

On finite edge-primitive and edge-quasiprimitive graphs*

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Abstract

Many famous graphs are edge-primitive, for example, the Heawood graph, the Tutte–Coxeter graph and the Higman–Sims graph. In this paper we systematically analyse edge-primitive and edge-quasiprimitive graphs via the O’Nan–Scott Theorem to determine the possible edge and vertex actions of such graphs. Many interesting examples are given and we also determine all G -edge-primitive graphs for G an almost simple group with socle $\text{PSL}(2, q)$.

1 Introduction

Let Γ be a finite connected graph and $G \leq \text{Aut}(\Gamma)$. We say that Γ is G -*edge-primitive* if G acts primitively on the set of edges of Γ , that is, if G preserves no nontrivial partition of the edge set. If Γ is $\text{Aut}(\Gamma)$ -edge-primitive we call Γ *edge-primitive*. The aim of this paper is to initiate a systematic study of edge-primitive graphs and the wider class of *edge-quasiprimitive* graphs, that is graphs with a group of automorphisms which acts quasiprimitively on edges. (A transitive permutation group is said to be *quasiprimitive* if every nontrivial normal subgroup is transitive).

The Atlas [3] notes many edge-primitive graphs with a sporadic simple group as a group of automorphisms. These include the Hoffman–Singleton and Higman–Sims graphs, and the rank three graphs of the sporadic simple groups J_2 , McL , Ru , Suz and Fi_{23} . Weiss [18] has determined all edge-primitive graphs of valency three. These are the complete bipartite graph $K_{3,3}$, the Heawood graph, the Biggs–Smith cubic distance-transitive graph on 102 vertices and the Tutte–Coxeter graph (also known as Tutte’s 8-cage or the Levi graph). All but the Biggs–Smith graph are bipartite. We say that Γ is s -*arc-transitive* if the automorphism group acts transitively on the set of s -*arcs* of

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Γ , that is, on the set of $(s + 1)$ -tuples (v_0, v_1, \dots, v_s) where v_i is adjacent to v_{i+1} and $v_i \neq v_{i+2}$. Of the four edge-primitive cubic graphs, $K_{3,3}$ is 3-arc-transitive, the next two are 4-arc-transitive while the Tutte–Coxeter graph is 5-arc-transitive.

Whereas any primitive permutation group with a nontrivial self-paired orbital gives rise to a vertex-primitive graph, the existence of edge-primitive graphs is far more restrictive. Given a group G there is a G -edge-primitive graph if and only if there exists a maximal subgroup E of G with an index two subgroup properly contained in some corefree subgroup H of G with $H \neq E$ (see Proposition 2.5 and Lemma 3.4).

One of the main motivations for our study of edge-primitive and edge-quasiprimitive graphs is the study of graph decompositions [9]. Given a graph Γ and a group of automorphisms G , we say that a partition \mathcal{P} of the edge set is a G -transitive decomposition if \mathcal{P} is G -invariant and G acts transitively on \mathcal{P} . A G -transitive decomposition \mathcal{P} of a graph Γ is called a *homogeneous factorisation* if the kernel of the action of G on \mathcal{P} is vertex-transitive. Homogeneous factorisations have been studied in [7, 8, 13]. Let Γ be a G -edge-transitive graph. Then Γ is G -edge-primitive if and only if Γ has no G -transitive decompositions. If G is edge-quasiprimitive then the G -transitive decompositions of Γ are not homogeneous factorisations. Conversely, if none of the G -transitive decompositions of Γ are homogeneous factorisations then the kernel of each G -transitive decomposition is vertex-intransitive.

If Γ is a bipartite graph with a vertex-transitive group of automorphisms G , then G has a normal subgroup G^+ of index two which fixes each of the bipartite halves setwise. We say that a transitive group G is *biprimitive* if it is imprimitive and all nontrivial systems of imprimitivity have precisely two parts, while we say that G is *biquasiprimitive* if G is not quasiprimitive and every normal subgroup has at most two orbits. We note here that some authors' definition of biprimitive as a transitive permutation group G with index two subgroup G^+ acting primitively on both of its orbits is not equivalent to ours. All our biprimitive groups are biprimitive in this sense but not all biprimitive groups in this alternative sense are biquasiprimitive. For example $S_n \times S_2$ acting imprimitively on $2n$ points for $n \geq 3$ has a system of imprimitivity with n parts of size 2 while the index two subgroup S_n acts primitively on each of its orbits. Given property P , we say that a graph Γ with a group of automorphisms G is G -locally P if for each vertex v , the vertex stabiliser G_v has property P on the set $\Gamma(v)$ of all vertices adjacent to v . In particular, Γ is called G -locally primitive if G_v acts primitively on $\Gamma(v)$ for all vertices v .

For any positive integer n and prime p , the star $K_{1,n}$ and the cycle C_p are both edge-primitive. We call these two examples *trivial*. Disconnected edge-primitive graphs are easily reduced to connected ones (see Lemma 3.1). We see in Lemma 3.4 that except for the trivial examples, edge-primitivity implies arc-transitivity.

Let Γ be a connected G -arc-transitive graph and let \mathcal{B} be a G -invariant partition of $V\Gamma$. We define the *quotient graph* $\Gamma_{\mathcal{B}}$ to be the graph with vertex set \mathcal{B} such that $B, C \in \mathcal{B}$ are adjacent if and only if Γ has an edge $\{v, w\}$ with $v \in B$ and $w \in C$. It easily follows that $\Gamma_{\mathcal{B}}$ is arc-transitive. We are interested in the special case where for an arc (B, C) of $\Gamma_{\mathcal{B}}$, there is only one arc (v, w) of Γ with $v \in B$ and $w \in C$. In this case we call Γ a *spread* of $\Gamma_{\mathcal{B}}$.

We will see in Lemma 3.5 that if G is edge-primitive and vertex-transitive then it is either vertex-quasiprimitive or vertex-biquasiprimitive on vertices. In fact we can reduce to the vertex-primitive or vertex-biprimitive cases.

Theorem 1.1. *Let Γ be a connected nontrivial G -edge-primitive graph. Then Γ is G -arc-transitive, and one of the following holds.*

1. Γ is G -vertex-primitive.
2. Γ is G -vertex-biprimitive.
3. Γ is a spread of a G -edge-primitive graph which is G -locally imprimitive.

Conversely, a G -edge-primitive, G -locally imprimitive graph Σ is a quotient graph of a larger G -edge-primitive graph Γ with $G^{E\Sigma} \cong G^{E\Gamma}$.

This reduces the study of edge-primitive graphs to those which are also vertex-primitive or vertex-biprimitive.

The actions of primitive permutation groups are described by the O’Nan–Scott Theorem. We follow the subdivision in [15] of primitive groups into 8 types and these are described in Section 4. By playing the edge-primitive action of G against the vertex-primitive action of G or G^+ we see that the possible actions for edge-primitive graphs are quite restrictive.

Theorem 1.2. *Let Γ be a connected nontrivial G -edge-primitive graph with $G^{E\Gamma}$ primitive of type X such that $G^{V\Gamma}$ is either primitive or biprimitive. Then one of the following holds.*

1. $\Gamma = K_{n,n}$.
2. $G^{V\Gamma}$ is primitive of type X and $X \in \{\text{AS}, \text{PA}\}$.
3. $G^{V\Gamma}$ is biprimitive and G^+ is primitive of type X on each orbit with $X \in \{\text{AS}, \text{PA}\}$.
4. $G^{E\Gamma}$ is of type SD or CD, Γ is bipartite and arises from Construction 5.6, and G^+ is primitive of type CD on each orbit.

We see in Sections 2 and 5 that examples exist in all cases. Moreover, we can find G -locally imprimitive examples in each case. A characterisation of all groups which act edge-primitively on $K_{n,n}$ is given in Theorem 3.7. We also see in Proposition 6.15 that the existence of G -edge-primitive graphs with G of type PA relies on the existence of edge-primitive graphs where the action on edges is of type AS.

We undertake much of our analysis in the context of vertex-quasiprimitive graphs and only specialise to the edge-primitive case when we are able to obtain stronger conclusions. There are however, a couple of notable differences between the two classes. There are many G -edge-quasiprimitive graphs with G not vertex-transitive, for example any bipartite graph with an edge-transitive simple group G of automorphisms is G -edge-quasiprimitive while G has two orbits on vertices. Vertex-transitive, edge-quasiprimitive graphs are still either vertex-quasiprimitive or vertex-biquasiprimitive but we are no longer able to reduce to the vertex-primitive or vertex-biprimitive cases. Theorem 6.12 is an analogue of Theorem 1.2 in the G -vertex-transitive, G -edge-quasiprimitive case.

It appears feasible to determine all edge-primitive graphs for certain families of almost simple groups, for example, for low rank groups of Lie type. We begin this process in Section 8 by determining all G -edge-primitive graphs where $\text{soc}(G) = \text{PSL}(2, q)$. The *socle* (denoted $\text{soc}(G)$) of a group G is the product of all of its minimal normal subgroups.

Table 1: G -edge primitive with $\text{soc}(G) = \text{PSL}(2, q)$

G	Γ
$\text{PGL}(2, 7)$	Heawood graph (Example 2.2)
$\text{PGL}(2, 7)$	co-Heawood graph (Example 2.2)
$\text{PGL}(2, 9), M_{10}$ or $\text{P}\Gamma\text{L}(2, 9)$	$K_{6,6}$
$\text{PGL}(2, 9), M_{10}$ or $\text{P}\Gamma\text{L}(2, 9)$	Tutte–Coxeter graph (Example 2.3)
$\text{PGL}(2, 11)$	$(H, E, E \cap H) = (A_5, D_{20}, D_{10})$
$\text{PSL}(2, 17)$	Biggs–Smith graph
	$(H, E, E \cap H) = (S_4, D_{16}, D_8)$
$\text{PSL}(2, 19)$	$(H, E, E \cap H) = (A_5, D_{20}, D_{10})$
$\text{PSL}(2, 25)$ or $\text{P}\Sigma\text{L}(2, 25)$	Example 2.4
$\text{PSL}(2, p), p \equiv \pm 1, \pm 9 \pmod{40}$	$(H, E, E \cap H) = (A_5, S_4, A_4)$
$\text{PGL}(2, p), p \equiv \pm 11, \pm 19 \pmod{40}$	$(H, E, E \cap H) = (A_5, S_4, A_4)$

Theorem 1.3. *Let Γ be a G -edge-primitive graph with $\text{soc}(G) = \text{PSL}(2, q)$, such that $q = p^f$ for some prime p and $q \neq 2, 3$. Then either Γ is complete and G is listed in Table 2, or Γ and G are given in Table 1.*

In some rows of Table 1 we just state the edge stabiliser E and vertex stabiliser H along with $H \cap E$ as by Proposition 2.5, a G -edge-transitive graph is uniquely determined by the vertex stabiliser and edge stabiliser. Note for the first two examples $\text{PGL}(2, 7) \cong \text{Aut}(\text{PSL}(3, 2))$, for the fourth example note $\text{P}\Gamma\text{L}(2, 9) \cong \text{Aut}(\text{PSp}(4, 2))$, while for the eighth example $\text{PSL}(2, 25) \cong \text{P}\Omega^-(4, 5)$. Apart from complete graphs and $K_{6,6}$, we get two infinite families and seven sporadic examples. All of the graphs listed in Table 1 are 2-arc-transitive except for the eighth one.

2 Some examples

If $G \leq S_n$ acts arc-transitively on K_n then G is 2-transitive on vertices. Moreover, G is edge-primitive if and only if G acts primitively on 2-subsets. The following theorem, which is essentially [17, Theorem 6], classifies all such G .

