

Symmetrical Covers, Decompositions and Factorisations of Graphs

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Abstract

This paper introduces three new types of combinatorial structures associated with group actions, namely symmetrical covers, symmetrical decompositions, and symmetrical factorisations of graphs. These structures are related to and generalise various combinatorial objects, such as 2-designs, regular maps, near-polygonal graphs, and linear spaces.

1 Introduction to the concepts

In this introductory section we fix our notation and introduce the concepts of cover, decomposition and factorisation of a graph and explain when we regard such configurations as symmetrical. The objective of this chapter is to develop the general theory of symmetrical covers, decompositions and factorisations of graphs. We will mainly concentrate on the arc-symmetrical case.

A graph $\Gamma = (V, E)$ consists of a vertex set V and a subset E of unordered pairs of vertices called edges. Its automorphism group, denoted $\text{Aut}(\Gamma)$, is the subgroup of all permutations of V that preserve E .

Let $\Gamma = (V, E)$ be a graph, and let P_1, \dots, P_k with $k \geq 2$ be subsets of the edge set E such that $E = P_1 \cup P_2 \cup \dots \cup P_k$. Then $\mathcal{P} = \{P_1, \dots, P_k\}$ is called a *cover* of Γ , and the P_i are called the *parts* of \mathcal{P} . A cover \mathcal{P} of Γ is called a λ -*uniform cover* if each edge of Γ is contained in a constant number λ of the P_i . We usually identify a part P_i with its induced subgraph $[P_i] = (V_i, P_i)$ of Γ where V_i is the set of vertices of Γ which lie on an edge in P_i .

The well known *cycle double cover* conjecture for graphs (see [17, 18]) asserts that every 2-edge connected graph has a 2-uniform cover by cycles. The λ -uniform covers of the complete graph K_n with parts isomorphic to K_k correspond to the 2 - (n, k, λ) *designs* (see for example [13]). The vertices of K_n correspond to the points of the design, while each block of the design is the set of vertices in some part of the cover. Since each edge lies in λ parts, it follows that each pair of points lies in λ blocks.

A 1-uniform cover, that is, a cover such that every edge of Γ is contained in precisely one part is called a *decomposition* of Γ . Under the correspondence described in the

previous paragraph, decompositions of a complete graph with parts isomorphic to K_k correspond to *linear spaces* with line size k . This is discussed further in Section 4.2. A decomposition is called a *factorisation* if each part is a spanning subgraph. (By spanning, we mean that for every vertex v of Γ there is an edge $\{v, w\}$ in the subgraph.)

For a decomposition $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ of a graph Γ , if the subgraphs induced by each of the P_i are all isomorphic to Σ , then the decomposition is called an *isomorphic decomposition*, and Σ is called a *divisor* of Γ . An isomorphic decomposition of a graph is called an *isomorphic factorisation* if it is a factorisation, and in this case the divisors are called *factors*. Decompositions of graphs have been widely studied, see for example [3, 13], as have isomorphic factorisations, for example [11, 12].

Let \mathcal{P} be a cover of Γ and let $G \leq \text{Aut}(\Gamma)$. If G preserves \mathcal{P} and the permutation group $G^{\mathcal{P}}$ induced on \mathcal{P} is transitive then we say that the cover (Γ, \mathcal{P}) is *G-transitive*. If further \mathcal{P} is a decomposition or a factorisation of Γ , then Γ is called a *G-transitive decomposition* or a *G-transitive factorisation*, respectively. By definition, a transitive cover, decomposition or factorisation is an isomorphic cover, decomposition or factorisation, respectively. Symmetries of decompositions have been studied in [30, 33]. In particular, Robinson conjectured that every finite group occurs as $G^{\mathcal{P}}$ where (Γ, \mathcal{P}) is an isomorphic factorisation of a complete graph Γ and G is the largest group of automorphisms of Γ preserving \mathcal{P} . He showed [30, Proposition 3] that every finite group does occur as a subgroup of some $G^{\mathcal{P}}$. To our knowledge this conjecture is still open.

In this paper, transitivity is required not only on the set of parts, divisors, or factors, but also on the graphs: namely on their vertices or edges or arcs. For any graph Γ we denote by $V\Gamma$, $E\Gamma$, $A\Gamma$ the set of vertices, edges, and arcs respectively.

Definition 1.1. Let Γ be a graph, and let \mathcal{P} be a G -transitive cover, decomposition, or factorisation, of Γ , where $G \leq \text{Aut}\Gamma$. Let G_P be the stabiliser in G of the part $P \in \mathcal{P}$ and let X , xxx be as in one of the columns of Table 1. If G is transitive on $X\Gamma$ and G_P is transitive on XP , then (Γ, \mathcal{P}) is called *G-xxx-symmetrical*.

X	V	E	A
xxx	vertex	edge	arc

Table 1: Possibilities for X and xxx

We remark that there are $3 \times 3 = 9$ different objects defined in this definition; for example, if Γ is G -arc-transitive and P is G_P -arc-transitive, then (Γ, \mathcal{P}) is a G -arc-symmetrical cover, decomposition or factorisation. We see in Lemma 4.4 that if (Γ, \mathcal{P}) is a G -transitive decomposition and Γ is G -arc-transitive (respectively G -edge-transitive) then (Γ, \mathcal{P}) is a G -arc-symmetrical (respectively G -edge-symmetrical) decomposition. The following simple examples show that neither implication is true for covers.

Example 1.2. Let $\Gamma = C_6$ with vertices labelled by the elements of \mathbb{Z}_6 and x adjacent to $x \pm 1 \pmod{6}$.

(1). Let

$$P_1 = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}\}$$

$$P_2 = \{\{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 0\}\}$$

$$P_3 = \{\{4, 5\}, \{5, 0\}, \{0, 1\}, \{1, 2\}\}$$

and $\mathcal{P} = \{P_1, P_2, P_3\}$. Then (Γ, \mathcal{P}) is a 2-uniform cover which is invariant under the group $G = D_6$ (the dihedral group of order 6). Now G acts transitively on \mathcal{P} and on the set of edges of Γ . However, $G_{P_1} \cong C_2$ is not transitive on the edges of P_1 . Hence (Γ, \mathcal{P}) is not G -edge-symmetrical.

(2). Let

$$P_1 = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}\}$$

and let $\mathcal{P} = P_1^{C_6}$. Then $|\mathcal{P}| = 6$ and is preserved by $G = D_{12}$ and (Γ, \mathcal{P}) is a G -transitive, 4-uniform cover. However, the group G acts transitively on the set of arcs of Γ while $G_{P_1} \cong C_2$ is not transitive on the arcs of P_1 . Hence (Γ, \mathcal{P}) is not G -arc-symmetrical.

Arc-symmetrical covers of graphs with no isolated vertices are both edge-symmetrical and vertex-symmetrical. Conversely, it is a consequence of Lemma 4.4 (see Remark 4.5) that if (Γ, \mathcal{P}) is a G -edge-symmetrical decomposition and G acts arc-transitively on Γ then (Γ, \mathcal{P}) is G -arc-symmetrical. Similarly, if (Γ, \mathcal{P}) is a G -vertex-symmetrical decomposition and G acts edge-transitively (respectively arc-transitively) on Γ , then (Γ, \mathcal{P}) is G -edge-symmetrical (respectively G -arc-symmetrical). We see in Examples 3.4 and 3.16 that the same is not true for covers.

In the literature, various special cases of symmetrical covers, decompositions and factorisations have been studied. Arc-symmetrical 1-factorisations of complete graphs are classified by Cameron and Korchmáros in [6]. Arc-symmetrical 1-factorisations of arc-transitive graphs are addressed in [10] where a characterisation of those for 2-arc-transitive graphs is given. Arc-symmetrical decompositions of complete graphs are studied in [31], and arc-symmetrical decompositions of rank three graphs are investigated in [1]. The G -arc-symmetrical decompositions of Johnson graphs where G acts primitively on the set of divisors of the decomposition are classified in [8].

If (Γ, \mathcal{P}) is a G -transitive decomposition and the kernel M of the action of G on \mathcal{P} is vertex-transitive, then (Γ, \mathcal{P}) is called a (G, M) -homogeneous factorisation. In particular, homogeneous factorisations are vertex-symmetrical. The study of homogeneous factorisations was initiated by the second and third authors who introduced in [21] homogeneous factorisations of complete graphs. General homogeneous factorisations were introduced and investigated in [14] and studied further in [15, 16].

