mapsto x^p$. Note that $M_\alpha \lhd G_\alpha \leqslant R_\alpha$. Now

$$\Gamma(\alpha) = \{\{0, x\}, \{1, x\} \mid x \in \mathrm{GF}(q) \setminus \{0, 1\}\}.$$

Note that if $x \in \mathrm{GF}(p) \setminus \{0, 1\}$ then $\{0, x\}^{R_\alpha} = \{\{0, x\}, \{1, 1 - x\}\}$.

Suppose first that M is 2-homogeneous but not 2-transitive. By [12, Theorem 1], q is odd and $|M|$ is odd. Thus M does not contain an element interchanging 0 and 1, and hence $\{0, -1\}^{R_\alpha} = \{\{0, -1\}, \{1, 2\}\}$ splits into two M_α-orbits. Now $m = t_{1,1}$ is a translation and so $m \in M$ and $\{0, -1\}^m = \{0, 1\} = \alpha$. Thus α^m lies in the paired M_α-orbit of $\{0, -1\}$. Since $\alpha^m = \{0, 1\}^m = \{1, 2\}$, the two M_α-orbits on $\{0, -1\}^{R_\alpha}$ are paired. Since each $E_i(\alpha)$ is self-paired it follows that $\{\{0, -1\}, \{1, 2\}\} \subseteq E_i(\alpha)$ for some i. However, $\{\{0, -1\}, \{1, 2\}\}$ is R_α-invariant and hence also G_α-invariant, contradicting Theorem 2.1. Thus M is 2-transitive. This implies that M_α contains an element interchanging 0 and 1 and so if $x \in \mathrm{GF}(p) \setminus \{0, 1\}$ then $\{0, x\}^{M_\alpha} = \{0, x\}^{R_\alpha}$. Thus if $p \neq 2$, there is an M_α-orbit invariant under G_α, again contradicting Theorem 2.1. Hence $p = 2$. It follows that $R_\alpha = \langle t_{1,1}, \sigma \rangle$ and since $t_{1,1}$ is a translation, $t_{1,1} \in M_\alpha$.

Suppose now that r is a prime dividing d and let $F = \mathrm{GF}(2^r)$. Then $\{\{0, x\}, \{1, x\} \mid x \in F\}$ is fixed setwise by R_α. For each $x \in F \setminus \{0, 1\}$, the orbit $O = \{0, x\}^{R_\alpha}$ is

$$\{\{0, x\}, \{1, 1 + x\}, \{0, x^2\}, \{1, 1 + x^2\}, \ldots, \{0, x^{2^{r-1}}\}, \{1, 1 + x^{2^{r-1}}\}\}$$

which has size $2r$. Since M_α contains $t_{1,1}$, M_α is normal in R_α, and G_α fixes no M_α-orbit, it follows that all the M_α-orbits on O have size 2 and G_α is transitive on O. Thus r divides $|G_\alpha|$ and G_α permutes the M_α-orbits in an r-cycle. Hence $|\mathcal{E}|$ divides r and so $|\mathcal{E}| = r$. As this holds for any prime dividing d it follows that $d = r^f$ for some prime r. Moreover, G_α projects onto $\langle \sigma \rangle$ since G_α acts nontrivially on F. Since G_α contains $t_{1,1}$ it follows that $G_\alpha = R_\alpha$. Thus by [16, Lemma 4.7], $G = \mathrm{A\Gamma L}(1, q)$. Let ω be a primitive element of $\mathrm{GF}(q)$. Then by [16, Lemma 4.4] there exist unique integers d, e and s such that $M_0 = \langle t_{\omega^d,0}, t_{\omega^e,0}\sigma^s \rangle$ with $0 \leq e < d$ and satisfying particular divisibility conditions. Without loss of generality we may suppose that M is the kernel of the action of G on $\mathcal{E}$ and so $M_0 \lhd G_0$. It follows from [16, Lemma 4.9] that d divides e. Hence $e = 0$.

Moreover, as M_0 is transitive on the nonzero elements of $\mathrm{GF}(q)$, [16, Lemma 4.7] implies $d = 1$. Thus $M = \langle \mathrm{AGL}(1,q), \sigma^s \rangle$ for some proper divisor s of r^f.