Theorem 2.1. *Let G be a 2-transitive subgroup of S_n such that G is primitive on 2-subsets. Then G and n are as in Table 2.*

Proof. By Burnside’s Theorem (see for example [5, Theorem 4.1B]), G is either almost simple or a subgroup of $\text{AGL}(d, p)$ with $n = p^d$ for some prime p . Sibley [17] classified all G -transitive decompositions of K_n for G a 2-transitive simple group and so this yields a classification of almost simple groups acting edge-primitively on K_n . Suppose now that $G \leq \text{AGL}(d, p)$ and let u, v be a pair of points of $\text{AG}(d, p)$. Then $\{u, v\}$ lies on a unique line l and so $G_{\{u,v\}} \leq G_l < G_B \leq G$, where B is the parallel class containing l . Thus for $d \geq 2$, G is not primitive on 2-subsets. Note that this includes A_4 and S_4 . When $d = 1$, there is a unique parallel class and $G_{\{u,v\}} \cong C_2$. In this case, G is primitive on 2-subsets if and only if $p = 2$ or 3 . Here $G \cong S_2, S_3$ respectively. \square

Table 2: 2-transitive groups which are primitive on 2-subsets

n	G	Conditions
n	S_n	$n \neq 4$
n	A_n	$n \geq 5$
$q + 1$	$\text{soc}(G) = \text{PSL}(2, q)$	$q \geq 7$ $G \neq \text{PSL}(2, 7), \text{PSL}(2, 9),$ $\text{P}\Omega\text{L}(2, 9) \text{ or } \text{PSL}(2, 11).$
$q^2 + 1$	$\text{soc}(G) = \text{Sz}(q)$	$q = 2^{2d+1}$
11	$\text{PSL}(2, 11)$	
11	M_{11}	
12	M_{11}	
12	M_{12}	
22	$M_{22}, \text{Aut}(M_{22})$	
23	M_{23}	
24	M_{24}	
176	HS	
276	Co_3	

There are many geometrical constructions of edge-primitive graphs with the following being just a couple.

Example 2.2. Let $T = \text{PSL}(d, q)$ for $d \geq 3$ and $G = \text{Aut}(T)$. Let Δ_1 be the set of r -dimensional subspaces of a d -dimensional vector space over $\text{GF}(q)$ with $1 \leq r < d/2$ and let Δ_2 be the set of $(d - r)$ -dimensional subspaces. We define Γ to be the bipartite graph with vertex set $\Delta_1 \cup \Delta_2$ with adjacency given by inclusion. Then $G \leq \text{Aut}(\Gamma)$ and acts biprimitively on vertices such that the stabiliser G^+ of the bipartition is equal to $\text{P}\Gamma\text{L}(d, q)$. Moreover, the stabiliser E of an edge is a maximal subgroup of G and so Γ is G -edge-primitive. When $(d, r) = (3, 1)$, the graph obtained is 4-arc transitive and when $(d, r, q) = (3, 1, 2)$, the graph obtained is the Heawood graph.

Alternatively, we can define an r -space to be adjacent to a $(d - r)$ -space if they are complementary. This also gives us a G -edge-primitive graph with G acting biprimitively on vertices and when $(d, r, q) = (3, 1, 2)$ we get the co-Heawood graph.

Example 2.3. Let V be a 4-dimensional vector space over $\text{GF}(q)$ with q even and let B be a nondegenerate alternating form. Let Δ_1 be the set of totally isotropic 1-spaces and Δ_2 be the set of totally isotropic 2-spaces. Define Γ to be the graph with vertex set $\Delta_1 \cup \Delta_2$ and adjacency defined by inclusion. Then $\text{PTSp}(4, q)$ is an edge-transitive group of automorphisms of Γ but has two orbits on vertices. Let τ be a duality of the polar space interchanging Δ_1 and Δ_2 . Then $G = \langle \text{PTSp}(4, q), \tau \rangle$ is an arc-transitive group of automorphisms of Γ which is vertex-biprimitive. Moreover, an edge stabiliser G_e is a maximal subgroup of G and so Γ is G -edge-primitive. When $q = 2$, Γ is the Tutte–Coxeter graph.

There are also many other geometrical constructions of infinite families of edge-primitive graphs involving sesquilinear or quadratic forms. We give one such example here.

Example 2.4. Let V be a vector space of dimension d over the field $\text{GF}(q)$, with $q = 3$ or 5 , and let Q be a nondegenerate quadratic form on V with associated bilinear form B . Let Γ be the graph whose vertex set is the set of all nonsingular 1-spaces upon which the quadratic form is a square with adjacency given by orthogonality with respect to B . By Witt's Lemma, the group $G = \text{PO}(d, q)$ of all isometries of Q is an arc-transitive automorphism group of Γ .

Let $e = \{\langle v \rangle, \langle w \rangle\}$ be an edge of Γ . If $q = 5$ then $\langle v, w \rangle$ is a hyperbolic line while if $q = 3$ then $\langle v, w \rangle$ is anisotropic. Moreover, in both cases $\langle v \rangle, \langle w \rangle$ are the only 1-spaces of $\langle v, w \rangle$ upon which Q is a square. Thus $G_e = G_{\langle v, w \rangle}$. By [10], it follows that if $q = 5$ then G_e is maximal in G except when $d = 4$ and Q is hyperbolic. Also, if $q = 3$ then G_e is maximal in G except when $d = 4$ or 5 .

Edge-primitive graphs can be defined via group theoretic means using the coset graph construction. Let G be a group with a core-free subgroup H . Let $g \in G$ such that g does not normalise H and $g^2 \in H$. We define the coset graph $\Gamma = \text{Cos}(G, H, HgH)$ to have vertex set, the set $[G : H]$ of right cosets of H in G with two vertices Hx, Hy being adjacent if and only if $xy^{-1} \in HgH$. The graph Γ is connected if and only if $\langle H, g \rangle = G$. Moreover, G acts as an arc-transitive group of automorphisms of Γ via right multiplication. The valency of Γ is $|H : H \cap H^g|$ while the stabiliser of the edge $\{H, Hg\}$ is $\langle H \cap H^g, g \rangle$. Conversely, suppose that Γ is a graph with adjacent vertices v and w . Let $G \leq \text{Aut}(\Gamma)$ be arc-transitive and let $g \in G$ interchange v and w . Then $\Gamma \cong \text{Cos}(G, G_v, G_v g G_v)$. We have the following characterisation of arc-transitive edge-primitive graphs.

Proposition 2.5. *Let G be a group with a maximal subgroup E . Then there exists a G -edge-primitive, arc-transitive graph Γ with edge stabiliser E if and only if E has a subgroup A of index two, and G has a corefree subgroup H such that $A < H \neq E$; in this case $\Gamma = \text{Cos}(G, H, HgH)$ for some $g \in E \setminus A$.*

Proof. Suppose first that G, E, A, H and g are as in the statement. Since E is maximal in G and H is not contained in E it follows that $E < \langle H, g \rangle = G$. As H is corefree in G we have that g does not normalise H . Let $\Gamma = \text{Cos}(G, H, HgH)$, let $v = H$, $w = Hg$ and $e = \{v, w\}$. Then Γ is connected, $G_v = H$, $G_w = H^g$, $G_{vw} = H \cap H^g$ and $G_e = \langle H \cap H^g, g \rangle \neq G$. Since g does not normalise H , but does normalise A we have $A \leq H \cap H^g < H$ and so $E \leq \langle H \cap H^g, g \rangle = G_e$. The maximality of E implies that $G_e = E$ and Γ is edge-primitive.

Conversely, suppose that Γ is a G -arc-transitive, G -edge-primitive graph. Let $e = \{v, w\}$ be an edge of Γ . Then $H = G_v$ is corefree in G . Since G is arc-transitive, there exists $g \in G$ such that $v^g = w$ and $w^g = v$. Moreover, $\Gamma \cong \text{Cos}(G, H, HgH)$. Now $G_{vw} = H \cap H^g$ which is an index two subgroup of $G_e = \langle H \cap H^g, g \rangle$. Since G is edge-primitive, $E = G_e$ is maximal in G and $A = H \cap E = G_{vw}$ has index 2 in E . \square

We also have the following lemma.

Lemma 2.6. *Let $\Gamma = \text{Cos}(G, H, HgH)$. Then for any subgroup $L < G$ such that $H \cap H^g < L < H$, the graph $\text{Cos}(G, L, LgL)$ is a spread of Γ .*

Proof. Let v be the vertex of $\Sigma = \text{Cos}(G, L, LgL)$ corresponding to L and w the vertex adjacent to v corresponding to Lg . Then $B = v^H$ is a block of imprimitivity for G on $V\Sigma$ containing v and the corresponding block containing w is B^g . Let $\mathcal{B} = \{B^k \mid k \in G\}$. Since Γ is G -arc-transitive, so is $\Sigma_{\mathcal{B}}$ and H is the stabiliser of the vertex of $\Sigma_{\mathcal{B}}$ given by the block B . Hence $\Sigma_{\mathcal{B}} = \text{Cos}(G, H, HgH) = \Gamma$. Now the stabiliser of the block B^g is H^g and $H \cap H^g < L$. Let (x, y) be an arc of Σ with $x \in B$ and $y \in B^g$. Then there exists $h \in G$ mapping v to x and w to y . Since B is a block of imprimitivity, $h \in H \cap H^g < L$ and so $h \in L \cap L^g$. Thus h fixes v and w and so $\{v, w\}$ is the only edge between the blocks B and B^g . Hence Σ is a spread of Γ . \square

One easy way of constructing edge-primitive graphs is to look for novelty maximal subgroups. Given a group G with a normal subgroup N , we say that a maximal subgroup E of G not containing N is a *novelty* if $E \cap N$ is not maximal in N . Thus if N is an index two subgroup of G , every novelty maximal subgroup E of G gives rise to a G -edge-primitive graph with edge stabiliser E , arc stabiliser $A = E \cap N$ and vertex stabiliser H , where H is a proper subgroup of N properly containing A . This phenomenon lies behind Examples 2.2 and 2.3. We also have the following example.

Example 2.7. Let T be the Mathieu group M_{12} and $G = \text{Aut}(T)$. From the Atlas [3, p 33], G has maximal subgroups $E \cong S_5$ and $H \cong \text{PGL}(2, 11)$ such that $A = E \cap H \cong A_5$ and $H \cap T = \text{PSL}(2, 11)$ is a maximal subgroup of T . The subgroup E is a novelty maximal. Let $g \in E \setminus A$. Then by Proposition 2.5, $\Gamma = \text{Cos}(G, H, HgH)$ is G -edge-primitive. As H is maximal in G it follows that G acts primitively on $V\Gamma$. Note that $A \leq T$ and so $TA \neq G$. Hence T acts transitively on vertices and edges but not on arcs. Moreover, as A is selfnormalising in T , we have $A < H \cap T < T$ and A is the stabiliser in T of an edge. Thus Γ is T -edge-quasiprimitive, but not T -edge-primitive. Moreover, Γ is G -locally imprimitive and letting $B = H \cap T$, we see that Γ is the quotient graph of the bipartite graph $\Sigma = \text{Cos}(G, B, BgB)$. The graph Σ is G -edge-primitive and $(G, 2)$ -arc-transitive such that $G^{E\Sigma} = G^{E\Gamma}$ and is G -vertex-biquasiprimitive, but not G -vertex-biprimitive. There is a partition \mathcal{P} of $V\Sigma$ into blocks of size two such that $\Sigma_{\mathcal{P}} = \Gamma$. Each block of \mathcal{P} has one vertex in each bipartite half of Σ , and there is at most one edge between any two blocks.

We have the following general construction of locally imprimitive, edge-primitive graphs.

Construction 2.8. Let E be an almost simple primitive permutation group of degree n such that E has an index 2 subgroup A which preserves a nontrivial partition of the n points into l parts of size k . Let $H = S_k \text{ wr } S_l$ and $G = S_n$. Suppose that E is a maximal subgroup of G and let $g \in E \setminus A$. Then by Proposition 2.5, the graph $\Gamma = \text{Cos}(G, H, HgH)$ is G -edge-primitive. If A is not maximal in H then Γ is G -locally imprimitive.

The requirements for A and E are often satisfied. An infinite family of examples is where $E = \text{Aut}(\text{PSL}(d, q))$ for $d \geq 3$ and $A = \text{P}\Gamma\text{L}(d, q)$. Let $n = (q^d - 1)(q^{d-1} - 1)/(q - 1)^2$, the number of point-hyperplane incident pairs. Then by [14], E is maximal in $G = S_n$. However, A is imprimitive and preserves a partition of $l = (q^d - 1)/(q - 1)$ parts of size $k = (q^{d-1} - 1)/(q - 1)$. Moreover, A is not maximal in $H = S_l \text{ wr } S_k$ since it is contained in $S_k \text{ wr } \text{P}\Gamma\text{L}(d, q)$. Thus Γ is G -locally imprimitive.

3 Initial Analysis

We begin by noting the following lemma.