The next section gives some fundamental notions and results on permutation groups that underpin an investigation of these symmetrical configurations. This is followed by three sections addressing basic examples and theory for covers, decompositions and factorisations respectively. In the final section we discuss the behaviour of covers and decompositions when we pass to the quotient graph.

2 Some fundamentals concerning permutation groups

In this section we introduce some permutation group notions needed later. The reader is referred to [9] for more details.

Given a permutation group G acting on a set Ω and $\alpha \in \Omega$, we let $G_\alpha = \{g \in G \mid \alpha^g = \alpha\}$, the stabiliser in G of α . Let $B = \{\alpha_1, \dots, \alpha_k\} \subseteq \Omega$. For $g \in G$, $B^g = \{\alpha^g \mid \alpha \in B\}$ and the *setwise stabiliser* of B in G is $G_B = \{g \in G \mid B^g = B\}$. The *pointwise stabiliser* of B in G is $G_{(B)} = \{g \in G \mid \alpha_1^g = \alpha_1, \alpha_2^g = \alpha_2, \dots, \alpha_k^g = \alpha_k\}$ and is also denoted by $G_{\alpha_1, \alpha_2, \dots, \alpha_k}$. The following lemma will be particularly useful.

Lemma 2.1. [9, Ex 1.4.1] *Let G be a transitive subgroup of $\text{Sym}(\Omega)$ and $H \leq G$. Then $G = HG_\alpha$ if and only if H is transitive on Ω .*

Let G be a transitive subgroup of $\text{Sym}(\Omega)$. A partition $\mathcal{B} = \{B_1, \dots, B_k\}$ of Ω is called a *system of imprimitivity* and its elements are called *blocks of imprimitivity*, if for each $g \in G$ and $B_i \in \mathcal{B}$, the image $B_i^g \in \mathcal{B}$. Trivial blocks of imprimitivity exist for any transitive group G , and are the singleton subsets $\{\alpha\}$ ($\alpha \in \Omega$) and the whole set Ω . All other blocks of imprimitivity are called nontrivial and a transitive group G is said to be *imprimitive* if there exists a nontrivial block of imprimitivity. Given $\alpha \in \Omega$, there is a one-to-one correspondence between the subgroups H with $G_\alpha \leq H \leq G$ and the blocks of imprimitivity B containing α , given by $B = \alpha^H$ and $H = G_B$. See for example [9, Theorem 1.5A]. In particular, note that the stabiliser in G of a block of imprimitivity B is transitive on B . We say that G is *primitive* if it has no nontrivial systems of imprimitivity. It follows from the correspondence between blocks and overgroups of G_α that a transitive group G on Ω is primitive if and only if G_α is maximal in G for some $\alpha \in \Omega$.

Every nontrivial normal subgroup of a primitive group is transitive, for otherwise, the set of orbits of an intransitive normal subgroup forms a system of imprimitivity. We say that a permutation group is *quasiprimitive* if every nontrivial normal subgroup is transitive. Thus every primitive group is quasiprimitive. However, not every quasiprimitive group is primitive. For example, the right multiplication action of a nonabelian simple group on the set of right cosets of a nonmaximal subgroup is quasiprimitive but not primitive.

Given two graphs Γ, Σ we define the *cartesian product* of Γ and Σ to be the graph denoted by $\Gamma \square \Sigma$ with vertex set $V\Gamma \times V\Sigma$ and $\{(u_1, u_2), (v_1, v_2)\}$ is an edge if and only if either $u_1 = v_1$ and $\{u_2, v_2\} \in E\Sigma$, or $\{u_1, v_1\} \in E\Gamma$ and $u_2 = v_2$. If $G \leq \text{Aut}(\Gamma)$ and $H \leq \text{Aut}(\Sigma)$ then $G \times H \leq \text{Aut}(\Gamma \square \Sigma)$. The cartesian product of graphs is associative and hence the cartesian product $\Gamma_1 \square \Gamma_2 \square \dots \square \Gamma_t$ for graphs $\Gamma_1, \Gamma_2, \dots, \Gamma_t$ is well defined for any $t \geq 2$.

3 Transitive covers and symmetrical covers

Our first lemma shows that many covers of edge-transitive graphs are uniform.

Lemma 3.1. *Let Γ be a G -edge-transitive graph and \mathcal{P} be a cover of Γ which is G -invariant. Then (Γ, \mathcal{P}) is a uniform cover.*

Proof. Let $\{u, v\}$ be an edge of Γ and suppose that $\{u, v\}$ is contained in precisely λ parts of \mathcal{P} . Since G is edge-transitive and \mathcal{P} is G -invariant, λ is independent of the choice of $\{u, v\}$. Thus \mathcal{P} is a λ -uniform cover. \square

In fact every edge-transitive graph has many transitive covers. Here by a subgraph Σ of a graph Γ we mean any graph (U, E_U) where $U \subseteq V$ and $E_U \subseteq E \cap (U \times U)$. Also, for a subset $P \subseteq E$, the edge-induced subgraph $[P]$ is the subgraph (V_P, P) where V_P is the subset of vertices incident with at least one edge of P .

Lemma 3.2. *An edge-transitive graph Γ has a transitive cover with parts Σ if and only if Γ has a subgraph isomorphic to Σ .*

Proof. Let Σ be a subgraph of Γ and $G \leq \text{Aut}(\Gamma)$ be edge-transitive. Let $P = E\Sigma$, and let $\mathcal{P} = P^G$. Then by definition, (Γ, \mathcal{P}) is a G -transitive cover with parts isomorphic to Σ . \square

Each edge-intransitive subgroup of an edge-transitive group gives rise to edge-symmetrical uniform covers and the parameters λ can be expressed group theoretically.

Lemma 3.3. *Let $\Gamma = (V, E)$ be a connected G -edge-transitive graph and let $H < G$ such that H is intransitive on $E\Gamma$. Let P be an H -orbit in $E\Gamma$ and $\mathcal{P} = P^G$. Then (Γ, \mathcal{P}) is a G -edge-symmetrical λ -uniform cover with*

$$\lambda = \frac{|G_{\{v,w\}} : H_{\{v,w\}}|}{|G_P : H|}$$

where $\{v, w\} \in P$. Moreover, if Γ is G -arc-transitive and for each $\{v, w\} \in P$ there exists $g \in H$ such that $(v, w)^g = (w, v)$, then (Γ, \mathcal{P}) is a G -arc-symmetrical cover.

Proof. By definition, $H \leq G_P$ and the edge-induced subgraph $[P]$ is H -edge-transitive. Since G is edge-transitive, every edge of Γ occurs in some image of P and so \mathcal{P} is a G -transitive cover. Then by Lemma 3.1, λ -uniform cover for some λ and since $[P]$ is H -edge-transitive (Γ, \mathcal{P}) is G -edge-symmetrical. Moreover, $G_P = HG_{\{v,w\},P}$, so $|P| = |G_P : G_{\{v,w\},P}| = |H : H_{\{v,w\}}|$. Since \mathcal{P} is a λ -uniform cover, we have

$$|H : H_{\{v,w\}}||G : G_P| = |P||\mathcal{P}| = \lambda|E| = \lambda|G : G_{\{v,w\}}| = \frac{\lambda|G : G_P||G_P : H||H : H_{\{v,w\}}|}{|G_{\{v,w\}} : H_{\{v,w\}}|}.$$

Hence $\lambda = |G_{\{v,w\}} : H_{\{v,w\}}|/|G_P : H|$. \square

As noted in the introduction, every G -arc-symmetrical cover of a graph with no isolated vertices is G -edge-symmetrical and G -vertex-symmetrical. The following is an example of a G -edge-symmetrical cover of a G -arc-transitive graph which is not G -arc-symmetrical. In particular, it shows that for an arc-transitive graph, spinning an edge will not necessarily give an arc-symmetrical cover. Moreover, it is an example of the construction underlying Lemma 3.2, and if we take H to be a subgroup C_{11} it also illustrates Lemma 3.3.

Example 3.4. Let $\Gamma = K_{11}$ and $G = M_{11}$. Then Γ is G -arc-transitive. Let Σ be an 11-cycle in Γ . Since $M_{11} \cap D_{22} = C_{11}$, it follows that $G_\Sigma = C_{11}$ which is edge-transitive and vertex-transitive on Σ , but not arc-transitive. Let $P = E\Sigma$ and $\mathcal{P} = P^G$. Then (Γ, \mathcal{P}) is a G -edge-symmetrical and G -vertex-symmetrical cover which is not G -arc-symmetrical.