If $r = 2$ then $O = \{\{0,x\}, \{1, 1+x\}, \{0, x^2\}, \{1, 1+x^2\}\}$. Now $m = t_{x^{-1},0} \in M$ and maps $\{0,x\}$ to $\{0,1\}$ while $\{0,1\}^m = \{0, x^{-1}\}$. Hence since $x^{-1} = x^2$, $\{0,x\}^{M_\alpha}$ is paired with $\{0,x^2\}^{M_\alpha}$. Thus $O \subseteq E_i(\alpha)$ for some i and so $E_i(\alpha)$ is G_α-invariant, which is a contradiction. Hence r is odd. Since $|\mathcal{E}| = r$, and Construction 3.4 yields all G-invariant partitions of $E\Gamma$ or size r, it follows that $\mathcal{E}$ is a partition yielded by Construction 3.4. ∎

In view of [12, Theorem 1], Remark 5.3 and Proposition 5.4, we may assume from now on that G, M are affine 2-transitive groups on X satisfying the same part of Hering's Theorem, namely one of the parts (2)–(9). We have already seen in Lemma 3.5 that item (3) does not give rise to any homogeneous factorisations.

Lemma 5.5. *There is no (G,M)-homogeneous factorisation of $\Gamma = J(q^a, 2)$ when M_0, G_0 belong to item* (2) *of Hering's Theorem.*

Proof. Let V be a 6-dimensional vector space over $\mathrm{GF}(q)$ for q even equipped with a nondegenerate alternating form. Let $M_0, G_0 \leqslant \mathrm{GL}(V)$ preserve the form such that $G_2(q)' \lhd M_0 \lhd G_0 \leqslant \mathrm{Aut}(G_2(q)) \times Z$ where Z is the set of scalar matrices. By [4, Lemma 5.2], $G_2(q)$ has two orbits on totally isotopic subspaces of V, only one of which, $\mathcal{L}$, has length $(q^6-1)/(q-1)$. (This is the set of lines of the classical generalised hexagon preserved by $G_2(q)$ whose points are the set of 1-spaces of V.) Hence $\mathcal{L}$ is also an $\mathrm{Aut}(G_2(q)) \times Z$ orbit. Let $v \in V \setminus \{0\}$ and $\alpha = \{0, v\}$. Let

$$O(\alpha) = \{\{0,w\} \mid \langle v,w\rangle \in \mathcal{L}\} \cup \{\{v,w\} \mid \langle v,w\rangle \in \mathcal{L}\}.$$

Given $\langle v,w\rangle \in \mathcal{L}$ we have that $G_2(q)_{\langle v,w\rangle}$ is a parabolic subgroup of $G_2(q)$ ([4, Lemma 5.2]). Since $G_2(q)$ acts flag-transitively on the classical generalised hexagon it follows that $G_2(q)_{\langle v,w\rangle}$ acts transitively on the set of 1-spaces of $\langle v,w\rangle$. Now $G_2(q)_{\langle v,w\rangle}$ is a 2-group extended by $\mathrm{GL}(2,q)$ and $G_2(q)_{\langle v,w\rangle}$ induces $\mathrm{GL}(2,q)$ on $\langle v,w\rangle$. Hence $G_2(q)_{\langle v,w\rangle,v}$ acts transitively on the set of 1-spaces contained in $\langle v,w\rangle \setminus \langle v\rangle$. Thus $G_2(q)_v$ acts transitively on $\{\{v,w\} \mid \langle v,w\rangle \in \mathcal{L}\}$. When $q > 2$ we have $G_2(q)' = G_2(q)$ while $G_2(2)'$ is an index 2 subgroup of $G_2(2)$. Moreover, $G_2(2)'_v$ also acts transitively on $\{\{v,w\} \mid \langle v,w\rangle \in \mathcal{L}\}$. Thus for all q, $O(\alpha)$ is an M_α-orbit. However, $(\mathrm{Aut}(G_2(q)) \times Z)_\alpha$ also preserves $\mathcal{L}$ and so $O(\alpha)$ is also a G_α-orbit. By Theorem 2.1, no (G,M)-homogeneous factorisation exists. ∎