Lemma 3.1. *If Γ is a disconnected G -edge-primitive graph then either Γ is a union of isolated vertices and single edges, or Γ is a union of isolated vertices and a connected G -edge-primitive graph.*

Proof. Each connected component which contains an edge forms a block of imprimitivity for G on edges. Thus either each connected component consists of zero or one edge, or there is a unique connected component with at least one edge. \square

Next we look at vertex-transitivity.

Lemma 3.2. *Let Γ be a connected G -edge-quasiprimitive graph. Then either G is vertex-transitive, or Γ is bipartite and G has two orbits on vertices. Moreover, in the latter case, either Γ is a star or G acts faithfully and quasiprimitively on each of its two orbits.*

Proof. Since G is edge-transitive, either G is vertex-transitive or Γ is bipartite and the two orbits Δ_1, Δ_2 of G on $V\Gamma$ are the two parts of the bipartition. Suppose that we are in the latter case and let N be a nontrivial normal subgroup of G . Then N acts transitively on $E\Gamma$ and so, since Γ is connected, N acts transitively on both Δ_1 and Δ_2 . Thus either $|\Delta_1| = 1$ and Γ is a star, or G acts faithfully and quasiprimitively on each of its two orbits. \square

In the edge-primitive case things are more restricted.

Lemma 3.3. *Let Γ be a connected G -edge-primitive graph. Then either Γ is a star or G is vertex-transitive.*

Proof. Suppose that G is vertex-intransitive. Then as G is edge-transitive, Γ is a bipartite graph with the orbits of G being the two bipartite halves Δ_1 and Δ_2 . Let $v \in \Delta_1$ and $B = \{\{v, w\} \mid w \in \Gamma(v)\}$. Then B forms a block of imprimitivity for G on edges. Thus either $|\Gamma(v)| = 1$ or $\Delta_1 = \{v\}$. Since Γ is connected, it follows that Γ is a star. \square

We can now show that all nontrivial edge-primitive graphs are arc-transitive.

Lemma 3.4. *Let Γ be a connected G -edge-primitive graph. Then one of the following holds:*

1. Γ is a star;
2. Γ is a cycle of prime length p , and G is a cyclic group of order p ;
3. Γ is G -arc-transitive.

Proof. By Lemma 3.3, either case (1) holds or G is vertex-transitive. Suppose now that G is vertex-transitive but not arc-transitive. Then for an edge $e = \{v, w\}$ we have $G_e = G_{vw} = G_v \cap G_w$. However, as G acts primitively on edges, G_e is a maximal subgroup of G . Thus $G_v = G_w$ for every pair of adjacent vertices. But Γ is connected, and so G_v fixes every vertex of Γ . This implies that $G_v = 1 = G_e$ and so G acts

regularly on vertices and on edges. Thus Γ has the same number of edges as vertices and so the connectivity of Γ implies that it is a cycle. Furthermore, as G is primitive on edges this cycle has a prime number of edges and hence vertices. Moreover, as G is not arc-transitive, G is cyclic. Thus either case (2) or (3) holds. \square

Lemma 3.4 does not hold for G -edge-quasiprimitive graphs. In particular, the graph Γ in Example 2.7 is T -edge-quasiprimitive, T -vertex-transitive but not T -arc-transitive.

Next we look at the action of G on vertices.

Lemma 3.5. *Let Γ be a connected G -vertex transitive, G -edge-quasiprimitive graph. Then G is either quasiprimitive or biquasiprimitive on the set of vertices of Γ .*

Proof. Let N be a nontrivial normal subgroup of G . Then N is transitive on edges and so is either transitive on vertices or Γ is bipartite and N has two orbits on the vertex set. Thus G is either quasiprimitive or biquasiprimitive on $V\Gamma$. \square

In the edge-primitive case we can actually reduce to the situation where G is either primitive or biprimitive on vertices.

Proof. (of Theorem 1.1) By Lemma 3.4 G is arc-transitive. Suppose that G is neither primitive nor biprimitive on $V\Gamma$. Then there exists a G -invariant partition \mathcal{B} of $V\Gamma$ with at least three parts. Since Γ is connected and edge-transitive, the edges of Γ occur between the parts of \mathcal{B} , that is, there are no edges within parts. Let $\Gamma_{\mathcal{B}}$ be the quotient graph of Γ with respect to the partition \mathcal{B} . Given $B_1, B_2 \in \mathcal{B}$ which are adjacent in $\Gamma_{\mathcal{B}}$, the set of edges of Γ between vertices of B_1 and vertices of B_2 forms a block of imprimitivity for G . Hence there is a unique edge in Γ between vertices of B_1 and vertices of B_2 . Thus Γ is a spread of $\Gamma_{\mathcal{B}}$ and $G^{E\Gamma} \cong G^{E\Gamma_{\mathcal{B}}}$. Moreover, if $g \in G$ fixes each part of \mathcal{B} , then g fixes each edge of Γ . Thus G acts faithfully on \mathcal{B} . Moreover, by choosing \mathcal{B} to be a maximal G -invariant partition with at least three parts, G is either primitive or biprimitive on the set of vertices of $V\Gamma_{\mathcal{B}}$. Let $v \in B_1$ and $w \in B_2$ be the unique pair of adjacent vertices in $B_1 \cup B_2$. Then $G_{B_1 B_2} = G_{vw} < G_v < G_{B_1}$, since G is arc-transitive and $|B_1| > 1$. Hence $\Gamma_{\mathcal{B}}$ is G -locally imprimitive.

Conversely, suppose that Σ is a G -edge-primitive, G -locally imprimitive graph. Let $\{\alpha, \beta\} \in E\Sigma$. Then there exists a subgroup H such that $G_{\alpha\beta} < H < G_{\alpha}$. Since G is arc-transitive, there exists $g \in G$ such that g interchanges α and β . Thus $H \cap H^g \leq G_{\alpha} \cap G_{\beta}$, but since g normalises $G_{\alpha\beta}$ we have $H \cap H^g = G_{\alpha\beta}$. Moreover, $g^2 \in G_{\alpha\beta} \leq H$. Thus we can define the graph $\Gamma = \text{Cos}(G, H, HgH)$. Let v be the vertex of Γ given by H and w the vertex given by the coset Hg . Then $e = \{v, w\}$ is an edge and $G_e = \langle H^g \cap H, g \rangle = \langle G_{\alpha\beta}, g \rangle = G_{\{\alpha, \beta\}}$. Hence $G^{E\Gamma} \cong G^{E\Sigma}$ and so Γ is G -edge-primitive. Since $H = G_v < G_{\alpha} < G$, it follows that $B_1 = v^{G_{\alpha}}$ is a block of imprimitivity for G on $V\Gamma$. Let \mathcal{B} be the corresponding system of imprimitivity. Now $v^{G_{\alpha}g} = v^{gg^{-1}G_{\alpha}g} = w^{G_{\beta}}$ and so $B_2 = w^{G_{\beta}}$ is the block of \mathcal{B} containing w . Moreover, (v, w) is the unique edge between the two blocks B_1 and B_2 . Then as $G_{\alpha} = G_{B_1}$ and g interchanges the edge $\{B_1, B_2\}$ of the quotient graph $\Gamma_{\mathcal{B}}$ we have that $\Gamma_{\mathcal{B}} \cong \text{Cos}(G, G_{\alpha}, G_{\alpha}gG_{\alpha}) \cong \Sigma$. \square

We also have the following lemma in the vertex-biquasiprimitive case.

Lemma 3.6. *Let Γ be a G -vertex-biquasiprimitive graph which is not complete bipartite. Then G^+ is faithful on each orbit.*

Proof. Let Δ_1 and Δ_2 be the two orbits of G^+ on vertices and suppose that G^+ is unfaithful on Δ_1 . Let K_1 be the kernel of the action of G^+ on Δ_1 and K_2 be the kernel of the action of G^+ on Δ_2 . Then as G is vertex-transitive, there exists $g \in G$ such that $K_1^g = K_2$. Moreover, $1 \neq K_1 \times K_2 \triangleleft G$. Since G is vertex-biquasiprimitive, it follows that K_1 is transitive on Δ_2 and K_2 is transitive on Δ_1 . Since K_1 fixes each vertex in Δ_1 , we have that each vertex of Δ_1 is adjacent to each vertex of Δ_2 . Thus Γ is complete bipartite. \square

We can determine all n and G such that $K_{n,n}$ is G -edge-primitive and G^+ acts faithfully on each bipartite half.

Theorem 3.7. *Let $\Gamma = K_{n,n}$ be a G -edge-primitive graph. Then one of the following holds:*

1. $n = 6^k$ and $\text{soc}(G^+) = A_6^k$.
2. $n = 12^k$ and $\text{soc}(G^+) = M_{12}^k$.
3. $n = (q^2(q^2 - 1)/2)^k$ and $\text{soc}(G^+) = \text{PSp}(4, q)$ with q even.
4. *There exists a primitive group H of degree n with a transitive but not regular normal subgroup K and automorphism ϕ such that $G^+ = \{(hk_1, h^\phi k_2) \mid k_1, k_2 \in K, h \in H\}$, and $(g, 1_H)(1, 2) \in G$ for some $g \in H$ interchanges the two G^+ orbits where ϕ^2 is conjugation by g .*

Proof. Let Δ_1 and Δ_2 be the two bipartite halves of Γ . Suppose that G^+ is imprimitive on Δ_1 and let \mathcal{P}_1 be a system of imprimitivity for G^+ on Δ_1 . Then there exists a system of imprimitivity \mathcal{P}_2 of G^+ on Δ_2 such that $\mathcal{P}_2 = \mathcal{P}_1^g$ for all $g \in G \setminus G^+$. Let $B_1 \in \mathcal{P}_1$ and $B_2 \in \mathcal{P}_2$. Then $C = \{(v, w) \mid v \in B_1, w \in B_2\}$ is a block of imprimitivity for G on $E\Gamma$. Hence G^+ is primitive on each bipartite half.

Let $v \in \Delta_1$ and $w \in \Delta_2$. By Lemma 3.4, G is arc-transitive. Thus G_v is transitive on Δ_2 and so $G^+ = G_v G_w$. Suppose first that G^+ is faithful on Δ_1 and Δ_2 . Since $G_w = G_v^g$ for some $g \in G$ with $g^2 \in G^+$, it follows that G^+, G_v and G_w are determined by [2, Theorem 1.1]. Either $G = \text{AGL}(3, 2) \text{ wr } K$ for some transitive subgroup K of S_k , or $\text{soc}(G^+) = T^k$ where T is one of $\text{P}\Omega^+(8, q)$, $\text{PSp}(4, q)$ $q > 2$ even, A_6 or M_{12} .

If $G = \text{AGL}(3, 2) \text{ wr } K$, then $G_{\{v,w\}} = \langle (C_7 \times C_3) \text{ wr } K, (\alpha, \dots, \alpha) \rangle$ where α is an automorphism of $\text{AGL}(3, 2)$ interchanging the two conjugacy classes of complements of C_2^3 . Hence $G_{\{v,w\}} < C_2^{3k} \rtimes G_{\{v,w\}} < G$ and so G is not edge-primitive.

If $N = \text{soc}(G^+) = \text{P}\Omega^+(8, q)^k$ then $G^+ \leq H^k$ where H is an extension of $\text{P}\Omega^+(8, q)$ by field automorphisms, $N_v = \text{P}\Omega(7, q)$, $N_{vw} = G_2(q)$ and $n = q^4(q^4 - 1)$ [2, Theorem 1.1]. Since $N_v^g = N_w$ for some $g \in G \setminus G^+$ such that $g^2 \in G^+$, it follows that g does not induce a triality automorphism of $\text{P}\Omega^+(8, q)$. Hence by [11], $G_{\{vw\}}$ is not maximal in G , and so G is not edge-primitive. Thus $\text{soc}(G^+)$ and n are as listed in the statement of the theorem.