We are often interested in λ -covers for small values of λ , so we propose the following problems.

- Problem 3.5.** (i) For small values of λ , characterise the arc-transitive graphs that have an arc-symmetrical λ -uniform cover.
- (ii) For a given arc-transitive graph Γ , find the smallest value of λ such that Γ has an arc-symmetrical λ -uniform cover.

To illustrate the theory, we will present briefly some examples of symmetrical covers of some well known graphs, and examples of symmetrical covers with given specified parts.

3.1 Covers of complete graphs

Lemma 3.3 has the following corollary.

Corollary 3.6. *For an edge-transitive graph Σ on m vertices, a complete graph K_n with $n \geq m$ has an edge-symmetrical cover with parts isomorphic to Σ .*

Lemma 3.3 and Corollary 3.6 lead to the following illustrative examples.

Example 3.7. Let $\Gamma = K_n$, a complete graph with n vertices.

- (i) Let $G = \text{Aut}\Gamma = S_n$, acting arc-transitively on Γ . Let P be a complete subgraph of Γ with m vertices, where $m < n$. Then $G_P = S_m \times S_{n-m}$, and acts arc-transitively on P . Let \mathcal{P} be the set of all complete subgraphs with m vertices. Since G is m -transitive on $V\Gamma$, G is transitive on \mathcal{P} . Thus (Γ, \mathcal{P}) is a G -arc-symmetrical cover of K_n . Further, \mathcal{P} is an $\binom{n-2}{m-2}$ -uniform cover.
- (ii) Let $n = q + 1 = p^d + 1$ with p prime, and let $G = \text{PGL}(2, q)$. Let P be a 3-cycle of Γ . Then $G_P = S_3$. Let \mathcal{P} be the set of all 3-cycles of Γ . Since G is 3-transitive on $V\Gamma$, the pair (Γ, \mathcal{P}) is a G -arc-symmetrical cover. It is an $(n - 2)$ -uniform cover.
- (iii) Let $n \geq 10$ and $G = \text{Aut}\Gamma = S_n$. Let $H \cong S_5$ be a subgroup of G acting transitively on a subset $\Delta \subset \Omega$ of size 10. Then there exist two vertices $v, w \in \Delta$ such that the induced subgraph $\Sigma := [\{v, w\}^H]$ is a Petersen graph. Let $\mathcal{P} = \Sigma^G$. Then (Γ, \mathcal{P}) is a G -arc-symmetrical λ -uniform cover with $\lambda = (n - 2)!/4(n - 10)!$ and parts the Petersen graph.
- (iv) Let $n = q + 1$ with $q = 3^f$ with f even. Let $G = \text{PSL}(2, q)$ and $H \leq G$ such that $H \cong A_5$. Then G is arc-transitive on Γ and by [7, Lemma 11], H has an orbit Δ on vertices of size 10. There exist two vertices $v, w \in \Delta$ such that the induced subgraph $\Sigma := [\{v, w\}^H]$ is a Petersen graph. Let $\mathcal{P} = \Sigma^G$. Then (Γ, \mathcal{P}) is a G -arc-symmetrical λ -uniform cover with $\lambda = \frac{q-1}{4}$. Note in particular, that when $q = 9$ then $\lambda = 2$.

3.2 Covers for complete multipartite graphs

For integers $m, n \geq 2$, a complete m -partite graph with part size n is denoted by $K_{m[n]} = K_{n,n,\dots,n}$. The following corollary to Lemma 3.3 for complete multipartite graphs is analogous to Corollary 3.6 for complete graphs.

Corollary 3.8. *Let Σ be an H -edge-transitive graph such that $V\Sigma$ has an H -invariant partition \mathcal{B} with block size b and $|\mathcal{B}| = l$. Then for each $m \geq l$ and $n \geq b$, $K_{m[n]}$ has an edge-symmetrical cover with parts isomorphic to Σ .*

Here are some examples.

Example 3.9. Let $\Gamma = K_{m[n]}$, and let $G = \text{Aut}\Gamma = S_n \text{ wr } S_m$.

- (i) For $m = 2$, let \mathcal{P} be the set of all induced subgraphs of Γ which are isomorphic to $K_{i,i}$ for $i < n$. For $P \in \mathcal{P}$, $H := (S_i \times S_{n-i}) \text{ wr } S_2$, acts arc-transitively on P . Further, G is transitive on \mathcal{P} , and (Γ, \mathcal{P}) is a G -arc symmetrical λ -uniform cover, where $\lambda = \binom{n-1}{i-1}^2$.
- (ii) For $m \geq 3$, let \mathcal{P} be the set of all induced subgraphs of Γ that are isomorphic to K_m . Let $P \in \mathcal{P}$. Then $G_P = S_{n-1} \text{ wr } S_m$ acts arc-transitively on P , and (Γ, \mathcal{P}) is a G -transitive n^{m-2} -uniform cover. Taking $m = 3$ and $G = S_n \text{ wr } S_3$, shows that the complete tri-partite graph $K_{n,n,n}$ is G -arc transitive and has a G -arc-symmetrical n -uniform 3-cycle cover.

3.3 Covers involving cliques

For $n > k$, the Johnson graph $J(n, k)$ is the graph with V the set of k -element subsets of an n -set with two subsets adjacent if they have $k - 1$ points in common. The valency of $J(n, k)$ is $k(n - k)$ and the group $G = S_n$ acts arc-transitively on $J(n, k)$. For an edge $\{v, w\}$, we have $G_v = S_k \times S_{n-k}$, and $G_{vw} = S_{k-1} \times S_{n-k-1}$.

Example 3.10. Let $\Gamma = J(n, k)$ and $G = S_n$. Let ℓ satisfy $1 \leq \ell < k$ and let I be the set of ℓ -element subsets of the n -set. For each $A \in I$, let $\Gamma_A = (V_A, E_A)$ where V_A consists of all the k -element subsets containing A , and E_A is the subset of E joining elements of V_A . Then $\Gamma_A \cong J(n - \ell, k - \ell)$, and each edge $\{B, C\}$ of Γ is an edge of each of the $\binom{k-1}{\ell}$ graphs Γ_A such that $A \subseteq B \cap C$. The stabiliser G_A of Γ_A induces $S_{n-\ell}$ on Γ_A . Thus $\mathcal{G} = \{\Gamma_A \mid |A| = \ell\}$ is an edge-symmetrical uniform cover with $\lambda = \binom{k-1}{\ell}$ and hence is a factorisation if $\ell = k - 1$. In this latter case the factors $J(n - k + 1, 1) \cong K_{n-k+1}$ are maximal cliques of Γ .

The previous example was pointed out to us by Michael Orrison who uses the case $\ell = k - 1$ in [25] for the analysis of unranked data. He also noticed that it is a special case of clique covers of graphs, that is, covers in which the parts are cliques (complete subgraphs). These arise naturally for edge-transitive graphs as follows. Let Γ be a G -edge transitive graph and let A be a maximal clique. Let $\mathcal{G} = A^G = \{A^g \mid g \in G\}$. Then (Γ, \mathcal{G}) is a uniform cover which is G -transitive. There are some graphs for which each edge lies in exactly one clique in the G -class of cliques \mathcal{G} . For these graphs \mathcal{G} is a G -edge-symmetrical decomposition (see Lemma 4.4).

3.4 Cycle covers, near polygonal graphs and rotary maps

Each finite arc-transitive graph of valency at least three contains cycles. The next example shows that such graphs have edge-symmetrical cycle covers. The method presented here has been used in [23] for constructing polygonal graphs and we discuss this below.

Construction 3.11. Let Γ be a regular graph of valency at least 3, and $G \leq \text{Aut}\Gamma$ be such that Γ is G -arc transitive. Then there exists a set \mathcal{P} of cycles such that (Γ, \mathcal{P}) is a G -edge-symmetrical cycle cover. The set \mathcal{P} is constructed as follows: For a pair of adjacent vertices v and w , let $g \in G \setminus G_v$ such that $v^g = w$ and $w^g \neq v$. Then the set of images of (v, w) under $\langle g \rangle$ forms a cycle C say. Let $\mathcal{P} = C^G$.

The fact that the partition \mathcal{P} produced in Construction 3.11 is a G -edge-symmetrical cover of Γ follows from [23, Lemmas 1.1 and 2.2], and further, if G_C^C is dihedral, then \mathcal{P} is a G -arc-symmetrical cover.