Lemma 5.6. *There is no (G,M)-homogeneous factorisation of $\Gamma = J(p^d, 2)$ where M_0, G_0 belong to item* (4) *of Hering's Theorem.*

Proof. If $p^d = 2^4$ then $M_0 \geqslant \mathrm{Sp}(4,2)' \cong A_6$ and M_0 has two orbits on $\Gamma(\alpha)$, of lengths 12 and 16. Hence G_0 fixes each of these orbits so by Theorem 2.1, no (G,M)-homogeneous factorisation exists in this case. Thus we may assume that $M_0 \geqslant \mathrm{Sp}(a,q)'$ with $(a,q) \neq (4,2)$. Let $U = \mathrm{GF}(q)^a$, with a even, $\{e_1, e_2, \ldots, e_{a/2}, f_1, f_2, \ldots, f_{a/2}\}$ be the standard basis for U and $\langle e_1\rangle^\perp = \langle e_1, \ldots, e_{a/2}, f_2, \ldots, f_{a/2}\rangle$. Let $\mathrm{Sp}(a,q)' \leqslant M_0 \lhd G_0 \leqslant \Gamma\mathrm{L}(a,q)$. Take $g \in M_0$ such that:

$$\begin{array}{llcll} g: & e_1 & \mapsto & e_1 & \\ & e_2 & \mapsto & e_2 + \lambda e_1 & \\ & f_1 & \mapsto & f_1 - \lambda f_2 & \\ & e_i & \mapsto & e_i, & 3 \le i \le a/2 \\ & f_i & \mapsto & f_i, & 2 \le i \le a/2 \end{array}$$

where λ is arbitrary. Then $g \in M_{0,e_1}$ and $(e_1 + e_2)^g = (1+\lambda)e_1 + e_2$.

p^d	i	$\lvert H_i \rvert$	$(H_i)_\alpha$ orbits in $\Gamma(\alpha)$
5^2	1	600	23 of length 2
	2	1200	3 of length 2, 10 of length 4
	3	2400	3 of length 2, 5 of length 8
7^2	1	2352	47 orbits of length 2
	2	7056	5 orbits of length 2, 14 of length 6
11^2	1	14520	all orbits have length 2
	2	29040	9 orbits of length 2 and 55 of length 4

Table 1: Details of orbits for groups in Hering's Theorem (5)

Let $u, u' \in \langle e_2, \dots, e_{a/2}, f_2, \dots, f_{a/2} \rangle \setminus \{0\}$ and let $h \in \mathrm{Sp}(a-2, q)$ such that $h : u \mapsto u'$. Extend h by defining $e_1^h = e_1, f_1^h = f_1$. Then $h \in M_{0,e_1}$ and $(\lambda e_1 + u)^h = \lambda e_1 + u'$. Hence, $(e_1 + e_2)^{M_{0,e_1}} = \langle e_1 \rangle^\perp \setminus \langle e_1 \rangle$. Since $M_0 \lhd G_0$ then G_{0,e_1} fixes $\langle e_1 \rangle^\perp \setminus \langle e_1 \rangle$ setwise.

If $\alpha = \{0, e_1\}$ then G_α and M_α also fix $\langle e_1 \rangle^\perp \setminus \langle e_1 \rangle$ setwise and by Theorem 2.1 no (G, M)-homogeneous factorisation may exist. ■

For the next lemma, we consider the affine groups M, G such that M_0, G_0 belong to item (5) of Hering's Theorem. Let $W = \mathrm{A\Gamma L}(d, p)$ and $T = (\mathbb{Z}_p)^d$. In proving the lemma, we obtained using MAGMA [1] a subgroup S of W_0 isomorphic to $\mathrm{SL}(2, 3)$ and found its normaliser N. In addition, we constructed all the subgroups X_0 of N that are transitive on $T \setminus \{0\}$ and contain $\mathrm{SL}(2, 3)$. Then the subgroups $X = \mathbb{Z}_p^2 \rtimes X_0 < \mathrm{AGL}(2, p)$ are the possibilities for M and G belonging to item (5). For each X, we constructed the stabiliser of $\alpha = \{x, y\} \in V\Gamma$ and found its orbits in $\Gamma(\alpha)$. The results are given in Table 1 and are used in our analysis of this case.