Suppose next that G^+ is unfaithful on Δ_1 and Δ_2 . Let K_1 be the kernel of the action of G^+ on Δ_1 and K_2 be the kernel of the action of G^+ on Δ_2 . Then $K_1 \times K_2 \triangleleft G$ and so is transitive on $E\Gamma$. Hence K_1 acts faithfully and transitively on Δ_2 and K_2 acts transitively and faithfully on Δ_1 . Let $H = (G^+)^{\Delta_1}$ and $K = (K_2)^{\Delta_1}$. Then H is

a primitive permutation group with transitive normal subgroup K . Now $G \leq H \text{ wr } S_2$ and $G = \langle G^+, (g, 1_H)(1, 2) \rangle$ for some $g \in H$. Then $K_2 = \{(1_H, k) \mid k \in K\}$ and $K_1 = K_2^{(g, 1_H)(1, 2)} = \{(k, 1_H) \mid k \in K\}$. Furthermore, there exists an automorphism ϕ of H such that $G^+ = \{(hk_1, h^\phi k_2) \mid h \in H, k_1, k_2 \in K\}$. Since $(g, 1_H)(1, 2)$ normalises G^+ it follows that ϕ^2 is conjugation by g . If K is regular then $H = K \rtimes H_v$ and so $G^+ = \langle K \times K \rangle \rtimes \{(h, h^\phi) \mid h \in H_v\}$. Moreover, $G_e = \langle \{(h, h^\phi) \mid h \in H_v\}, (g, 1_H)(1, 2) \rangle < \langle \{(h, h^\phi) \mid h \in H\}, (g, 1_H)(1, 2) \rangle < G$, contradicting G_e being maximal in G . Thus K is not regular. \square

4 Primitive and quasiprimitive types

In this section we describe the subdivision of primitive and quasiprimitive groups into 8 types given in [15]. This description is in terms of the action of the minimal normal subgroups. If N is a minimal normal subgroup of a group G then $N \cong T^k$ for some finite simple group T . Moreover, if G is quasiprimitive then G has at most two minimal normal subgroups.

HA: A quasiprimitive group is of type HA if it has a unique minimal normal subgroup N and N is elementary abelian. If $|N| = p^d$ for some prime p , then G can be embedded in $\text{AGL}(d, p)$ in its usual action on a d -dimensional vector space over $\text{GF}(p)$ with N identified as the group of all translations.

HS and HC: These two classes consist of all quasiprimitive groups with two minimal normal subgroups. In both cases, the two minimal normal subgroups are regular and nonabelian. For type HS, the two minimal normal subgroups are simple, while for type HC the two minimal normal subgroups are isomorphic to T^k for some $k \geq 2$ and T nonabelian simple.

All quasiprimitive groups of type HA, HS and HC are in fact primitive. For the remaining five types the groups may or may not be primitive.

AS: This class consists of all groups G such that $T \leq G \leq \text{Aut}(T)$ for some finite nonabelian simple group, that is, G is an almost simple group. Note that any action of an almost simple group with T transitive is quasiprimitive.

TW: This type consists of all quasiprimitive groups G with a unique minimal normal subgroup $N \cong T^k$, for some finite nonabelian simple group T and positive integer $k \geq 2$, such that N is regular. Thus $G = N \rtimes G_\omega$ and can be constructed as a twisted wreath product (see [1]). If G is primitive then G_ω normalises no nontrivial proper subgroup of N . The following lemma gives us a necessary and sufficient condition for a quasiprimitive TW group to be primitive.

Lemma 4.1. [1, Lemmas 3.1 and 3.2] *Let $N \cong T^k$ for some finite nonabelian simple group T and $G = N \rtimes P$. Let Q be the normaliser in P of a simple direct factor of N and $\varphi : Q \rightarrow \text{Aut}(T)$ be the homomorphism induced by the action of Q on this factor. Then P is maximal in G if and only if $\text{Inn}(T) \leq \varphi(Q)$ and there is no subgroup H of P with a homomorphism $\hat{\varphi}$ from H to $\text{Aut}(T)$ which extends φ .*

Before describing the remaining three types of quasiprimitive groups we need some definitions. Let $N = T_1 \times \cdots \times T_k$ for some nontrivial groups T_1, \dots, T_k . For each $i = 1, \dots, k$, let $\pi_i : N \rightarrow T_i$ be the natural projection map. Given a subgroup K of N , we say that K is a *subdirect product* of N if $\pi_i(K) = T_i$ for each $i = 1, \dots, k$, while we say that K is a *diagonal subgroup* of N if K is isomorphic to each of its projections, that is, $K \cong \pi_i(K)$ for all $i = 1, \dots, k$. If $T_1 = T_2 = \cdots = T_k$ and $\pi_i(g) = \pi_j(g)$ for all $g \in K$, we call K a *straight diagonal subgroup*. A *full diagonal subgroup* of N is a subgroup which is both a subdirect product and a diagonal subgroup.

We call K a *strip* of N if there exists some subset J of $\{1, \dots, k\}$ such that $\pi_i(K) \cong K$ for all $i \in J$ while $\pi_i(K) = 1$ for all $i \notin J$. We refer to J as the *support* of K . Note that a strip is a diagonal subgroup of $\prod_{i \in J} T_i$. We call K a *full strip* if it is a full diagonal subgroup of $\prod_{i \in J} T_i$, while we say that it is *nontrivial* if $|J| > 1$. We say that two strips are *disjoint* if their supports are disjoint. Note that disjoint strips commute.

If $N = T_1 \times \cdots \times T_k$, where the T_i are pairwise isomorphic nonabelian simple groups, a well known lemma (see for example [16]) says that if K is a subdirect product of N then K is the direct product of pairwise disjoint full strips. The set of supports of these strips is a partition \mathcal{P} of $\{1, \dots, k\}$. Note that if K is normalised by a group G , then G preserves \mathcal{P} and if G acts transitively by conjugation on the set $\{T_1, \dots, T_k\}$, then G acts transitively on \mathcal{P} and so the parts of \mathcal{P} all have the same size.

SD: A quasiprimitive group G acting on a set Ω is of type SD if G has a unique minimal normal subgroup N , $N \cong T^k$ for some nonabelian simple group T , $k \geq 2$ and given $\omega \in \Omega$, the point stabiliser N_ω is a full diagonal subgroup of N . Conjugating G , if necessary, by an element of $\text{Sym}(\Omega)$ we may assume that $N_\omega = \{(t, \dots, t) \mid t \in T\}$ and $G_\omega \leq \{(t, \dots, t) \mid t \in \text{Aut}(T)\} \times S_k$. Since N is a minimal normal subgroup of G and $G = NG_\omega$, it follows that G_ω acts transitively by conjugation on the set of k simple direct factors of N . A quasiprimitive group G of type SD is primitive, if and only if G acts primitively on the set of k simple direct factors of N .

CD: A quasiprimitive group G acting on a set Ω is of type CD if G has a unique minimal normal subgroup N , $N \cong T^k$ for some nonabelian simple group T , $k \geq 2$ and given $\omega \in \Omega$, N_ω is a product of $\ell \geq 2$ full strips of N , that is, $N_\omega \cong T^\ell$ and is a subdirect product of N . Note that G acts transitively by conjugation on the set of k simple direct factors of N and preserves a partition \mathcal{P} of $\{1, \dots, k\}$ given by the set of supports of the full strips. The group G is a subgroup of $H \text{ wr } S_\ell$ acting on $\Omega = \Delta^\ell$, for some quasiprimitive group H of type SD on Δ with unique minimal normal subgroup $T^{k/\ell}$. In fact, given $P \in \mathcal{P}$, the group G_P induces H on Δ . Moreover, G is primitive if and only if H is primitive and so G is primitive if and only if for $P \in \mathcal{P}$, G_P acts primitively on \mathcal{P} .

Given two partitions $\mathcal{P}_1, \mathcal{P}_2$ of a set Ω , we say that \mathcal{P}_1 *refines* \mathcal{P}_2 if each $P \in \mathcal{P}_2$ is a union of elements of \mathcal{P}_1 . This defines a partial order on the set of all partitions of Ω and we can define $\mathcal{P}_1 \vee \mathcal{P}_2$ to be the smallest partition of Ω refined by both \mathcal{P}_1 and \mathcal{P}_2 . The following lemma will be very handy in our analysis of SD and CD groups.

Lemma 4.2. *Let $N = T_1 \times \cdots \times T_k$ for some nontrivial groups T_i and let K_1, K_2 be subgroups of N . For each $i = 1, 2$, suppose that K_i is a product of strips such that the set*

of supports of these strips is the partition \mathcal{P}_i of $\{1, \dots, k\}$. Then $K_1 \cap K_2$ is a product of strips such that the set of supports of these strips is $\mathcal{P}_1 \vee \mathcal{P}_2$.

Proof. For each $P \in \mathcal{P}_1 \vee \mathcal{P}_2$, let

$$K_P = \{g \in K_1 \cap K_2 \mid \pi_i(g) = 1 \text{ for all } i \notin P\}.$$

Then $X = \prod_{P \in \mathcal{P}_1 \vee \mathcal{P}_2} K_P$ is a subgroup of $K_1 \cap K_2$.

Let $g \in K_1 \cap K_2$ such that $g \neq 1$, and let J be the set of all $i \in \{1, \dots, k\}$ such that $\pi_i(g) \neq 1$. Since $g \in K_1$ it follows that J is a union of parts of \mathcal{P}_1 and since $g \in K_2$ it follows that J is a union of parts of \mathcal{P}_2 . Hence J is a union of ℓ parts of $\mathcal{P}_1 \vee \mathcal{P}_2$ for some $\ell \geq 1$. Thus each K_P is a strip. If $\ell = 1$ then $g \in K_P$ for some P and so $g \in X$. If $\ell > 1$, let P be one of the parts contained in J . Since $g \in K_1$ and K_1 is a product of strips, there exists $k_1 \in K_1$ such that $\pi_i(k_1) = \pi_i(g)$ for all $i \in P$ while $\pi_i(k_1) = 1$ for all $i \notin P$. Similarly, there exists $k_2 \in K_2$ such that $\pi_i(k_2) = \pi_i(g)$ for all $i \in P$ while $\pi_i(k_2) = 1$ for all $i \notin P$. Hence $k_1 = k_2 \in K_P \leq X$. Moreover, $gk_1^{-1} \in K_1 \cap K_2$ and has support $J \setminus P$, a union of $\ell - 1$ parts of $\mathcal{P}_1 \vee \mathcal{P}_2$. It follows that $g \in X$ and so $K_1 \cap K_2$ is a product of the strips K_P , whose supports are the parts of $\mathcal{P}_1 \vee \mathcal{P}_2$. \square

PA: A quasiprimitive group G acting on a set Ω is of type PA if G has a unique minimal normal subgroup N , $N \cong T^k$ for some nonabelian simple group T , $k \geq 2$ and given $\omega \in \Omega$, N_ω is a subdirect product of R^k for some $R < T$. The following two lemmas will be useful for determining primitivity. See for example, [5, Lemma 2.7A] for a proof of the first.

Lemma 4.3. *Let B be a group with subgroup $H \neq 1$. Then for each positive integer k , $H \text{ wr } S_k$ is maximal in $B \text{ wr } S_k$ if and only if H is maximal in B .*

Lemma 4.4. *Let T be a nonabelian simple group and let $T \leq A \leq B \leq \text{Aut}(T)$. Suppose that H is a maximal subgroup of B such that $B = TH$ and $T \cap H \neq 1$. Let*

$$G = \langle A^k, (b, \dots, b) \mid b \in B \rangle \rtimes S_k$$

and

$$L = \langle (A \cap H)^k, (h, \dots, h) \mid h \in H \rangle \rtimes S_k.$$

Then L is a maximal subgroup of G .

Proof. Let M be a subgroup of G containing L and let $X = M \cap B^k$. Since $S_k \leq M$ it follows that $\pi_i(X) \cong \pi_j(X)$ for all i and j . Since $L \leq M$ we have $H \leq \pi_i(X)$ and since H is maximal in B it follows that $\pi_i(X) = B$ for all i . Hence $X \cap T^k$ is a subdirect product of T^k . However, $H \cap T \neq 1$ and $(H \cap T)^k \leq X$. Thus $X \cap T^k = T^k$. Since $B = TH$ we also have $A = T(A \cap H)$. Then as $(A \cap H)^k \leq X$ it follows that $A^k \leq X$. Thus $X = G \cap B^k$ and so $M = G$, that is, L is maximal. \square

5 Constructions

All the examples in Section 2 had G an almost simple group. In this section we provide some general constructions for G -edge-quasiprimitive graphs where G is not of type AS.

Our first construction takes a B -edge-primitive graph where B is an almost simple group such that $B \neq \text{soc}(B)$, and builds a G -edge-primitive graph where G is primitive of type PA on edges and primitive of type PA on vertices.