Example 3.12. Let $\Gamma = K_4$, the complete graph on 4 vertices, and let $G = \text{Aut}\Gamma = S_4$. Let (v, w) be an arc. Let \mathcal{P} be a cover of Γ produced by Construction 3.11. If $g \in G$ is of order 3, then \mathcal{P} contains 4 triangles while if $g \in G$ is of order 4, then \mathcal{P} contains 3 cycles of length 4. In both cases (Γ, \mathcal{P}) is a G -arc-symmetrical 2-uniform cycle cover.

A *2-arc* in a graph Γ is a triple (u, v, w) such that $u \neq w$ and both (u, v) and (v, w) are arcs. Following [27], a graph Γ is called a *near-polygonal graph* if there is a collection \mathcal{C} of m -cycles in Γ such that each 2-arc of Γ is contained in exactly one cycle in \mathcal{C} . Suppose that Γ is such a graph of valency k and that $G \leq \text{Aut}(\Gamma)$ preserves \mathcal{C} and is transitive on the set of 2-arcs of Γ . Then G_{vw} is transitive on $\Gamma(v) \setminus \{w\}$ and for each of the $k - 1$ vertices $u \in \Gamma(v) \setminus \{w\}$, the 2-arc (u, v, w) lies in a unique cycle of \mathcal{C} . Moreover, these cycles are pairwise distinct, by definition of \mathcal{C} , and they are the only cycles of \mathcal{C} containing (w, v) since each such cycle must contain (u, v, w) for some $u \in \Gamma(v) \setminus \{w\}$. Thus the edge $\{v, w\}$ lies in exactly $k - 1$ cycles in \mathcal{C} , that is, \mathcal{C} is a $(k - 1)$ -uniform cycle cover and is G -arc-symmetrical. Examples of infinite families of near-polygonal graphs can be found in [23, 27, 28]. It is shown in [23] that each 2-arc-regular graph (that is, $\text{Aut}(\Gamma)$ is regular on the 2-arcs of Γ) is a near-polygonal graph, so each 2-arc-regular graph of valency k has an arc-symmetrical $(k - 1)$ -uniform cycle cover. In particular, 2-arc-regular cubic graphs have a 2-uniform cycle cover.

A *map* on a surface (2-manifold) is a 2-complex of the surface. The 0-cells, 1-cells and 2-cells of the 2-complex are called vertices, edges and faces of the map, respectively. Incidence between these objects is defined by inclusion. A map \mathcal{M} may be viewed as a 2-cell embedding of the underlying graph Γ into the supporting surface. A vertex-edge incident pair is called a *dart*, and a pairwise incident vertex-edge-face triple is called a *flag*. A permutation of flags of a map \mathcal{M} preserving the incidence relation is an *automorphism* of \mathcal{M} , and the set of all automorphisms of \mathcal{M} forms the map automorphism group $\text{Aut}\mathcal{M}$. A map \mathcal{M} is said to be *rotary* or *regular* if $\text{Aut}\mathcal{M}$ acts transitively on the darts or on the flags of \mathcal{M} , respectively. Further, a rotary map is called *chiral* if it is not regular.

Example 3.13. Let \mathcal{M} be a map with underlying graph Γ , and let $G = \text{Aut}\mathcal{M}$. Let \mathcal{P} be the set of cycles which are boundaries of faces of \mathcal{M} . Then \mathcal{P} is a 2-uniform cycle

cover of the underlying graph Γ . If \mathcal{M} is regular, then \mathcal{P} is a G -arc-symmetrical cycle cover; if \mathcal{M} is chiral, then \mathcal{P} is a G -edge-symmetrical but not G -arc-symmetrical cover.

3.5 Vertex-symmetrical covers

First we note that not every vertex-symmetrical cover is a uniform cover, as seen in the following example.

Example 3.14. Let $\Gamma \cong K_2 \square K_4$ be the graph with vertex set such that $\{1, 2\} \times \{1, 2, 3, 4\}$ and (u_1, v_1) is adjacent to (u_2, v_2) if and only if $u_1 = u_2$ or $v_1 = v_2$. We saw in Example 3.12, that K_4 has an S_4 -arc-symmetrical 2-uniform cover $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ consisting of four 3-cycles. For each $P_i \in \mathcal{P}$, let Q_i be the set of edges $\{(u_1, v_1), (u_2, v_2)\}$ such that $v_1 = v_2$, or $u_1 = u_2$ and $\{v_1, v_2\} \in P_i$. Then $\mathcal{Q} = \{Q_1, Q_2, Q_3, Q_4\}$ is a cover of Γ with edges of the form $\{(u_1, v), (u_2, v)\}$ lying in all four parts while edges of the form $\{(u, v_1), (u, v_2)\}$ lie in precisely two parts. The group $G = S_2 \times S_4$ is vertex-transitive on Γ , preserves \mathcal{Q} and $G^{\mathcal{Q}}$ is transitive. Moreover, for $Q \in \mathcal{Q}$, $G_Q = S_2 \times S_3$ is transitive on the vertices in Q and so \mathcal{Q} is a G -vertex-symmetrical cover.

We have the following general construction of vertex-symmetrical covers.

Construction 3.15. Let $\Gamma = (V, E)$ be a G -vertex-transitive graph. Let $H < G$ and let V_0 be an orbit of H on vertices. Suppose that there exists an H -invariant nonempty subset $P \neq E$ of the edge set of the induced subgraph $[V_0]$ such that P contains an edge from each G -orbit on E , and let $\mathcal{P} = P^G$. Then each edge of Γ lies in some P^g . Also, as V_0 is an H -orbit and P is H -invariant, each vertex of V_0 lies in some edge of P . Thus $V_P = V_0$ and so $H \leq G_P$ is transitive on VP . Hence (Γ, \mathcal{P}) is a G -vertex-symmetrical cover.

Every G -arc-symmetrical cover is G -vertex-symmetrical and Example 3.14 shows that the converse is not true in general. Moreover, the following example shows that even if Γ is G -arc-transitive, a G -vertex-symmetrical cover is not necessarily G -edge-symmetrical.

Example 3.16. Let $\Gamma = K_{12}$ and $G = \text{PSL}(2, 11)$. Then Γ is G -arc-transitive. Moreover, there exists a set V_0 of 5 vertices such that $G_{V_0} \cong C_5$. Let P be the complete graph on V_0 and $\mathcal{P} = P^G$. Then as seen in Construction 3.15, (Γ, \mathcal{P}) is a G -vertex-symmetrical cover. Since G_{V_0} is not edge-transitive on P , (Γ, \mathcal{P}) is not G -edge-symmetrical.

We have already seen in Section 1 that uniform covers correspond to 2-designs. This leads to the following lemma.

Lemma 3.17. *A uniform cover (Γ, \mathcal{P}) of a complete graph $\Gamma = (V, E)$ with complete subgraph parts is G -vertex-symmetrical if and only if (V, \mathcal{P}) is a G -flag-transitive 2-design.*

4 Transitive decompositions

By definition, an xxx-symmetrical decomposition is an xxx-symmetrical 1-uniform cover for each $\text{xxx} \in \{\text{vertex, edge, arc}\}$. Any G -arc-transitive graph has a G -arc-symmetrical

decomposition with each divisor consisting of a single edge. Such a decomposition is called *trivial*. We have the following existence criterion for nontrivial transitive decompositions of edge-transitive graphs.

Lemma 4.1. *Let Γ be a G -edge-transitive graph. Then Γ has a non-trivial G -transitive decomposition if and only if G acts on $E\Gamma$ imprimitively. More precisely, a subgraph Σ of Γ is a divisor of a G -transitive decomposition if and only if $E\Sigma$ is a block of imprimitivity for G acting on $E\Gamma$.*

Proof. By definition a partition \mathcal{P} of $E\Gamma$ forms a G -transitive decomposition of Γ precisely if \mathcal{P} is G -invariant, that is to say, \mathcal{P} is a system of imprimitivity for G on $E\Gamma$. \square

This leads to the following general construction.

Construction 4.2. Let Γ be a G -edge-transitive graph. Suppose that $\{v, w\}$ is an edge of Γ and $G_{\{v, w\}} < H < G$. Let $P = \{v, w\}^H$. Then $\mathcal{P} = P^G$ is a G -transitive decomposition of Γ .

In fact, every transitive decomposition of an edge-transitive graph arises in this way.