Lemma 5.7. *There is no (G, M)-homogeneous factorisation of $\Gamma = J(p^d, 2)$ where M_0, G_0 belong to item (5) of Hering's Theorem.*

Proof. For $p = 23$ there is only one 2-transitive group in item (5). Thus this does not give rise to a (G, M)-homogeneous factorisation by Theorem 2.1. For $p = 5, 7$ and 11 there are two or three 2-transitive groups. Moreover, for each value of p the possibilities can be labelled H_i such that $H_i < H_{i+1}$.

Table 1 collates information about the orbits of each of the groups. Note that for all values of p, $(H_1)_\alpha$ has an orbit on $\Gamma(\alpha)$ which is fixed setwise by $(H_i)_\alpha$ for each $i \geq 2$. If $(\Gamma, \mathcal{E})$ were a (G, M)-homogeneous factorisation with M and G from item (5) then we would have $M_\alpha = (H_1)_\alpha$ or $(H_2)_\alpha$ and $G_\alpha = (H_2)_\alpha$ or $(H_3)_\alpha$. Hence there exists an M_α-orbit on $\Gamma(\alpha)$ preserved by G_α, and so there are no such (G, M)-homogeneous factorisations by Theorem 2.1. ■

Lemma 5.8. *There is no (G, M)-homogeneous factorisation for $\Gamma = J(81, 2)$ where M_0, G_0 belong to item (6) of Hering's Theorem.*

Proof. Let $W = \mathrm{A\Gamma L}(4, 3)$. If $(\Gamma, \mathcal{E})$ is a (G, M)-homogeneous factorisation and M_0, G_0 belong to item (6) then M_0/E and G_0/E are one of C_5, D_{10}, $\mathrm{AGL}(1, 5)$, A_5, or S_5. As $\mathrm{AGL}(1, 5)$ and S_5 are self-normalising in S_5, neither of these can be M_0/E.

If $M_0/E \cong \mathbb{Z}_5$ then $G_0/E \cong D_{10}$ or $\mathrm{AGL}(1, 5)$. Similarly, if $M_0/E \cong D_{10}$ then $G_0/E \cong \mathrm{AGL}(1, 5)$, and if $M_0/E \cong A_5$ then $G_0/E \cong S_5$. Calculating in MAGMA, we find that G_α fixes an M_α-orbit of length 2 in each case, contradicting Theorem 2.1. ■

We now consider the groups belonging to item (7). For the proof of this lemma, we calculate using MAGMA and proceed as in Lemma 5.7.

q^d	i	$\lvert H_i\rvert$	$(H_i)_\alpha$ orbits in $\Gamma(\alpha)$
9^2	1	19440	7 of length 2, 24 of length 6
	2	38880	7 of length 2, 12 of length 12
	3	77760	1 of length 2, 3 of length 4, 4 of length 12 and 4 of length 24
11^2	1	14520	all have length 2
	2	72600	9 of length 2 and the rest of length 10
29^2	1	706440	all have length 2
	2	1412880	27 of length 2 and the rest of length 4

Table 2: Details of orbits for groups in Hering's Theorem (7)

Lemma 5.9. *There is no (G, M)-homogeneous factorisation of* $\Gamma = J(p^d, 2)$ *where* M_0, G_0 *belong to item* (7)*,* (8) *or* (9) *of Hering's Theorem.*

Proof. When $q = 19$ or 59 there is only one 2-transitive group in item (7) and so they do not give rise to homogeneous factorisations. For $q = 9, 11$ and 29 there are two or three 2-transitive groups in item (7). Moreover, for each value of p the possibilities can be labelled H_i such that $H_i < H_{i+1}$. Table 2 collates information about the orbits of each of the groups. Note that for all values of q, $(H_1)_\alpha$ has an orbit on $\Gamma(\alpha)$ which is fixed setwise by $(H_i)_\alpha$ for each $i \geq 2$. If $(\Gamma, \mathcal{E})$ were a (G, M)-homogeneous factorisation with M_0 and G_0 from item (7) then we would have $M_\alpha = (H_1)_\alpha$ or $(H_2)_\alpha$ and $G_\alpha = (H_2)_\alpha$ or $(H_3)_\alpha$. Hence there exists an M_α-orbit on $\Gamma(\alpha)$ preserved by G_α, contradicting Theorem 2.1.