Construction 5.1. (Primitive PA on vertices and primitive PA on edges) Let Σ be a B -edge-primitive, B -vertex-primitive graph such that B is an almost simple group with socle $T < B$. Note that Example 2.7 is such a graph. Then there exist a maximal subgroup H of B and $g \in B \setminus H$, such that $g^2 \in H$ and $\Sigma \cong \text{Cos}(B, H, HgH)$. Let

$$G = \langle T^k, (b, \dots, b) \mid b \in B \rangle \rtimes S_k$$

and

$$L = \langle (T \cap H)^k, (h, \dots, h) \mid h \in H \rangle \rtimes S_k.$$

Then letting $\sigma = (g, \dots, g)$ we define the coset graph $\Gamma = \text{Cos}(G, L, L\sigma L)$.

Lemma 5.2. *The graph $\Gamma = \text{Cos}(G, L, L\sigma L)$ given by Construction 5.1 is G -edge-primitive of type PA and G -vertex-primitive of type PA. Moreover, if Σ given in Construction 5.1 is B -locally imprimitive then Γ is G -locally imprimitive.*

Proof. Since $T < B$ and B is primitive, it follows that $H \neq 1$. Then as H is a maximal subgroup of B , Lemma 4.4 implies that the action of G on $V\Gamma = [G : L]$ is primitive of type PA. Let v be the vertex given by the coset L and w be the adjacent vertex given by $L\sigma$. Then $G_v = L^\sigma$ and

$$G_v \cap G_w = \langle (T \cap H \cap H^g)^k, (h, \dots, h) \mid h \in H \cap H^g \rangle \rtimes S_k.$$

Furthermore,

$$\begin{aligned} G_{\{v,w\}} &= \langle G_v \cap G_w, \sigma \rangle \\ &= \langle (T \cap H \cap H^g)^k, (h, \dots, h) \mid h \in \langle H \cap H^g, g \rangle \rangle \rtimes S_k \end{aligned}$$

which by Lemma 4.4, is a maximal subgroup of G since $\langle H \cap H^g, g \rangle$ is a maximal subgroup of B . Hence G acts primitively on $E\Gamma$ of type PA.

If Σ is B -locally imprimitive there exists a subgroup R such that $H \cap H^g < R < H$. It follows that G_{vw} is not maximal in L and so Γ is G -locally imprimitive. \square

We have the following construction which takes a B -edge-primitive bipartite graph such that B is almost simple and B^+ is primitive on each bipartite half, and builds a G -edge-primitive bipartite graph with G primitive of type PA on edges and G^+ primitive of type PA on each of the bipartite halves.

Construction 5.3. (Primitive PA on edges and biprimitive on vertices with G^+ primitive of type PA) Let Σ be a bipartite connected B -edge-primitive graph such that B is an almost simple group with socle T such that B^+ acts primitively on each bipartite

half. Then there exist a corefree maximal subgroup H of B^+ and $g \in B \setminus B^+$ such that $g^2 \in H$ and $\Sigma = \text{Cos}(B, H, HgH)$. Let $\sigma = (g, \dots, g)$,

$$G = \langle (B^+)^k, \sigma \rangle \rtimes S_k,$$

and $L = H^k \rtimes S_k$. Define $\Gamma = \text{Cos}(G, L, L\sigma L)$.

Lemma 5.4. *The connected bipartite graph $\Gamma = \text{Cos}(G, L, L\sigma L)$ yielded by Construction 5.3 is G -edge-primitive of type PA and G -biprimitive on vertices such that G^+ acts primitively of type PA on both of its vertex orbits. Moreover, Γ is G -locally primitive if and only if Σ is B -locally primitive.*

Proof. Since Σ is connected we have $\langle H, g \rangle = B$. It follows that $\langle L, \sigma \rangle = G$ and so Γ is connected. The stabiliser in G of the edge $e = \{L, L\sigma\}$ is $\langle (H \cap H^g)^k, \sigma \rangle \rtimes S_k$ which by Lemma 4.4 is a maximal subgroup of G since $\langle H \cap H^g, g \rangle$, the stabiliser in B of an edge in Σ , is a maximal subgroup of B . Hence G acts primitively of type PA on $E\Gamma$. The index two subgroup $G^+ = B^+ \text{ wr } S_k$ of G has two orbits on $V\Gamma$. Hence Γ is bipartite. Moreover, since H is a maximal subgroup of B^+ it follows from Lemma 4.3 that G^+ acts primitively of type PA on each of the bipartite halves.

Since $\langle H \cap H^g, g \rangle$ is maximal in the almost simple group B^+ , we have $H \cap H^g \neq 1$. Thus by Lemma 4.3 $(H \cap H^g)^k \rtimes S_k$ is maximal in $H^k \rtimes S_k$ if and only if $H \cap H^g$ is maximal in H , and Γ is G -locally primitive if and only if Σ is B -locally primitive. \square

Remark 5.5. Suppose that in Construction 5.3, we let $k = 2$ and let $\overline{G} = (B^+)^2 \rtimes \langle (g, g)(1, 2) \rangle \leq G$. Then $\overline{G}_e = (H \cap H^g)^2 \rtimes \langle (g, g)(1, 2) \rangle$, which is a maximal subgroup of \overline{G} . Thus \overline{G} is edge-primitive of type PA and biquasiprimitive on vertices. Moreover, $(\overline{G})^+ = (B^+)^2$ and $\overline{G}_v = H^2$. Hence $(\overline{G})^+$ is not quasiprimitive on each bipartite half of Γ .

We now give a general construction of G -edge-quasiprimitive graphs for which the action of G on edges is of type SD or CD and G is vertex-transitive.

Construction 5.6. (Quasiprimitive SD or CD on edges and vertex-transitive) Let G be a quasiprimitive group on a set Ω of type SD or CD with socle $N = T^k$. Let $\omega \in \Omega$ and let \mathcal{P} be the G -invariant partition of $\{1, \dots, k\}$ given by the set of supports of the full strips of N_ω . If G is of type SD then $\mathcal{P} = \{\{1, \dots, k\}\}$ while if G is of type CD then \mathcal{P} is a nontrivial system of imprimitivity for G . Suppose that G has an index two subgroup G^+ which leaves invariant two distinct partitions \mathcal{P}_1 and \mathcal{P}_2 of $\{1, \dots, k\}$ which are interchanged by G , and such that $\mathcal{P}_1 \vee \mathcal{P}_2 = \mathcal{P}$.

Let $L = G_\omega$. Conjugating by a suitable element of $\text{Sym}(\Omega)$ we may assume that each $h \in L$ is of the form $(t_1, \dots, t_k)\sigma$ where $t_i \in \text{Aut}(T)$, $\sigma \in S_k$, σ preserves \mathcal{P} , and if i, j belong to the same part of \mathcal{P} then $t_i = t_j$. Since $L^{\{1, \dots, k\}} = G^{\{1, \dots, k\}}$, it follows that L has an index two subgroup L^+ which leaves \mathcal{P}_1 and \mathcal{P}_2 invariant. Moreover, $L = \langle L^+, g \rangle$ for some element $g = (t_1, \dots, t_k)\sigma \in G$, where σ interchanges \mathcal{P}_1 and \mathcal{P}_2 . For a subset I of $\{1, \dots, k\}$, let T_I be the straight full strip of N whose support is I . Let $N_1 = \prod_{I \in \mathcal{P}_1} T_I$ and let $H = N_G(N_1)$. Then as L^+ leaves \mathcal{P}_1 invariant and G^+ is the stabiliser of \mathcal{P}_1 in G , we have $L^+ \leq H \leq G^+$. Moreover, since $G^+ = NL^+$, if $nl \in H$ with $n \in N$ and $l \in L$ then $n \in N_N(N_1) = N_1$. Thus $H = N_1 L^+$. Furthermore, $H^g = N_2 L^+$ where $N_2 = \prod_{I \in \mathcal{P}_2} T_I$. Since $g^2 \in L^+$ it follows that $g^2 \in H$ and we can define $\Gamma = \text{Cos}(G, H, HgH)$.

Lemma 5.7. *The graph $\Gamma = \text{Cos}(G, H, HgH)$ obtained from Construction 5.6 is G -edge-quasiprimitive of type SD or CD such that G is vertex-biquasiprimitive. Moreover, Γ is G -locally primitive if and only if \mathcal{P}_1 is the coarsest L^+ -invariant partition of $\mathcal{P}_1 \vee \mathcal{P}_2$ refined by \mathcal{P}_1 .*

Proof. Let v be the vertex corresponding to the coset H and w be the vertex corresponding to Hg . Then $e = \{v, w\}$ is an edge and $G_{vw} = H \cap H^g = (N_1 \cap N_2)L^+$. Elements of $N_1 \cap N_2$ are constant on the parts of \mathcal{P}_1 and the parts of \mathcal{P}_2 , hence are constant on the parts of $\mathcal{P}_1 \vee \mathcal{P}_2 = \mathcal{P}$. Thus $N_1 \cap N_2 = L \cap N$. Hence $G_{vw} = L^+$ and $G_e = L$. It follows that $G^\Omega \cong G^{E\Gamma}$ and so Γ is G -edge-quasiprimitive with $G^{E\Gamma}$ of type SD or CD. Moreover, G^+ has two orbits on $V\Gamma$ and so Γ is bipartite. Since N is the unique minimal normal subgroup of G and has two vertex orbits it follows that G is vertex-biquasiprimitive. Further, $G_{vw} = L^+$ is maximal in $G_v = H$ if and only if \mathcal{P}_1 is the coarsest L^+ -invariant partition of $\mathcal{P}_1 \vee \mathcal{P}_2$ refined by \mathcal{P}_1 . Hence the statement regarding local primitivity follows. \square

We now demonstrate the various vertex actions which can be yielded by Construction 5.6.

Example 5.8. A suitable choice for G primitive of type SD in Construction 5.6, is $N \rtimes K$, where $K = S_d \text{ wr } S_2$ for some $d \geq 3$, $N = T^{d^2}$ and $G^+ = N \rtimes S_d^2$. Here \mathcal{P}_1 corresponds to the set of orbits of $1 \times S_d$ on the d^2 simple direct factors of N (that is, the ‘‘horizontal’’ blocks) while \mathcal{P}_2 corresponds to set of orbits of $S_d \times 1$ (that is, the ‘‘vertical blocks’’). Note that G^+ is primitive of type CD on each of its vertex orbits and G is vertex-biprimitive.

Example 5.9. Let $G = T^4 \rtimes \langle (1, 3, 2, 4) \rangle$. Here $\mathcal{P}_1 = \{\{1, 4\}, \{2, 3\}\}$, $\mathcal{P}_2 = \{\{1, 3\}, \{2, 4\}\}$, and $\mathcal{P} = \{\{1, 2, 3, 4\}\}$. Then G is quasiprimitive but not primitive of type SD on edges and $G^+ = T^4 \rtimes \langle (1, 2)(3, 4) \rangle$ is primitive of type HC on each vertex orbit.

Example 5.10. A suitable choice for G primitive of type CD is $G = N \rtimes K$ where $N = T^{d^2m}$ and $K = (S_d^2)^m \cdot 2 \rtimes S_m$ such that K preserves the partition \mathcal{P} of m blocks of size d^2 with $d \geq 3$. Here K has an index two subgroup $K_1 = S_d^2 \text{ wr } S_m$ with two systems of imprimitivity \mathcal{P}_1 and \mathcal{P}_2 with dm parts of size d , interchanged by K . The partition \mathcal{P}_1 is the set of orbits of $(1 \times S_d)^m$ on the set of d^2m simple direct factors of N (the set of horizontal blocks in each part of \mathcal{P}) while \mathcal{P}_2 is the set of orbits of $(S_d \times 1)^m$ (the set of vertical blocks of each part of \mathcal{P}). Moreover, $\mathcal{P}_1 \vee \mathcal{P}_2 = \mathcal{P}$. Note that G^+ is primitive of type CD on each of its orbits on $V\Gamma$ and G is vertex-biprimitive.