Lemma 4.3. *Let (Γ, \mathcal{P}) be a G -transitive decomposition with G acting edge-transitively on Γ . Then (Γ, \mathcal{P}) arises from Construction 4.2 using $H = G_P$, where P is the divisor of \mathcal{P} containing $\{v, w\}$.*

Proof. By Lemma 4.1, P is a block of imprimitivity for G on $E\Gamma$. Thus $G_{\{v, w\}} < G_P$ and $P = \{v, w\}^{G_P}$. \square

We further note that when studying transitive decompositions, if G acts imprimitively on \mathcal{P} then there is a partition \mathcal{Q} of $E\Gamma$ refined by \mathcal{P} such that G acts primitively on \mathcal{Q} . Moreover, (Γ, \mathcal{Q}) is also a G -transitive decomposition. For some families of graphs the most reasonable approach is to study G -transitive decompositions (Γ, \mathcal{Q}) such that $G^{\mathcal{Q}}$ is primitive. For example, this was done for the Johnson graphs in [8] giving a classification of such decompositions.

4.1 Edge-symmetrical and arc-symmetrical decompositions

First we observe that in the edge-transitive and arc-transitive cases, transitive decompositions are symmetrical decompositions.

Lemma 4.4. *Let (Γ, \mathcal{P}) be a G -transitive decomposition. If G is edge-transitive on Γ then (Γ, \mathcal{P}) is G -edge-symmetrical; if G is arc-transitive on Γ then (Γ, \mathcal{P}) is G -arc-symmetrical.*

Proof. If G is edge-transitive on Γ , Lemma 4.1 implies that \mathcal{P} is a system of imprimitivity for G on $E\Gamma$. Thus for $P \in \mathcal{P}$, G_P is transitive on P and so (Γ, \mathcal{P}) is a G -edge-symmetrical decomposition. Moreover, if G is also arc-transitive then \mathcal{P} is a system of imprimitivity for G on $A\Gamma$ and it follows that (Γ, \mathcal{P}) is a G -arc-symmetrical decomposition. \square

Remark 4.5. Since G -vertex-symmetrical decompositions are G -transitive decompositions it follows that G -vertex-symmetrical decompositions of G -edge-transitive graphs (respectively G -arc-transitive graphs) are also G -edge-symmetrical (respectively G -arc-symmetrical). Similarly, G -edge-symmetrical decompositions of G -arc-transitive graphs are G -arc-symmetrical.

By Lemmas 4.4 and 4.3, all edge-symmetrical decompositions and arc-symmetrical decomposition arise from Construction 4.2.

We give two examples of arc-symmetrical decompositions to illustrate how Construction 4.2 may be applied to two important families of graphs.

Example 4.6. Let Γ be the Petersen graph, and let $G = A_5$. Let $\{u, v\}$ be an edge of Γ . Then $G_{\{u,v\}} = C_2^2 < A_4 < G$. Hence letting $H = A_4$ we obtain a G -arc-symmetrical decomposition of Γ . Each part consists of three disjoint edges.

This example generalises as follows. Let $\Gamma = O_k$, an odd graph of degree k , that is the graph with vertex set the set of all k -subsets of a $(2k + 1)$ -set such that two k -sets are adjacent if and only if they are disjoint. Then $G = S_{2k+1} \leq \text{Aut}\Gamma$, and acts transitively on the set of arcs of Γ . The graph Γ has $\binom{2k+1}{k}$ vertices, and is of valency $k + 1$, so Γ has $\binom{2k+1}{k}(k + 1)/2$ edges. The group $G = S_{2k+1}$ is imprimitive on $E\Gamma$. For two adjacent vertices v, w , the vertex stabiliser $G_v = S_{k+1} \times S_k$, and the edge stabiliser satisfies $G_{\{v,w\}} = (S_k \times S_k).2 < H := S_{2k}$. Let $P = \{(v, w)^g \mid g \in H\}$, and let $\mathcal{P} = P^G$. Then (Γ, \mathcal{P}) is a G -transitive decomposition. The graph $[P]$ induced by P has vertices all k -sets not containing i where i is the unique point not in $v \cup w$ and two vertices are adjacent if and only if they are disjoint. Hence $[P]$ consists of $\binom{2k}{k}/2$ disjoint edges.

Example 4.7. Let $\Gamma = H(d, n) = K_n \square K_n \square \dots \square K_n = K_n^{\square d}$ and $G = S_n \text{ wr } S_d$. Then Γ can be decomposed into edge disjoint maximal cliques K_n , giving a G -arc-symmetrical decomposition as follows: vertices $v = (1, \dots, 1)$ and $w = (2, 1, \dots, 1)$ are adjacent and $G_{\{v,w\}} = (S_{n-2}.2) \times (S_{n-1} \text{ wr } S_{d-1}) < H := S_n \times (S_{n-1} \text{ wr } S_{d-1})$. Let $P = \{v, w\}^H$. Then $[P] \cong K_n$ and $\mathcal{P} = P^G$ is a G -arc-symmetrical decomposition.

4.2 Link with linear spaces

We now consider decomposing complete graphs into complete subgraphs.

A *linear space* (Ω, \mathcal{L}) is an incidence geometry with point set Ω and line set \mathcal{L} where each line is a subset of Ω , $|\mathcal{L}| \geq 2$, and each pair of points lies on exactly one line. For a linear space (Ω, \mathcal{L}) with $n = |\Omega|$, let $\Gamma \cong K_n$ be its point graph, that is, the complete graph with vertex set Ω , and let \mathcal{P} be the set of subgraphs of Γ such that $P \in \mathcal{P}$ if and only if P is the complete graph whose vertex set consists of all points on some line. Then (Γ, \mathcal{P}) is a decomposition of Γ . Moreover,

- (i) (Ω, \mathcal{L}) is G -line transitive if and only if (Γ, \mathcal{P}) is G -transitive. See [26].
- (ii) (Ω, \mathcal{L}) is G -flag-transitive if and only if (Γ, \mathcal{P}) is G -vertex-symmetrical;
- (iii) G acts 2-transitively on the points of (Ω, \mathcal{L}) if and only if (Γ, \mathcal{P}) is G -arc-symmetrical.

The linear spaces in (iii), with a group of automorphisms acting 2-transitively on points, were determined by Kantor [19] while all flag-transitive linear spaces for which G is not a 1-dimensional affine group were classified in [5] and subsequent papers. Thus vertex-symmetrical decompositions of complete graphs with complete divisors are essentially known. Moreover, the arc-symmetrical decompositions of complete graphs with arbitrary divisors were characterised in [31], extending the classification in [6] for the case where the divisors are 1-factors. Sibley's characterisation has been made more explicit both in [20] for homogeneous factorisations of K_n , and in [1] to provide input decompositions for a series of general decomposition constructions for products and cartesian products of complete graphs.

4.3 Vertex-symmetrical decompositions

Now we consider vertex-symmetrical decompositions of vertex-transitive graphs. Let Γ be a G -vertex-transitive graph. If Γ is disconnected, then the set of connected components forms a G -vertex-symmetrical decomposition of Γ . Moreover, since the connected components are isomorphic (because Γ is G -vertex-transitive), each G -vertex-symmetrical decomposition $(\Gamma_0, \mathcal{P}_0)$ of a connected component of Γ , where $G_0 = G_{\Gamma_0}$, leads to the G -vertex-symmetrical decomposition $(\Gamma, \mathcal{P}_0^G)$ of Γ . The next example illustrates that not all vertex-symmetrical decompositions of disconnected graphs arise in this way. Nevertheless we will confine our further discussion to the case where Γ is connected.

Example 4.8. Let Γ be the vertex disjoint union of the two 3-cycles $\{1, 2\}, \{2, 3\}, \{3, 1\}$ and $\{4, 5\}, \{5, 6\}, \{6, 4\}$, and let $G = S_3 \times S_2 \leq \text{Aut}(\Gamma)$ acting transitively on $V\Gamma$. Let $P = \{\{1, 2\}, \{4, 5\}\}$ and $\mathcal{P} = P^G$. Then $G_P = \langle (1, 2)(4, 5) \rangle \times S_2$ and so (Γ, \mathcal{P}) is a G -vertex-symmetrical decomposition.

If Γ is G -edge-transitive, then a G -vertex-symmetrical decomposition of Γ is also a G -edge-symmetrical decomposition and hence arises from Construction 4.2. We give below a general construction for G -vertex-symmetrical decompositions of connected graphs when G is not edge-transitive.