In items (8) and (9), M_0 is self-normalising in $\mathrm{GL}(d, p)$, contradicting Proposition 2.11. ■

Based on Proposition 5.4 and Lemmas 5.5 to 5.9 along with Lemma 3.5, we have now proved Theorem 5.1.

6 On $J(n, r)$ for $r > 2$

This section classifies all homogeneous factorisations of $J(n, r)$ for $r > 2$. We conclude in Lemmas 6.7 and 6.8 that no homogeneous factorisation exists for $J(n, r)$ when $r \geq 4$, and in Theorem 6.5, we find that the only homogeneous factorisations for $J(n, 3)$ are those given in Constructions 3.7 and 3.10.

6.1 Factorisations of $J(n, 3)$

If $(\Gamma, \mathcal{E})$ is a (G, M)-homogeneous factorisation of $\Gamma = J(n, 3)$ then M is 3-homogeneous. From Kantor's Theorem (see [12]), M is either 3-transitive, or $\mathrm{PSL}(2, q) \leqslant M \leqslant \mathrm{P\Gamma L}(2, q)$ where $n - 1 = q \equiv 3 \pmod 4$, or $M = \mathrm{AGL}(1, 8), \mathrm{A\Gamma L}(1, 8)$ or $\mathrm{A\Gamma L}(1, 32)$.

Lemma 3.8 classifies the (G, M)-homogeneous factorisations of $J(8, 3)$ when $M = \mathrm{AGL}(1, 8)$. We now consider the other 3-homogeneous but not 3-transitive groups.

Lemma 6.1. *There is no (G, M)-homogeneous factorisation of* $J(q + 1, 3)$ *with* $\mathrm{PSL}(2, q) \leqslant M \leqslant \mathrm{P\Gamma L}(2, q)$ *and* $q \equiv 3 \pmod 4$*.*

Proof. Let $q = p^r \equiv 3 \pmod 4$ and let $X = \mathrm{GF}(q) \cup \{\infty\}$. Let $\Gamma = J(q + 1, 3)$ with $V\Gamma = X^{\{3\}}$ and let $\alpha = \{0, 1, \infty\} \in V\Gamma$.

Suppose $M = \mathrm{PSL}(2, q)$. Then $M_\alpha = \{t_{1,0,0,1}, t_{1,-1,1,0}, t_{0,1,-1,1}\} \cong C_3$. These are the permutations taking z to z, $\frac{z-1}{z}$, $\frac{1}{-z+1}$ respectively.

If $G = \mathrm{PGL}(2,q)$ then $G_\alpha = M_\alpha \cup \{t_{0,1,1,0}, t_{1,0,1,-1}, t_{1,-1,0,-1}\}$. The latter set consists of the permutations taking z to $\frac{1}{z}, \frac{z}{z-1}, -z+1$ respectively. Thus, taking $\beta = \{0, \infty, -1\} \in \Gamma(\alpha)$ we find that

$$\beta^{M_\alpha} = \{\{0,\infty,-1\}, \{\infty,1,2\}, \{1,0,2^{-1}\}\}$$

is invariant under G_α. Further, let $\sigma \in \mathrm{Aut}(\mathrm{GF}(q))$ such that $\sigma : z \mapsto z^p$ and let σ map ∞ to itself. Then $G \cdot \langle\sigma\rangle = \mathrm{P\Gamma L}(2,q)$ and $\sigma \in \mathrm{P\Gamma L}(2,q)_\alpha$. Moreover, σ fixes β so $\mathrm{P\Gamma L}(2,q)_\alpha$ fixes the M_α-orbit containing β. Thus by Theorem 2.1, no (G,M)-homogeneous factorisation exists with $\mathrm{PSL}(2,q) \leqslant M \leqslant \mathrm{P\Gamma L}(2,q)$, $q \equiv 3 \pmod 4$. ■

We now consider the 3-transitive groups. These are S_n, A_n, the Mathieu groups, the 3-transitive subgroups of $\mathrm{P\Gamma L}(2,q)$ (see [7]) and $\mathrm{AGL}(d,2)$ for $d \geq 2$. First we consider the 3-transitive groups other than the subgroups of $\mathrm{P\Gamma L}(2,q)$.