Example 5.11. Let $G = T^8 \rtimes K$ where

$$K = \langle (1, 2)(3, 4)(5, 6)(7, 8), (1, 5)(2, 6)(3, 7)(4, 8), (1, 3, 2, 4)(5, 8, 6, 7) \rangle \cong D_8.$$

Then K has an index 2 subgroup $K_1 = \langle (1, 2)(3, 4)(5, 6)(7, 8), (1, 5)(2, 6)(3, 7)(4, 8) \rangle$ which preserves the two partitions $\mathcal{P}_1 = \{\{1, 4\}, \{2, 3\}, \{5, 8\}, \{6, 7\}\}$ and $\mathcal{P}_2 = \{\{1, 3\}, \{2, 4\}, \{5, 7\}, \{6, 8\}\}$. Moreover, $\mathcal{P}_1 \vee \mathcal{P}_2 = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}$. Thus $G^+ = T^8 \rtimes K_1$ is primitive of type HC on each vertex orbit while G is quasiprimitive but not primitive of type CD on edges.

Example 5.12. If G is edge-quasiprimitive but not edge-primitive it is not even necessary for G^+ to be quasiprimitive on each orbit. For example, let $G = N \rtimes K$ where $N = T^{4d}$ and $K = (S_d \text{ wr } S_2) \text{ wr } S_2$ such that K preserves the partition $\{\{1, \dots, d\}, \{d+1, \dots, 2d\}, \{2d+1, \dots, 3d\}, \{3d+1, \dots, 4d\}\}$. Now K has an index two subgroup $K^+ = (S_d \text{ wr } S_2)^2$ which has two orbits of size $2d$ on $\{1, \dots, 4d\}$ and acts imprimitively on each orbit. Then with $G^+ = N \rtimes K^+$, and the two partitions $\mathcal{P}_1 = \{\{1, \dots, 2d\}, \{2d+1, \dots, 3d\}, \{3d+1, \dots, 4d\}\}$ and $\mathcal{P}_2 = \{\{1, \dots, d\}, \{d+1, \dots, 2d\}, \{2d+1, \dots, 4d\}\}$, Construction 5.6 yields a G -edge-quasiprimitive graph such that G^+ is not quasiprimitive on either of its orbits (since the strips of N_v are not all of equal length).

6 Analysing the quasiprimitive and primitive types

In this section we determine all the possible types of edge and vertex actions of edge-quasiprimitive graphs (Theorem 6.12). From this, after a bit more work we deduce Theorem 1.2. By Lemmas 3.2, 3.5 and 3.6 there are three types of vertex actions for G -edge-quasiprimitive graphs to consider:

- G -vertex-intransitive where G acts faithfully and quasiprimitively on both orbits;
- G -vertex-quasiprimitive;
- G -vertex-biquasiprimitive and G^+ faithful on each orbit.

We go through each of the 8 types of quasiprimitive groups as possibilities for the edge action and determine if there is a suitable vertex action in each case.

Lemma 6.1. *Let Γ be a connected G -edge-quasiprimitive graph such that G is of type HA on edges. Then Γ is either a cycle of prime length or a complete bipartite graph.*

Proof. The unique minimal normal subgroup N of G is elementary abelian and G is in fact edge-primitive. Since N is edge-transitive, it is either vertex-transitive or has two orbits. If N is vertex-transitive, then since N is abelian it acts regularly on $V\Gamma$ and so $|V\Gamma| = |E\Gamma|$. Hence Γ is a cycle and by the primitivity of G on $E\Gamma$ it follows that Γ has prime length. If N has two orbits, then G is biquasiprimitive on vertices and so by Lemma 3.6, either Γ is complete bipartite or N acts faithfully on each orbit. In the latter case, N acts regularly on each orbit and so there are twice as many vertices as edges. This contradicts the fact that Γ is connected and so Γ is complete bipartite. \square

Lemma 6.2. *Let Γ be a connected G -edge-quasiprimitive graph such that G is quasiprimitive of type HS or HC on edges. Then Γ is a complete bipartite graph.*

Proof. Let N_1 and N_2 be the two minimal normal subgroups of G . Since G is of type HS or HC on edges, it is edge-primitive and so by Lemma 3.3 either Γ is a star (and hence complete bipartite), or G is vertex-transitive. We may assume that G is vertex-transitive. Then by Lemma 3.5, $G^{V\Gamma}$ is either quasiprimitive or biquasiprimitive. If $G^{V\Gamma}$ is quasiprimitive then since G has two minimal normal subgroups, $G^{V\Gamma}$ is of type HS or HC, respectively. Hence N_1 and N_2 are vertex-regular and so $|E\Gamma| = |V\Gamma|$. Thus Γ is a cycle, contradicting N_1 being insoluble. Thus $G^{V\Gamma}$ is biquasiprimitive. Suppose

that Γ is not complete bipartite. Since neither N_1 nor N_2 has an index two subgroup, it follows that $N_1, N_2 \leq G^+$ and by Lemma 3.6, both act transitively and faithfully on each G^+ orbit. Since N_1 centralises N_2 , it follows that N_1 and N_2 act regularly on each G^+ orbit ([5, Theorem 4.2A]). This implies that there are twice as many vertices as edges, contradicting Γ being connected. Hence Γ is a complete bipartite graph. \square

Lemma 6.3. *Let Γ be a G -edge-quasiprimitive graph which is of type AS on edges. Then either G is quasiprimitive of type AS on vertices or Γ is bipartite and G^+ acts faithfully and quasiprimitively of type AS on both parts of the bipartition.*

Proof. Noticing that any nontrivial normal subgroup of G is almost simple, the result follows by comparing isomorphism types and Lemmas 3.2 and 3.5. \square

Before dealing with the SD and CD cases we need the following lemma.

Lemma 6.4. *Let Γ be a connected G -edge-quasiprimitive graph and let N be a normal subgroup of G such that $N \cong T^k$ for some finite nonabelian simple group T . Let $e = \{v, w\}$ be an edge of Γ . Then $N_e \neq N_v$.*

Proof. Suppose that $N_e = N_v$. Since N is edge-transitive, it has at most two orbits on vertices. If N is vertex-transitive then $|V\Gamma| = |E\Gamma|$ and so Γ is a cycle. This contradicts $N \leq \text{Aut}(\Gamma)$. Hence N has two orbits on vertices and Γ is bipartite. Let Δ_1 be the bipartite half containing v . Then $|\Delta_1| = |E\Gamma|$. This contradicts Γ being connected and so $N_e \neq N_v$. \square

Next we deal with the SD and CD cases.

Proposition 6.5. *Let Γ be a G -edge-quasiprimitive, G -vertex-transitive connected graph which is not complete bipartite and such that G is quasiprimitive of type SD or CD on edges. Let $N \cong T^k$ be the unique minimal normal subgroup of G , let $e = \{v, w\}$ be an edge and let \mathcal{P} be the partition of the set of k simple direct factors of N given by the set of supports of the full strips of N_e . Then the following all hold.*

1. Γ is bipartite and G^+ acts faithfully on each bipartite half.
2. There exists a nontrivial G^+ -invariant partition \mathcal{P}_1 of $\{1, \dots, k\}$ such that N_v is the product of full strips whose supports are the parts of \mathcal{P}_1 .
3. There exists a nontrivial G^+ -invariant partition \mathcal{P}_2 of $\{1, \dots, k\}$ such that N_w is the product of full strips whose supports are the parts of \mathcal{P}_2 .
4. $\mathcal{P}_1 \vee \mathcal{P}_2 = \mathcal{P}$ and G interchanges \mathcal{P}_1 and \mathcal{P}_2 .
5. Γ is isomorphic to the graph yielded by Construction 5.6 using G , \mathcal{P}_1 and \mathcal{P}_2 .

Proof. Since $N_e \cong T^l$ for some divisor l of k , it does not have an index two subgroup and so $N_e = N_{vw}$. Thus $\pi_i(N_v) = T$ for each i , and so by a well known lemma, (see for example [16, p 328]) there exists a partition \mathcal{P}_1 of $\{1, \dots, k\}$ such that N_v is the product of full strips whose supports are the parts of \mathcal{P}_1 . Similarly, $\pi_i(N_w) = T$ for each i and so there exists a partition \mathcal{P}_2 of $\{1, \dots, k\}$ such that N_w is the product of

full strips whose supports are the parts of \mathcal{P}_2 . Since $N_e = N_v \cap N_w$, Lemma 4.2 implies that $\mathcal{P}_1 \vee \mathcal{P}_2 = \mathcal{P}$. By Lemma 6.4 $N_e \neq N_v$, and so $\mathcal{P}_1, \mathcal{P}_2 \neq \mathcal{P}$, hence $\mathcal{P}_1 \neq \mathcal{P}_2$. Thus N is vertex-intransitive and $\mathcal{P}_1, \mathcal{P}_2$ are nontrivial partitions of $\{1, \dots, k\}$. Since N is edge-transitive, it follows that Γ is bipartite with the two bipartite halves being N -orbits. By Lemmas 3.2 and 3.6, G^+ is faithful on each bipartite half and so (1) holds. Moreover, $G^+ = NG_v = NG_w$ and so G^+, G_v and G_w all induce the same permutation group on the set of k simple direct factors of N . Hence \mathcal{P}_1 and \mathcal{P}_2 are G^+ -invariant and so parts (2) and (3) hold. Furthermore, since G is vertex-transitive there exists $g \in G$ such that $v^g = w$. Thus G interchanges \mathcal{P}_1 and \mathcal{P}_2 and so part (4) holds. It remains to prove part (5).

Conjugating by a suitable element of $\text{Sym}(VT)$ we may assume that N_v is a product of straight full strips corresponding to the parts of \mathcal{P}_1 . Thus N_v is the subgroup N_1 constructed in Construction 5.6. Since G interchanges \mathcal{P}_1 and \mathcal{P}_2 , it follows that $G_v \leq N_G(N_v) \leq G^+$. Since $G^+ = NG_v$ and N_v is selfnormalising in N , it follows that $G_v = N_G(N_v)$. Thus G_v is the subgroup H given in Construction 5.6. Letting $g \in G_e$ which interchanges v and w and hence $\mathcal{P}_1, \mathcal{P}_2$, it follows that $\Gamma \cong \text{Cos}(G, H, HgH)$, the graph constructed in Construction 5.6. Thus part (5) holds. \square

We have the following corollaries if G is edge-primitive.

Corollary 6.6. *Let Γ be a connected G -edge-primitive graph which is not complete bipartite such that G is primitive of type SD on edges. Then Γ is bipartite and G^+ is faithful and quasiprimitive of type CD on each bipartite half.*

Proof. Since $G^{E\Gamma}$ is primitive of type SD it follows that $\mathcal{P} = \{\{1, \dots, k\}\}$ and G acts primitively on the set of k simple direct factors of N . Since $G^+ \triangleleft G$ it follows that G^+ acts transitively on the the set of simple direct factors of N . Hence N is a minimal normal subgroup of G^+ and so G^+ acts faithfully and quasiprimitively on each orbit. By Lemma 6.4, $N_e < N_v$ and so this action is of type CD. \square

Corollary 6.7. *Let Γ be a connected G -edge-primitive graph which is not complete bipartite such that G is primitive of type CD on edges. Then Γ is bipartite and G^+ is faithful and quasiprimitive of type CD on each bipartite half.*

Proof. Let $N \cong T^k$ be the unique minimal normal subgroup of G . Let e be an edge and let \mathcal{P} be the symstem of imprimitivity of $\{1, \dots, k\}$ given by the set of supports of the strips of N_e . Since G is primitive of type CD on edges it follows that for $P \in \mathcal{P}$, G_P acts primitively on P . Also $|G_P : G_P^+| \leq 2$. If $|G_P : G_P^+| = 1$ then G_P^+ acts primitively on P . However, by Proposition 6.5, $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2$ where \mathcal{P}_1 and \mathcal{P}_2 are preserved by G^+ . Hence P is a union of blocks of \mathcal{P}_1 , contradicting G_P^+ acting primitively on P . Thus $|G_P : G_P^+| = 2$ and so G^+ is transitive on \mathcal{P} . Moreover, as G_P is primitive on P it follows that G_P^+ is transitive on P and so G^+ is transitive on the set of k simple direct factors on N . Hence N is a minimal normal subgroup of G^+ and so G^+ acts faithfully and quasiprimitively on each orbit. By Lemma 6.4, $N_e < N_v$ and so this action is of type CD. \square

Next we investigate the case where G is of type PA on edges.