Construction 4.9. Let Γ be a connected G -vertex-transitive graph with G intransitive on edges, and let E_1, E_2, \dots, E_r be the orbits of G acting on the edge set $E\Gamma$. Then each induced subgraph $[E_i]$ is a G -edge-transitive spanning subgraph of Γ . Assume that each $[E_i]$ has a G -vertex-symmetrical decomposition $\mathcal{P}_i = \{P_{i1}, P_{i2}, \dots, P_{ik}\}$ such that for each $i, j \in \{1, \dots, r\}$ and $s \in \{1, \dots, k\}$ we have $VP_{is} = VP_{js}$. Let $P_j = P_{1j} \cup P_{2j} \cup \dots \cup P_{rj}$, and $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$. Note that for each j , $VP_j = VP_{1j}$ and so $G_{P_j} = G_{P_{1j}}$ is transitive on VP_j . Hence (Γ, \mathcal{P}) is a G -vertex-symmetrical decomposition.

Lemma 4.10. *Let (Γ, \mathcal{P}) be a G -vertex-symmetrical decomposition of a connected graph Γ with G intransitive on edges. Then (Γ, \mathcal{P}) can be obtained from Construction 4.9.*

Proof. Let E_1, \dots, E_r be the orbits of G on $E\Gamma$. Since G is vertex-transitive, each $[E_i]$ is a spanning subgraph of Γ . Let $\mathcal{P} = \{P_1, \dots, P_k\}$ and for each $i \in \{1, \dots, r\}$ and $s \in \{1, \dots, k\}$ let $Q_{is} = E_i \cap P_s$. Then for $i \in \{1, \dots, r\}$, $\mathcal{Q}_i = \{Q_{is} \mid s \in \{1, \dots, k\}\}$ is a G -transitive decomposition of $[E_i]$. Moreover, for each $s \in \{1, \dots, k\}$, $G_{Q_{is}} = G_{P_s}$.

Since (Γ, \mathcal{P}) is G -vertex-symmetrical, for each $s \in \{1, \dots, k\}$, G_{P_s} is transitive on VP_s . It follows, that for each $i \in \{1, \dots, r\}$, $VQ_{is} = VP_s$ and $G_{Q_{is}}$ is transitive on VQ_{is} . Thus $([E_i], \mathcal{Q}_i)$ is a G -vertex-symmetrical decomposition. Hence \mathcal{P} may be obtained from Construction 4.9. \square

A natural problem in this area is the following.

Problem 4.11. Characterise the vertex-transitive graphs which arise as vertex-symmetrical divisors of a complete graph.

5 Transitive factorisations

Let (Γ, \mathcal{P}) be a factorisation with $\mathcal{P} = \{P_1, \dots, P_k\}$. For $v \in V\Gamma$ and each $i \in \{1, \dots, k\}$ we can define $P_i(v) = \{w \in \Gamma(v) \mid \{v, w\} \in P_i\}$ and $\mathcal{P}(v) = \{P_1(v), \dots, P_k(v)\}$. Since \mathcal{P} is a partition of $E\Gamma$, it follows that $\mathcal{P}(v)$ is a partition of $\Gamma(v)$ and as each P_i is a spanning subgraph of Γ , each $P_i(v)$ is nonempty. If $G \leq \text{Aut}(\Gamma)$ preserves \mathcal{P} then G_v preserves $\mathcal{P}(v)$.

This local correspondence allows us to see that transitive factorisations of graphs are naturally connected to group factorisations. This fact can be used very effectively to study transitive factorisations for various classes of graphs or classes of groups.

Lemma 5.1. *Let Γ be a G -arc-transitive graph, and let (Γ, \mathcal{P}) be a G -transitive factorisation of Γ . Then for $P \in \mathcal{P}$ and $v \in V\Gamma$, $G = G_v G_P$, G_P is vertex-transitive on Γ and G_v is transitive on \mathcal{P} .*

Proof. Since G_v acts transitively on $\Gamma(v)$, it follows that G_v is transitive on $\mathcal{P}(v)$ and hence also on \mathcal{P} . Thus by Lemma 2.1, $G = G_v G_P$ and so again by Lemma 2.1, G_P acts transitively on $V\Gamma$. \square

The following lemma follows immediately from Lemma 5.1 and implies that G_P has index at most a subdegree of G .

Lemma 5.2. *If (Γ, \mathcal{P}) is a G -arc-symmetrical factorisation and $H = G_P$ for some $P \in \mathcal{P}$, then $|G : H| = |G_v : H_v|$ divides the valency of Γ .*

We now give two general constructions and show that all symmetrical factorisations arise from them.

Construction 5.3. (Edge-symmetrical and arc-symmetrical factorisations) Let $\Gamma = (V, E)$ be a G -edge-transitive graph. Assume that there is a subgroup H containing $G_{\{v, w\}}$ for some edge $\{v, w\}$, such that either $G = HG_v = HG_w$, or $HG_v = HG_w$ is an index two subgroup of G . Let $P = \{v, w\}^H$, and let $\mathcal{P} = P^G$. If $G = HG_v$ and Γ is G -vertex-transitive, then by Lemma 2.1, Γ is H -vertex-transitive and so $[P]$ is a spanning subgraph containing the edge $\{v, w\}$. On the other hand, if $G = HG_v = HG_w$ and Γ is not G -vertex-transitive, or if $HG_v = HG_w$ is an index two subgroup of G , then Γ is bipartite and H is transitive on each bipartite half. Again $[P]$ is a spanning subgraph. In all these cases, since $G_{\{v, w\}} < H < G$, \mathcal{P} is a system of imprimitivity for G on E and so (Γ, \mathcal{P}) is a G -edge-symmetrical factorisation. Moreover, if Γ is G -arc-transitive, then $G_{\{v, w\}}$ contains an element interchanging v and w , and hence so does H . Thus H is arc-transitive on $[P]$ and so (Γ, \mathcal{P}) is a G -arc-symmetrical factorisation.

The following example shows that edge-symmetrical factorisations exist with $G = HG_v$ and G either vertex-transitive or vertex-intransitive, and with $HG_v = HG_w$ an index two subgroup of G . Note that for arc-symmetrical factorisations only the case $G = HG_v$ and G -vertex-transitive occurs as Γ is both G - and H -vertex-transitive in this case.

Example 5.4. Let $\Gamma = C_6$ with vertices labelled by the elements of \mathbb{Z}_6 and x adjacent to $x \pm 1 \pmod{6}$, and let $h = (0, 1, 2, 3, 4, 5)$, $g = (1, 5)(2, 4) \in \text{Aut}(\Gamma)$. Let $v = 0$ and $w = 1$ so that $\{v, w\} \in E\Gamma$.

- (1). Let $G = \langle g, h \rangle = \text{Aut}(\Gamma) \cong D_{12}$. Then $G_v = \langle g \rangle$ and $G_{\{v, w\}} = \langle (0, 1)(2, 5)(3, 4) \rangle \cong C_2$. Let $H = \langle G_{\{v, w\}}, h^2 \rangle \cong D_6$. Then $G = G_v H$ and so we can use Construction 5.3 to obtain a G -edge-symmetrical factorisation. In particular, $P = \{v, w\}^H = \{\{0, 1\}, \{2, 3\}, \{4, 5\}\}$ and $\mathcal{P} = P^G$. Moreover, (Γ, \mathcal{P}) is G -arc-symmetrical.
- (2). Let $G = \langle h \rangle \cong C_6$. Then $G_v = G_w = 1 = G_{\{v, w\}}$. Let $H = \langle h^2 \rangle \cong C_3$. Then $HG_v = HG_w$ has index two in G and so we can use Construction 5.3 to obtain a G -edge-symmetrical factorisation. We again have $P = \{v, w\}^H = \{\{0, 1\}, \{2, 3\}, \{4, 5\}\}$.
- (3). Let $G = \langle h^2, g \rangle \cong D_6$ which is vertex-intransitive. Then $G_v = \langle g \rangle$ and $G_{\{v, w\}} = 1$. Let $H = \langle h^2 \rangle$. Then $G = G_v H$ and so we can again use Construction 5.3 to obtain a G -edge-symmetrical factorisation. Once again $P = \{v, w\}^H = \{\{0, 1\}, \{2, 3\}, \{4, 5\}\}$.