Lemma 6.2. *There is no (G,M)-homogeneous factorisation of $\Gamma = J(n,3)$ with M equal to S_n, A_n, one of the Mathieu groups or $\mathrm{AGL}(d,2)$ for $d \geq 3$, or $C_2^4 \rtimes A_7$.*

Proof. Let $V\Gamma = X^{\{3\}}$ where $|X| = n$. Since $n \geq 2r = 6$ the groups S_n, A_n, M_{11}, M_{12}, M_{23} and M_{24} are at least 4-transitive so by Lemma 2.8 no (G,M)-homogeneous factorisation exists when M is one of these groups.

Let $\Gamma = J(22,3)$ and $\alpha \in V\Gamma$. Then $|\Gamma(\alpha)| = 57$. Let $M = M_{22}$. Calculating in GAP [10], we find that M_α has 2 orbits in $\Gamma(\alpha)$, one of length 48 and the other of length 9. Thus, if $G = \mathrm{Aut}(M)$ then G_α fixes each M_α-orbit in $\Gamma(\alpha)$ and by Theorem 2.1, no factorisation exists for $M = M_{22}$.

If $M = \mathrm{AGL}(d,2)$ for $d \geq 3$ then M is self-normalising in S_{2^d}, so by Proposition 2.11, there is no group G that would give a (G,M)-homogeneous factorisation. Similarly for $M = C_2^4 \rtimes A_7$. ■

Lemma 6.3. *Let $(\Gamma, \mathcal{E})$ be a (G,M)-homogeneous factorisation of $\Gamma = J(q+1,3)$ with $\mathrm{PGL}(2,q) \leqslant M \lhd G \leqslant \mathrm{P\Gamma L}(2,q)$. Then $q = 2^{r^f}$ for some odd prime r, $G = \mathrm{P\Gamma L}(2,q)$, $M = \langle \mathrm{PGL}(2,q), \sigma^s\rangle$ for some proper divisor s of r^f and $\mathcal{E}$ is a partition yielded by Construction 3.10.*

Proof. Let $V\Gamma = X^{\{3\}}$ where $X = \mathrm{GF}(q) \cup \{\infty\}$, and suppose that $(\Gamma,\mathcal{E})$ is a (G,M)-homogeneous factorisation of index k. Let $\alpha = \{0,1,\infty\}$ and $\sigma \in \mathrm{Aut}(\mathrm{GF}(q))$ such that $\sigma : z \mapsto z^p$. Extend σ to X by mapping ∞ to itself. Then $\mathrm{P\Gamma L}(2,q) = \langle \mathrm{PGL}(2,q), \sigma\rangle$ and σ fixes α.

Suppose first that q is odd and let $\beta = \{0,-1,\infty\}$. Since σ fixes β it follows that $\mathrm{P\Gamma L}(2,q)_\alpha$ fixes the orbit of β under $\mathrm{PGL}(2,q)_\alpha$, contradicting Theorem 2.1. Thus $q = 2^n$ for some positive integer n.

By Lemma 3.9, $\mathrm{PGL}(2,q)_\alpha$ has $\frac{q-2}{2}$ self-paired orbits of length 6 in $\Gamma(\alpha)$ and each of these contains an element of the form $\{0,1,x\}$ for some $x \in X \setminus \alpha$.

Now $M = \langle \mathrm{PGL}(2,q), \sigma^s\rangle$ for some divisor s of n and $G = \mathrm{PGL}(2,q) \cdot \langle\phi\rangle$ where $\phi = \sigma^t$ for some t dividing s. Also $t \neq s$ since $G \neq M$.

Choose a prime r dividing n. Let F denote the subfield of $\mathrm{GF}(q)$ of order 2^r. Then G_α fixes setwise the union Y of M_α-orbits containing points of the form $\{0,1,x\}$ with $x \in F \setminus \{0,1\}$. This set Y is contained in $\Gamma(\alpha)$. Hence, by Theorem 2.1, G_α fixes no M_α-orbit in Y and it follows that ϕ acts nontrivially on F. Since $|\mathrm{Aut}(F)| = r$ then r divides $|\phi|$ and ϕ permutes the M_α-orbits in Y in cycles of length r. Applying Theorem 2.1 again, we find that k divides r and hence $k = r$.