Lemma 6.8. *Let Γ be a G -edge-quasiprimitive connected graph such that G is of type PA on edges. Let N be the unique minimal normal subgroup of G . Then $N_v \neq 1$.*

Proof. Since G is quasiprimitive of type PA on edges we have that $N_e \neq 1$. Suppose that $N_v = 1$. Then $|V\Gamma| \geq |N| > |E\Gamma|$, contradicting Γ being connected. Thus $N_v \neq 1$. \square

Corollary 6.9. *Let Γ be a G -edge-quasiprimitive connected graph such that G is of type PA on edges. Suppose that G is vertex-quasiprimitive. Then the quasiprimitive type of $G^{V\Gamma}$ is SD, CD or PA.*

Proof. Let N be the unique minimal normal subgroup of G . By Lemma 6.8, $N_v \neq 1$ and so G is not of type TW on vertices. Since G has a unique minimal normal subgroup which is not elementary abelian or simple, it follows that $G^{V\Gamma}$ is of type SD, CD or PA. \square

Corollary 6.10. *Let Γ be a G -edge-quasiprimitive connected graph such that G is of type PA on edges. Suppose that G is vertex-intransitive. Then the quasiprimitive type of G on each of its orbits is SD, CD or PA.*

Corollary 6.11. *Let Γ be a G -edge-quasiprimitive connected graph such that G is of type PA on edges. Suppose that G is vertex-biquasiprimitive and G^+ is quasiprimitive on each orbit. Then the quasiprimitive type of G^+ on each of its orbits is HS, HC, SD, CD or PA.*

Collecting together our results we have the following two theorems. We split the statements into the vertex-transitive and vertex-intransitive cases.

Theorem 6.12. *Let Γ be a G -edge-quasiprimitive, G -vertex-transitive connected graph of valency at least three such that $G^{E\Gamma}$ is of type X . Then one of the following holds.*

1. Γ is a complete bipartite graph.
2. $X \in \{\text{SD}, \text{CD}\}$ and Γ can be obtained from Construction 5.6.
3. $X = \text{PA}$ and G is quasiprimitive on $V\Gamma$ of type SD, CD or PA.
4. $X = \text{PA}$ and Γ is bipartite, such that G^+ is faithful and quasiprimitive on each of its orbits of type $Y \in \{\text{HS}, \text{HC}, \text{SD}, \text{CD}, \text{PA}\}$.
5. $X = \text{PA}$, Γ is bipartite, and G^+ is not quasiprimitive on either orbit.
6. $X = \text{AS}$ and either $G^{V\Gamma}$ is quasiprimitive of type AS or Γ is bipartite and G^+ is faithful and quasiprimitive of type AS on each of its orbits.
7. $X = \text{TW}$.

Moreover, examples occur in all cases.

Examples 5.8 and 5.10 provide edge-primitive examples for case (2), Construction 5.1 gives examples for case (3) where G is primitive of type PA on vertices, Construction 5.3 gives examples where G^+ is primitive of type PA on each orbit and Section 2 gives many example for case (6). An edge-primitive example for case (5) is given by Remark 5.5. Examples of edge-quasiprimitive but not edge-primitive are given in Section 7.

If G is edge-primitive we can sometimes deduce more information. For example, we can eliminate $X = \text{TW}$.

Proposition 6.13. *Let Γ be a G -edge-primitive graph such that G is of type TW on edges. Then Γ is a complete bipartite graph.*

Proof. Let Γ be a G -edge-primitive graph such that G is of type TW on edges. Let N be the unique minimal normal subgroup of G . Then $N = T_1 \times \cdots \times T_k$ with each $T_i \cong T$ for some finite nonabelian simple group T and $G = N \rtimes G_e$. Moreover, G_e acts transitively by conjugation on the set of k simple direct factors of N . Let $(G_e)_1$ be the normaliser in G_e of T_1 and $\varphi : (G_e)_1 \rightarrow \text{Aut}(T)$ be the homomorphism induced by the action of $(G_e)_1$ on T_1 by conjugation. By Lemma 4.1, since G_e is maximal in G we have that $\text{Inn}(T) \leq \varphi((G_e)_1)$ and φ extends to no overgroup of $(G_e)_1$ in G_e . Since G is arc-transitive it follows that G_{vw} is an index two subgroup of G_e . There are two cases to consider: $(G_e)_1 \cap G_{vw}$ is an index two subgroup of $(G_e)_1$, or $(G_e)_1 \leq G_{vw}$.

Suppose that $(G_e)_1 \cap G_{vw}$ is an index two subgroup of $(G_e)_1$. Then G_{vw} acts transitively on the set of k simple direct factors of N . Since $\text{Inn}(T)$ does not have an index two subgroup, it follows that $\text{Inn}(T) \leq \varphi(G_{vw})$. Suppose that there exists R with $(G_e)_1 \cap G_{vw} \leq R \leq G_{vw}$ such that φ extends to R . Then φ would extend to $\langle (G_e)_1, R \rangle \leq G_e$. Since φ does not extend to any overgroup of $(G_e)_1$ in G_e it follows that $R \leq (G_e)_1$ and so $R = (G_e)_1 \cap G_{vw}$. Thus by Lemma 4.1, $(G_e)_1 \cap G_{vw}$ is maximal in $N \rtimes ((G_e)_1 \cap G_{vw})$. Since G_{vw} normalises N_v and N_w , it follows that $N_v = N_w = 1$. Thus $|V\Gamma| = |N|$ or $2|N|$. However, $|E\Gamma| = |N|$ and so $|V\Gamma| = |N|$ and Γ is a cycle. This contradicts G being insoluble and so $(G_e)_1 \leq G_{vw}$.

Since $(G_e)_1 \leq G_{vw}$ it follows that G_{vw} has two equal sized orbits on the set of k simple direct factors of N . Without loss of generality we may suppose that these are $\{T_1, \dots, T_{k/2}\}$ and $\{T_{k/2+1}, \dots, T_k\}$ and note that they are interchanged by elements of G_e not in G_{vw} . Moreover, $(G_e)_1$ normalises N_v . Since $\varphi((G_e)_1)$ contains $\text{Inn}(T)$ it follows that the projection of N_v onto the first simple direct factor of N is either trivial or equal to T . Thus N_v is a subdirect product of either $T_{k/2+1} \times \cdots \times T_k$ or N . If $N_v \leq T_{k/2+1} \times \cdots \times T_k$ then $N_w \leq T_1 \times \cdots \times T_{k/2}$. Moreover, G_e normalises $\langle N_v, N_w \rangle$ and so by the maximality of G_e in G we have $\langle N_v, N_w \rangle = N$. Thus $N_v = T_{k/2+1} \times \cdots \times T_k$, and so N has two orbits on vertices and is unfaithful on each. Hence by Lemma 3.6, Γ is a complete bipartite graph. Thus we are left to consider the case where N_v is a subdirect product of N . Thus there exists a partition \mathcal{P} of $\{1, \dots, k\}$ such that $N_v = \prod_{I \in \mathcal{P}} T_I$ where T_I is a diagonal subgroup of $\prod_{i \in I} T_i$. Since $(G_e)_1 \leq G_{vw} \leq G_e$ it follows from Lemma 4.1 that G_{vw} is a maximal subgroup of $(T_1 \times \cdots \times T_{k/2}) \rtimes G_{vw}$. Hence $N_v \cap (T_1 \times \cdots \times T_{k/2}) = 1$. Similarly, G_{vw} is maximal in $(T_{k/2+1} \times \cdots \times T_k) \rtimes G_{vw}$ and so $N_v \cap (T_{k/2+1} \times \cdots \times T_k) = 1$. It follows that each $I \in \mathcal{P}$ is split equally between $\{1, \dots, k/2\}$ and $\{k/2 + 1, \dots, k\}$. However, since G_{vw} normalises N_v and $M = T_1 \times \cdots \times T_{k/2}$ it follows that G_{vw} normalises the projection of N_v onto M . Thus $|I| = 2$, as G_{vw} normalises no proper nontrivial subgroup of M . Hence $|N_v| = |T|^{k/2}$ and $|V\Gamma| = |T|^{k/2}$ or $2|T|^{k/2}$. The first case is not possible

as $|V\Gamma|^2 = |E\Gamma|$, a contradiction. Hence we have the second. This implies that Γ is complete bipartite and we are done. \square

We can also deduce more information when $X = \text{PA}$.

Lemma 6.14. *Let Γ be a G -edge-primitive graph such that G is of type PA on edges and Γ is not complete bipartite. Then one of the following holds:*

1. G is quasiprimitive on vertices of type PA;
2. G is biquasiprimitive and G^+ is quasiprimitive of type PA on each bipartite half;
3. G is biquasiprimitive and G^+ is not quasiprimitive on either bipartite half.

Proof. Let N be the unique minimal normal subgroup of G . Then $N = T^k$ for some finite nonabelian simple group T and $k \geq 2$. Also given an edge $e = \{v, w\}$ we have $N_e = R^k$ for some proper nontrivial subgroup R of T . Since $G^{E\Gamma}$ is primitive, there exists an almost simple group A with socle T and maximal subgroup H such that $H \cap T = R$. Suppose that $|R| = 2$. Then $H = C_A(z)$ and $R = C_T(z)$, where z is the involution which generates R . However, 4 divides $|T|$ and so either z is contained in a cyclic group of order 4 or an elementary abelian group of order 4, a contradiction. Thus $|R| > 2$. It follows that N_e does not have an index 2 subgroup and so $N_{vw} = N_e$. Hence $R^k \leq N_v$. Thus for each i such that $\pi_i(N_v) = T$, we have that N_v contains the i^{th} factor of N . Since Γ is not complete bipartite, Lemma 3.6 implies that N is faithful on each of its orbits on $V\Gamma$, and so N cannot contain any of its simple direct factors. Thus $\pi_i(N_v) \neq T$ for all i . Hence if G is quasiprimitive on $V\Gamma$, this implies that G is of type PA on vertices and we have case (1). If G is biquasiprimitive on vertices and G^+ is transitive on the set of simple direct factors of N then we have that G^+ is quasiprimitive of type PA on each of its orbits and we have case (2). If G^+ has two orbits on the set of simple direct factors of N then G^+ has two minimal normal subgroups contained in N . Since N_v does not project onto T in any coordinate, it follows that G^+ is not quasiprimitive on either orbit and so case (3) holds. \square

Note that when G is biprimitive on vertices, G^+ is primitive on each bipartite half. Hence Lemma 6.14 combined with Theorem 6.12, Corollaries 6.6 and 6.7, and Proposition 6.13 yields Theorem 1.2.

We complete this section by reducing the study of edge-primitive graphs of type PA to the study of edge-primitive graphs of type AS. Before doing so we need to establish some notation. Let T be a nonabelian simple group and let G be a subgroup of $\text{Aut}(T) \text{ wr } S_k$ for some $k \geq 2$, which contains $N = T_1 \times \cdots \times T_k$ where each $T_i \cong T$, and such that G induces a transitive subgroup of S_k on the set of k simple direct factors of N . Let G_1 be the normaliser in G of T_1 . Then $G_1 = G \cap (\text{Aut}(T) \times (\text{Aut}(T) \text{ wr } S_{k-1}))$ and there exists a projection $\pi_1 : G_1 \rightarrow \text{Aut}(T)$. Let $B = \pi_1(G_1)$. By [12, (2.2)], conjugating by a suitable element of $\text{Aut}(T) \text{ wr } S_k$ we may have chosen G such that $G \leq B \text{ wr } S_k$. We call B the *group induced by G* .

Proposition 6.15. *Suppose that Γ is a G -edge-primitive graph such that $G^{E\Gamma}$ is of type PA, and let $e = \{v, w\}$ be an edge. Let $N = \text{soc}(G) \cong T^k$ for some finite nonabelian simple group T and k a positive integer at least two. Suppose that G induces the primitive*

almost simple group B with socle T , and that G_e and G_v induce the subgroups E and H of B respectively. Then there exists a B -edge-primitive graph with edge-stabiliser E and vertex-stabiliser H .