Lemma 5.5. *Let (Γ, \mathcal{P}) be a G -edge-symmetrical factorisation. Then (Γ, \mathcal{P}) arises from Construction 5.3 using $H = G_P$ for $P \in \mathcal{P}$.*

Proof. By Lemma 4.1, \mathcal{P} is a block system of the G -action on E . Thus $H = G_P$ contains the edge stabiliser $G_{\{v, w\}}$ and $P = \{v, w\}^H$. Since H is transitive on the edges of the factor P , either Γ is H -vertex-transitive, or Γ is bipartite and H has two orbits on $V\Gamma$, these being the two bipartite halves. It follows from Lemma 2.1 that in the first case $G = HG_v$. In the second case, the stabiliser G^+ in G of each bipartite half has index at most two in G and Lemma 2.1 implies that $G^+ = HG_v = HG_w$. Thus (Γ, \mathcal{P}) arises from Construction 5.3 \square

Note that if (Γ, \mathcal{P}) is a G -arc-symmetrical factorisation then it is also G -edge-symmetrical and hence by Lemma 5.5 arises from Construction 5.3.

If (Γ, \mathcal{P}) is a G -vertex-symmetrical factorisation with G transitive on $E\Gamma$, then (Γ, \mathcal{P}) is an edge-symmetrical factorisation. We have the following general construction in the edge-intransitive case.

Construction 5.6. (Vertex-symmetrical factorisations) Let $\Gamma = (V, E)$ be a G -vertex-transitive graph and let E_1, \dots, E_r be the G -orbits on E . Suppose there is a subgroup H such that $G = HG_v$ for some vertex v and for each orbit E_i of G on E there exists $\{v_i, w_i\} \in E_i$ such that $G_{\{v_i, w_i\}} \leq H$. Let $P = \{\{v_1, w_1\}, \dots, \{v_r, w_r\}\}^H$ and $\mathcal{P} = P^G$. Since $G = HG_v$, Γ is H -vertex-transitive and so $[P]$ is a spanning subgraph containing each edge $\{v_i, w_i\}$. Also, since $G_{\{v_i, w_i\}} < H < G$ for each i , the partition $\mathcal{P}_i = \{P_j \cap E_i \mid P_j \in \mathcal{P}\}$ is a system of imprimitivity for G on E_i . Moreover, the action of G on \mathcal{P}_i is

equivalent to the action of G on the set of right cosets of H and hence $G^{\mathcal{P}_i} \cong G^{\mathcal{P}_j}$ for all $i \neq j$. Thus \mathcal{P} is indeed a factorisation of Γ and so (Γ, \mathcal{P}) is a G -vertex-symmetrical factorisation.

Example 5.7. Let Γ be the graph with vertices labelled by the elements of \mathbb{Z}_8 and x adjacent to $x \pm 1, x \pm 3 \pmod{8}$. Let $G = D_{16} \leq \text{Aut}(\Gamma)$. Then G has two orbits E_1, E_2 on the set of edges of Γ , with E_1 being the 8-cycle with adjacency $x \sim x \pm 1 \pmod{8}$ and E_2 the 8-cycle with adjacency $x \sim x \pm 3 \pmod{8}$. Now $\{0, 1\} \in E_1$ and $\{3, 6\} \in E_2$. Moreover, $G_{\{0,1\}} = G_{\{3,6\}} = \langle (0, 1)(2, 7)(3, 6)(4, 5) \rangle$. Now $G = G_0H$ where $H = \langle (0, 1)(2, 7)(3, 6)(4, 5), (0, 2, 4, 6)(1, 3, 5, 7) \rangle$ and H contains $G_{\{0,1\}} = G_{\{3,6\}}$. Thus we can use Construction 5.6 to find a G -vertex-symmetrical factorisation. The part $\mathcal{P} = \{\{0, 1\}, \{3, 6\}\}^H$ gives $[P] = 2C_4$ with components $(0, 1, 4, 5)$ and $(2, 3, 6, 7)$.

Lemma 5.8. *Let (Γ, \mathcal{P}) be a G -vertex-symmetrical factorisation. Then (Γ, \mathcal{P}) arises from Construction 5.6 using $H = G_P$ for $P \in \mathcal{P}$.*

Proof. Suppose that (Γ, \mathcal{P}) is a G -vertex-symmetrical factorisation and let $P \in \mathcal{P}$ contain the edge $\{v, w\}$. Then the subgraph P is spanning, and $H = G_P$ is transitive on V . Hence $G = HG_v$. Let E_1, \dots, E_r be the G -orbits on E . For $i \in \{1, \dots, r\}$, we have $\mathcal{P} \cap E_i := \{P_j \cap E_i \mid P_j \in \mathcal{P}\}$ is a G -edge-symmetrical factorisation of the induced subgraph $[E_i]$. By Lemma 4.1, $P \cap E_i$ is a block of imprimitivity for G acting on E_i , and so the block stabiliser $G_{P \cap E_i} = H$ properly contains $G_{\{v_i, w_i\}}$ for some edge $\{v_i, w_i\} \in E_i$. Moreover, $P \cap E_i = \{v_i, w_i\}^H$ and $P = \{\{v_1, w_1\}, \dots, \{v_r, w_r\}\}$. Hence (Γ, \mathcal{P}) is as obtained by Construction 5.6. \square

5.1 A link between transitive covers and homogeneous factorisations

There is an interesting situation that arises for G -transitive covers (Γ, \mathcal{P}) for vertex-quasiprimitive groups G . These are permutation groups G for which all nontrivial normal subgroups are vertex-transitive. We propose the general study of G -transitive uniform covers (Γ, \mathcal{P}) where G is arc-transitive and vertex-quasiprimitive.

Construction 5.9. For a cover (Γ, \mathcal{P}) define the following family $\mathcal{Q}(P)$ of sets as follows: For each $e \in E\Gamma$, let $\mathcal{P}_e = \{P \in \mathcal{P} \mid e \in P\}$ and let $Q_e = \bigcap_{P \in \mathcal{P}_e} P$. Then define $\mathcal{Q}(P) = \{Q_e \mid e \in E\Gamma\}$.

Lemma 5.10. *If (Γ, \mathcal{P}) is a G -transitive λ -uniform cover such that the kernel $N = G_{(\mathcal{P})}$ is vertex-transitive and Γ is G -edge-transitive, then $(\Gamma, \mathcal{Q}(P))$ is a (G, N) -homogeneous factorisation.*

(Homogeneous factorisations were defined at the end of Section 1.)

Proof. Let $e \in E\Gamma$. Since $e \in Q_e$ and N fixes Q_e setwise, it follows that Q_e is a spanning subgraph. Now G preserves \mathcal{Q} and since G is edge-transitive, it follows that G acts transitively on \mathcal{Q} . Moreover, for each $e \in E\Gamma$, there is a unique part of \mathcal{Q} , namely Q_e , which contains e . Hence (Γ, \mathcal{Q}) is a (G, N) -homogeneous factorisation. \square

This link can sometimes occur rather naturally and we demonstrate this phenomenon in the next lemma.

Lemma 5.11. *Let (Γ, \mathcal{P}) be a (G, M) -homogeneous factorisation such that G is 2-transitive on \mathcal{P} . Let $\mathcal{R} = \{P_i \cup P_j \mid i \neq j, P_i, P_j \in \mathcal{P}\}$. Then (Γ, \mathcal{R}) is a G -transitive $(|\mathcal{P}| - 1)$ -uniform cover. Moreover, the homogeneous factorisation obtained from (Γ, \mathcal{R}) using Construction 5.9 is (Γ, \mathcal{P}) .*

Proof. Since G is 2-transitive on \mathcal{P} , it acts transitively on \mathcal{R} . Moreover, as each edge lies in a unique element of \mathcal{P} , it lies in precisely $|\mathcal{P}| - 1$ elements of \mathcal{R} . Thus (Γ, \mathcal{R}) is a G -transitive $(|\mathcal{P}| - 1)$ -uniform cover. Given an edge e of Γ , if P is the unique part of \mathcal{P} containing e , then P is the intersection of all the parts of \mathcal{R} containing e . Hence (Γ, \mathcal{P}) is the homogeneous factorisation obtained from (Γ, \mathcal{R}) using Construction 5.9. \square

An explicit example of a homogeneous factorisation satisfying the conditions of Lemma 5.11 is $G = \text{AGL}(d, q)$, $\Gamma = K_{q^d}$, with \mathcal{P} the partition of edges into parallel classes. Application of Construction 5.9 arises most naturally when the group G involved is quasiprimitive on vertices.

Lemma 5.12. *Let (Γ, \mathcal{P}) be a G -transitive uniform cover of a G -edge-transitive, G -vertex-quasiprimitive graph Γ . Then either*

- (1). G acts faithfully on \mathcal{P} , or
- (2). Construction 5.9 yields a (G, N) -homogeneous factorisation (Γ, \mathcal{Q}) with $N = G_{(\mathcal{P})}$.