Since the above argument holds for any prime r dividing n, $n = r^f$ for some r. By the argument above, ϕ acts nontrivially on F, where $|F| = 2^r$, and hence $|\phi| = r^f$ and so $G = \mathrm{P\Gamma L}(2,q)$ and $s > 1$.

Now, each M_α-orbit in $\Gamma(\alpha)$ contains two elements of the form $\{0,1,x\}$ with $x \in X \setminus \{0,1\}$. If $r = 2$ then $|F| = 4$ and Y consists of exactly one M_α-orbit, say $E_i(\alpha)$, and so ϕ, and hence also G_α, fixes this $E_i(\alpha)$ setwise. This is a contradiction. Hence r is odd. Since $|\mathcal{E}| = r$, and Construction

3.10 yields all G-invariant partitions of $E\Gamma$ of size r, it follows that $\mathcal{E}$ is a partition yielded by Construction 3.10. ∎

By [7, Theorem 2.1], a 3-transitive subgroup of $\mathrm{P\Gamma L}(2,q)$ either contains $\mathrm{PGL}(2,q)$ or is equal to a group $M(s,q)$. Let $X = \mathrm{GF}(q) \cup \{\infty\}$ and let $\sigma \in \mathrm{Aut}(\mathrm{GF}(q))$ such that $\sigma : z \mapsto z^p$ and σ maps ∞ to itself. Then $M(s,q) = \langle \mathrm{PSL}(2,q), \sigma^s t_{\omega,0,0,1}\rangle$ where s is some divisor of $\frac{r}{2}$ and ω is a primitive element of $\mathrm{GF}(q)$. We then have the following lemma:

Lemma 6.4. *There is no (G,M)-homogeneous factorisation of $\Gamma = J(q+1,3)$ with M equal to the 3-transitive subgroup $M(s,q) = \langle \mathrm{PSL}(2,q), \sigma^s t_{\omega,0,0,1}\rangle$ of $\mathrm{P\Gamma L}(2,q)$.*

Proof. Let $q = p^r$ with $q > 3$, p odd and r even. Let $\Gamma = J(q+1,3)$ with $V\Gamma = X^{\{3\}}$ where $X = \mathrm{GF}(q) \cup \{\infty\}$. Let $\alpha = \{0,1,\infty\} \in V\Gamma$. Then $\mathrm{PSL}(2,q)_\alpha = \{t_{1,0,0,1}, t_{1,-1,1,0}, t_{0,1,-1,1}\}$. Let $M = M(s,q)$ where s is a divisor of $\frac{r}{2}$. Then by [7, Corollary 2.2], $M_{\infty 01} = \langle \sigma^{2s}\rangle$. Hence, $M_\alpha = \langle \mathrm{PSL}(2,q)_\alpha, \sigma^{2s}\rangle$.

Consider the M_α-orbit O in $\Gamma(\alpha)$ containing $\beta = \{0,\infty,-1\}$ and take $G = \langle M, \sigma\rangle = \mathrm{P\Gamma L}(2,q)$. Then $\sigma \in G_\alpha$ and G_α leaves the M_α-orbit O invariant. By Theorem 2.1, no homogeneous factorisation exists. ∎

We have now considered all 3-homogeneous groups. Lemmas 3.8 and 6.1–6.4 yield the following result:

Theorem 6.5. *The only homogeneous factorisations $(M,G,\Gamma,\mathcal{E})$ for $J(n,3)$ are those given in Construction 3.7 and Construction 3.10.*

6.2 Factorisations of $J(n,r)$ for $r \geq 4$

We now consider the possible factorisations of the Johnson graph $\Gamma = J(n,r)$ for r at least 4. We start with $r = 4$. If $(\Gamma,\mathcal{E})$ is a (G,M)-homogeneous factorisation then M and G are 4-homogeneous groups. First we consider the 4-homogeneous but not 4-transitive group $M = \mathrm{PSL}(2,8)$.