Proof. Let N be the kernel of the action of G on \mathcal{P} . If $N = 1$ then G acts faithfully on \mathcal{P} and we have case (1). On the other hand, if $N \neq 1$, since Γ is G -vertex-quasiprimitive, N is transitive on $V\Gamma$. Hence Construction 5.9 yields a (G, N) -homogeneous factorisation (Γ, \mathcal{Q}) . \square

6 Quotients

In this final section we discuss the behaviour of covers and decompositions when we pass to a quotient graph. Let Γ be a G -arc-transitive connected graph and \mathcal{B} a G -invariant partition of $V\Gamma$. The *quotient graph* $\Gamma_{\mathcal{B}}$ is the graph with vertex set \mathcal{B} such that two blocks B_1, B_2 are adjacent if and only if there exist $v \in B_1$ and $w \in B_2$ such that v and w are adjacent in Γ . The quotient $\Gamma_{\mathcal{B}}$ has no loops and is connected, and G acts arc-transitively (see [29]). If the G -invariant partition \mathcal{B} is the set of orbits of a normal subgroup N of G then we denote $\Gamma_{\mathcal{B}}$ by Γ_N and $\mathcal{P}_{\mathcal{B}}$ by \mathcal{P}_N .

We say that Γ *covers* the quotient graph $\Gamma_{\mathcal{B}}$ if the subgraph of Γ induced between two adjacent blocks is a perfect matching, that is, given two adjacent blocks B_1, B_2 , for all $v \in B_1$, we have $|\Gamma(v) \cap B_2| = 1$. This is an unfortunate re-use of the term ‘cover’. However, both uses of this word are standard in the graph theory literature. The context should make it clear which one is intended.

Given a cover \mathcal{P} of Γ (as in Section 1), for each $P \in \mathcal{P}$ let $P_{\mathcal{B}}$ be the set of all arcs (B, C) of $\Gamma_{\mathcal{B}}$ such that there exists $u \in B$ and $v \in C$ with $(u, v) \in P$. This allows us to

define $\mathcal{P}_{\mathcal{B}} = \{P_{\mathcal{B}} \mid P \in \mathcal{P}\}$. If the G -invariant partition \mathcal{B} is the set of orbits of a normal subgroup N of G then we denote $\Gamma_{\mathcal{B}}$ by Γ_N and $\mathcal{P}_{\mathcal{B}}$ by \mathcal{P}_N .

The following lemma records properties of (Γ, \mathcal{P}) that are inherited by $(\Gamma_{\mathcal{B}}, \mathcal{P}_{\mathcal{B}})$. Case (c) involves the condition that Γ covers its quotient graph Γ_N , a condition that always holds if Γ is G -locally primitive, see our comments after Theorem 6.2 below.

Lemma 6.1. *Let Γ be a G -arc-transitive connected graph and let \mathcal{B} be a G -invariant partition of $V\Gamma$.*

- (a) *If \mathcal{P} is a cover of Γ then $\mathcal{P}_{\mathcal{B}}$ is a cover of $\Gamma_{\mathcal{B}}$.*
- (b) *If (Γ, \mathcal{P}) is a G -transitive λ -uniform cover, then $(\Gamma_{\mathcal{B}}, \mathcal{P}_{\mathcal{B}})$ is a G -transitive μ -uniform cover for some $\mu \geq \lambda$.*
- (c) *Let N be a normal subgroup of G which acts trivially on \mathcal{P} and has at least three orbits on vertices, and suppose that Γ covers Γ_N . Then $(\Gamma_N, \mathcal{P}_N)$ is a (G/N) -transitive λ -uniform cover, and for each $P \in \mathcal{P}$, P covers P_N .*

Proof. (a) Let (B, C) be an arc of $\Gamma_{\mathcal{B}}$. As noted above there are no loops in $\Gamma_{\mathcal{B}}$, and so there exists $(u, v) \in A\Gamma$ such that $u \in B$ and $v \in C$. Since \mathcal{P} is a cover of Γ , there exists $P \in \mathcal{P}$ such that $(u, v) \in P$. Hence $(B, C) \in P_{\mathcal{B}}$ and so $\mathcal{P}_{\mathcal{B}}$ is a cover of $\Gamma_{\mathcal{B}}$.

(b) As noted above G acts arc-transitively on $\Gamma_{\mathcal{B}}$. Since \mathcal{P} is G -invariant it follows from the definition of $\mathcal{P}_{\mathcal{B}}$ that $\mathcal{P}_{\mathcal{B}}$ is also G -invariant, and since G is transitive on \mathcal{P} it is also transitive on $\mathcal{P}_{\mathcal{B}}$. Thus by part (a), $(\Gamma_{\mathcal{B}}, \mathcal{P}_{\mathcal{B}})$ is a G -transitive μ -uniform cover for some μ . Since an arc (u, v) with $u \in B$ and $v \in C$ is contained in λ parts of \mathcal{P} , it follows that $\mu \geq \lambda$.

(c) Let (B, C) be an arc of Γ_N . Since Γ is a cover of Γ_N , the subgraph induced between B and C is a complete matching and N acts transitively on the set of arcs from B to C . Thus if P_1, \dots, P_{λ} are the λ parts of \mathcal{P} containing the arc (u, v) with $u \in B$ and $v \in C$, then all arcs from B to C are contained in each P_1, \dots, P_{λ} . Thus (B, C) is contained in precisely λ parts of \mathcal{P}_N and so $(\Gamma_N, \mathcal{P}_N)$ is a (G/N) -transitive λ -uniform cover. Moreover, if $P \in \mathcal{P}$ contains an arc joining some (B, C) then since N acts transitively on B and fixes P , for each $b \in B$, there exists $c \in C$ such that $(b, c) \in P$. Since Γ covers Γ_N , c is unique and hence P covers P_N . □

Let (Γ, \mathcal{P}) be a G -transitive factorisation and let M be the kernel of the action of G on \mathcal{P} . Recall that if M is vertex-transitive, then (Γ, \mathcal{P}) is a (G, M) -homogeneous factorisation. If Γ is bipartite and both G and M fix the two parts of the bipartition and act transitively on both, then (Γ, \mathcal{P}) is called a (G, M) -bihomogeneous factorisation. In either of these cases if we replace \mathcal{P} by a G -invariant partition of $E\Gamma$ refined by \mathcal{P} the kernel will also be transitive on $V\Gamma$, or in the second case will have at most two vertex-orbits. We have a useful result about quotients for G -transitive decompositions (Γ, \mathcal{P}) in the case where G is primitive on \mathcal{P} , a property that may be obtained by replacing \mathcal{P} with a maximal invariant partition refined by \mathcal{P} . For a bipartite graph Γ which is

G -vertex-transitive, we denote by G^+ the index two subgroup of G which fixes setwise each of the two bipartite halves.

We have the following theorem.

Theorem 6.2. *Let (Γ, \mathcal{P}) be a G -transitive decomposition of the G -arc-transitive connected graph Γ . Suppose that G acts primitively on \mathcal{P} and let N be the kernel of the action of G on \mathcal{P} . Then one of the following holds.*

- (1). (Γ, \mathcal{P}) is a (G, N) -homogeneous factorisation.
- (2). (Γ, \mathcal{P}) is a (G^+, N) -bihomogeneous factorisation.
- (3). $(\Gamma_N, \mathcal{P}_N)$ is a (G/N) -transitive decomposition with G/N faithful on \mathcal{P}_N .
- (4). N has at least three vertex orbits and Γ does not cover Γ_N .

Proof. If N is vertex-transitive then we are clearly in the first case. If N has two orbits on $V\Gamma$, then Γ is bipartite with the two N -orbits being the two parts of the bipartition. Since $G^{\mathcal{P}}$ is primitive, it follows that G^+ acts transitively on \mathcal{P} and so (Γ, \mathcal{P}) is a (G^+, N) -bihomogeneous factorisation.

Suppose now that N has at least three orbits on vertices. If Γ is a cover of Γ_N , then Lemma 6.1(c) implies that $(\Gamma_N, \mathcal{P}_N)$ is a (G/N) -transitive decomposition. Since N is the kernel of the action of G on \mathcal{P} , G/N is faithful on \mathcal{P}_N . \square

When Γ is G -locally primitive, [29] implies that Γ is a cover of Γ_N , so case (4) of Theorem 6.2 does not arise in this case. Thus Theorem 6.2 suggests that for G -locally primitive graphs important G -transitive decompositions to study are those for which G acts faithfully on the decomposition.

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