Lemma 6.6. *There is no (G,M)-homogeneous factorisation of $J(9,4)$ with $M = \mathrm{PSL}(2,8)$.*

Proof. Let $X = \mathbb{F}_8 \cup \{\infty\}$ and ω be a primitive element of $\mathbb{F}_8$. Let $M = \mathrm{PSL}(2,8)$, $\Gamma = J(9,4)$ with $V\Gamma = X^{\{4\}}$, and take $\alpha = \{0,\omega,\omega^2,\omega^4\} \in V\Gamma$. Then $M_\alpha = \{t_{1,0,0,1}, t_{1,\omega,0,1}, t_{1,\omega^2,0,1}, t_{1,\omega^4,0,1}\}$ and $G = N_{S_9}(M) = M.\langle\sigma\rangle = \mathrm{P\Gamma L}(2,8)$.

Let O be the M_α-orbit containing $\beta = \{\omega,\omega^2,\omega^4,\infty\}$. Now $\sigma \in G_{\alpha,\beta}$ and hence G_α leaves the M_α-orbit O invariant. Therefore by Theorem 2.1, no (G,M)-homogeneous factorisation exists. ∎

We now make the following general statement for $J(n,4)$.

Lemma 6.7. *There are no (G,M)-homogeneous factorisations of $J(n,4)$.*

Proof. Let $(\Gamma,\mathcal{E})$ be a (G,M)-homogeneous factorisation of $\Gamma = J(n,4)$. Then M must be transitive on $V\Gamma$ and hence 4-homogeneous on X, where $V\Gamma = X^{\{4\}}$.

From Kantor's Theorem (see [12]), M is either 4-transitive or $M = \mathrm{PSL}(2,8)$, $\mathrm{P\Gamma L}(2,8)$ or $\mathrm{P\Gamma L}(2,32)$. From Lemma 6.6, no factorisation exists for $M = \mathrm{PSL}(2,8)$. Also no factorisations exist for $M = \mathrm{P\Gamma L}(2,8)$ or $\mathrm{P\Gamma L}(2,32)$ as here $\mathrm{Aut}(M) = M$ and this contradicts Proposition 2.11.

If M is 4-transitive then M is one of $A_n, S_n, M_{11}, M_{12}, M_{23}$ or M_{24}. Since $n \geq 2r = 8$, S_n, A_n, M_{12} and M_{24} are at least 5-transitive so by Lemma 2.8 no homogeneous factorisation exists when M is one of these groups. If $M = M_{11}$ or M_{23} then $\mathrm{Aut}(M) = M$ and from Proposition 2.11, there is no group G that would give a (G,M)-homogeneous factorisation . ∎

Finally we show that there are no (G,M)-homogeneous factorisations for the Johnson graphs $J(n,r)$ with $r \geq 5$.

Lemma 6.8. *There are no (G, M)-homogeneous factorisations of $J(n, r)$ with $r \geq 5$.*

Proof. Let $(\Gamma, \mathcal{E})$ be a (G, M)-homogeneous factorisation for $J(n, r)$ where $r \geq 5$. Then M, G are 5-homogeneous groups. Since all 5-homogeneous groups are also 5-transitive, the only possibilities for M, G are S_n, A_n and, if $r = 5$, also M_{12} and M_{24}. Since $n \geq 2r$ then M cannot be S_n or A_n by Lemma 2.8. If $M = M_{24}$ then $\mathrm{Aut}(M) = M$ and by Proposition 2.11, no homogeneous factorisation may exist. Similarly, since $\mathrm{Aut}(M_{12}) \cap S_{12} = M_{12}$ we do not have $M = M_{12}$. ■

7 Summary

We now prove Theorem 1.1: All the homogeneous factorisations of the Johnson graphs $J(n, r)$ are known.

Proof. From Lemmas 6.7 and 6.8, we find that there are no homogeneous factorisations for $J(n, r)$ when $r \geq 4$ while Theorem 6.5 states that the only homogeneous factorisations for $J(n, 3)$ are those found in Constructions 3.7 and 3.10.

When $r = 2$ and G is almost simple, by Theorem 4.1 there are no homogeneous factorisations for $J(n, r)$ except for the ones given in Construction 3.2.

When $r = 2$ and G is affine, by Theorem 5.1 the only homogeneous factorisations arise from Construction 3.4. This completes the proof. ■

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