Proof. Since G is a primitive group of type PA on $E\Gamma$, we have that $G \leq B \text{ wr } S_k$ and $G_e = G \cap (E \text{ wr } S_k)$ where $\pi_1((G_1)_e) = E$ is a maximal subgroup of B . Let $A = \pi_1((G_1)_{vw})$ and $H = \pi_1((G_1)_v)$. Note that $H \cap E = A$ and H is a proper subgroup of B . Since G is arc-transitive, $|G_e : G_{vw}| = 2$ and so $|(G_1)_e : (G_1)_{vw}| \leq 2$. Thus $|E : A| \leq 2$. If $E = A$ then $E \leq H$. However, by the maximality of E this implies that $E = H$ and so G_v is contained in some G -conjugate of G_e . This contradicts the fact that there are more edges than vertices and so $|E : A| = 2$. For the same reason $A < H$. Let $\sigma \in (G_1)_e \setminus (G_1)_{vw}$. Then $g = \pi_1(\sigma) \in E \setminus A$ and $\text{Cos}(G, H, HgH)$ is a G -edge-primitive graph with edge stabiliser E and vertex stabiliser H . \square

7 Quasiprimitive examples

In this section we construct examples of edge quasiprimitive graphs where the types of actions do not occur in the edge primitive case.

Example 7.1. (Quasiprimitive PA on edges and primitive SD on vertices) Let $G = T \text{ wr } S_2$ for some finite nonabelian simple group T and let $H = \{(t, t) \mid t \in T\} \times \langle \sigma \rangle$, where σ interchanges the two simple direct factors of $N = T^2 \triangleleft G$. Let $x \in T$ be of order two and let $g = (1, x) \in G$. Then

$$H^g = \{(t, t^x) \mid t \in T\} \times \langle (x, x)\sigma \rangle = \{(t, t^x) \mid t \in T\} \rtimes \langle \sigma \rangle$$

and $H \cap H^g = \{(t, t) \mid t \in C_T(x)\} \times \langle \sigma \rangle$. Let $\Gamma = \text{Cos}(G, H, HgH)$. Then G is vertex-primitive of type SD. Let $e = \{H, Hg\}$, an edge of Γ . Then $G_e = \langle H \cap H^g, g \rangle$. Since $x \in C_T(x)$, it follows that $G_e = \{(x^i t, x^j t) \mid t \in C_T(x); i, j \in \{0, 1\}\} \rtimes \langle \sigma \rangle$ and so G is quasiprimitive of type PA on edges.

Example 7.2. (Quasiprimitive PA on edges and primitive CD on vertices) Let $\sigma = (1, 2, 3, 4)$ and $G = T^4 \rtimes \langle \sigma \rangle$ for some finite nonabelian simple group T . Let $H = \{(t, s, t, s) \mid s, t \in T\} \rtimes \langle \sigma \rangle$ and $g = (x, x, 1, 1)$ where $x \in T$ has order two. Then $g^2 \in H$, $g \notin N_G(H)$ and $\langle H, g \rangle = G$. Let $\Gamma = \text{Cos}(G, H, HgH)$. Then G is primitive of type CD on vertices. Let v be the vertex corresponding to H and w the vertex corresponding to Hg . Then $G_w = H^g = \{(t^x, s^x, t, s) \mid t, s \in T\} \rtimes \langle \sigma \rangle$ and so for the edge $e = \{v, w\}$ we have $G_e = \{(tx^i, sx^j, tx^k, sx^l) \mid t, s \in C_T(x); i + j + k + l \equiv 0 \pmod{2}\} \rtimes \langle \sigma \rangle$. Thus G is quasiprimitive of type PA on edges.

We now give examples in the bipartite case.

Example 7.3. (Quasiprimitive PA on edges and G^+ primitive HS on each vertex orbit) Let T be a nonabelian simple group and let $x \in T$ have order 2. Let $G = T \text{ wr } S_2$ and $H = \{(t, t) \mid t \in T\}$. Let $g = (x, 1)\sigma$ where σ interchanges the two simple direct factors of $N = T^2 \triangleleft G$. Then $g^2 \in H$, $g \notin N_G(H)$ and $\langle H, g \rangle = G$. Let $\Gamma = \text{Cos}(G, H, HgH)$. Then $G^+ = N$ has two orbits on vertices. Let v be the vertex corresponding to H and

w be the vertex corresponding to Hg . Then $G_v = H$ and $G_w = H^g = \{(t, t^x) \mid t \in T\}$. Thus G^+ is primitive of type HS on each orbit. Moreover, $e = \{v, w\}$ is an edge and $G_e = \{(t, t) \mid t \in C_T(x)\} \times \langle g \rangle$. Thus G is quasiprimitive of type PA on edges.

Example 7.4. (Quasiprimitive PA on edges and G^+ primitive HC on each vertex orbit) Let $\sigma = (1, 2, 3, 4)$ and let $G = T \text{ wr} \langle \sigma \rangle$ for some finite nonabelian simple group T . Let $H = \{(t, t, s, s) \mid t, s \in T\} \rtimes \langle \sigma^2 \rangle$ and let $x \in T$ of order 2. Let $g = (1, x, 1, x)\sigma$. Then $g^2 \in H$, $g \notin N_G(H)$ and $\langle H, g \rangle = G$. Let $\Gamma = \text{Cos}(G, H, HgH)$, let v be the vertex corresponding to H and w the vertex corresponding to Hg . Then Γ is bipartite with $G^+ = T^4 \rtimes \langle \sigma^2 \rangle$ and $e = \{v, w\}$ is an edge. Moreover, G^+ is primitive of type HC on each orbit. Now $G_v = H$ and $G_w = H^g = \{(t, t^x, s, s^x) \mid t, s \in T\} \rtimes \langle \sigma^2 \rangle$. Thus $G_e = \{(t, t, s, s) \mid t, s \in C_T(x)\} \rtimes \langle g \rangle$. Hence G is quasiprimitive of type PA on edges.

Example 7.5. (Quasiprimitive PA on edges and G^+ primitive SD on each vertex orbit) Let T be a nonabelian simple group with outer automorphism τ of order two. Let $G = (T \times T) \rtimes \langle (1, \tau), \sigma \rangle$ where σ interchanges the two minimal normal subgroups of $N = T^2$. Let $H = \{(t, t) \mid t \in \langle T, \tau \rangle\} \times \langle \sigma \rangle$ and let $g = (1, \tau)$. Then $g^2 \in H$, $g \notin N_G(H)$ and $\langle H, g \rangle = G$. Thus we can define the graph $\Gamma = \text{Cos}(G, H, HgH)$. Let v be the vertex corresponding to H and w the adjacent vertex corresponding to Hg . Then $G_v = H$ and $G_w = H^g = \{(t, t^\tau) \mid t \in \langle T, \tau \rangle\} \times \langle \sigma \rangle$. Hence $G^+ = T^2 \rtimes \langle (\tau, \tau), \sigma \rangle$ acts primitively of type SD on each orbit. Let $e = \{v, w\}$. Then $G_e = \{(t, t) \mid t \in C_T(\tau)\} \rtimes \langle (1, \tau), \sigma \rangle$ and so G is quasiprimitive of type PA on edges.

Example 7.6. (Quasiprimitive PA on edges and G^+ primitive CD on each vertex orbit) Let $\sigma = (1, 2, 3, 4)$, T be a finite nonabelian simple group and τ an outer automorphism of T of order two. Let $G = T^4 \rtimes \langle (\tau, \tau, 1, 1), \sigma \rangle$ and $H = \{(t, s, t, s) \mid t, s \in T\} \rtimes \langle (\tau, 1, \tau, 1), \sigma \rangle$. Then letting $g = (\tau, \tau, 1, 1)$ we see that $g^2 \in H$, $g \notin N_G(H)$ and $\langle H, g \rangle = G$. Thus we can define the graph $\text{Cos}(G, H, HgH)$. Then $G^+ = T^4 \rtimes \langle (\tau, 1, \tau, 1), \sigma \rangle$ acts primitively of type CD on each vertex orbit. Let v be the vertex corresponding to H and w be the adjacent vertex corresponding to Hg . Then $G_v = H$ and $G_w = H^g = \{(t^\tau, s^\tau, t, s) \mid t, s \in T\} \rtimes \langle (\tau, 1, \tau, 1), \sigma \rangle$. Thus $G_e = \{(t, s, t, s) \mid t, s \in C_T(\tau)\} \rtimes \langle (\tau, \tau, 1, 1), \sigma \rangle$ and so G acts quasiprimitively of type PA on edge.

Construction 7.7. (Quasiprimitive of type TW on edges and G^+ primitive of type PA on both orbits.) Let T be a finite nonabelian simple group with maximal subgroup R and suppose that there exists an outer automorphism τ of order two such that $R \cap R^\tau = 1$. A suitable choice of T and R is $\text{PSL}(2, 29)$ and A_5 respectively. Let $G = \langle T^k, (\tau, \dots, \tau) \rangle \rtimes S_k$ and $H = R^k \rtimes S_k$. Then if $g = (\tau, \dots, \tau)$ we have $g^2 \in H$, $g \notin N_G(H)$ and $\langle H, g \rangle = G$. Hence $\Gamma = \text{Cos}(G, H, HgH)$ is a G -arc-transitive connected graph. Moreover, $\langle g \rangle \times S_k$ is the stabiliser of an edge. Thus letting $N = \text{soc}(G) = T^k$ we have that N acts regularly on $E\Gamma$ and so G is quasiprimitive of type TW on edges. Note that $G^{E\Gamma}$ is not primitive as an edge stabiliser is not maximal. Furthermore, Γ is bipartite with $G^+ = T^k \rtimes S_k$ acting primitively of type PA on both orbits.

8 Edge-primitive groups with socle $\text{PSL}(2, q)$

The following theorem of Dickson [4] determines the maximal subgroups of $\text{PSL}(2, q)$.

Theorem 8.1. *Let p be a prime, f a positive integer and $q = p^f$. Then the conjugacy classes of maximal subgroups of $\text{PSL}(2, q)$ are as follows:*

1. *one class of subgroups isomorphic to $[q] \rtimes C_{(q-1)/(2, q-1)}$,*
2. *one class of subgroups isomorphic to $D_{2(q-1)/(2, q-1)}$, if $q \notin \{5, 7, 9, 11\}$,*
3. *one class of subgroups isomorphic to $D_{2(q+1)/(2, q-1)}$, if $q \notin \{7, 9\}$,*
4. *two classes of subgroups isomorphic to A_5 , if $q \equiv \pm 1 \pmod{10}$, and $\mathbb{F}_q = \mathbb{F}_p[\sqrt{5}]$,*
5. *two classes of subgroups isomorphic to S_4 , if $q = p \equiv \pm 1 \pmod{8}$,*
6. *one class of subgroups isomorphic to A_4 , if $q = p \equiv 3, 5, 13, 27, 37 \pmod{40}$.*
7. *two classes of subgroups isomorphic to $\text{PGL}(2, p^{f/2})$ when p odd,*
8. *one class of subgroups isomorphic to $\text{PSL}(2, p^m)$ where f/m an odd prime or $p = 2$ and $m \geq 2$.*

We also have the following theorem about maximal subgroups of almost simple groups with socle $\text{PSL}(2, q)$.

Theorem 8.2. *[6, Theorem 1.1] Let $T = \text{PSL}(2, q) \leq G \leq \text{P}\Gamma\text{L}(2, q)$ and let E be a maximal subgroup of G which does not contain T . Then either $E \cap T$ is maximal in T , or we have one of the following cases.*

1. $G = \text{PGL}(2, 7)$ and $E = N_G(D_6) = D_{12}$.
2. $G = \text{PGL}(2, 7)$ and $E = N_G(D_8) = D_{16}$.
3. $G = \text{PGL}(2, 9)$, M_{10} or $\text{P}\Gamma\text{L}(2, 9)$ and $E = N_G(D_{10})$
4. $G = \text{PGL}(2, 9)$, M_{10} or $\text{P}\Gamma\text{L}(2, 9)$ and $E = N_G(D_8)$.
5. $G = \text{PGL}(2, 11)$ and $E = N_G(D_{10}) = D_{20}$.
6. $G = \text{PGL}(2, q)$, $q = p \equiv \pm 11, \pm 19 \pmod{40}$ and $E = N_G(A_4) = S_4$.

The following lemma shows that except for the 8 exceptions in Theorem 8.2, we can restrict our attention to searching for G -arc-transitive G -edge primitive graphs with $G = \text{PSL}(2, q)$.

Lemma 8.3. *Let Γ be a nontrivial G -edge-primitive connected graph with $T = \text{PSL}(2, q) \triangleleft G \leq \text{P}\Gamma\text{L}(2, q)$. Let E be the stabiliser in G of an edge of Γ . If $E \cap T$ is maximal in T then T is arc-transitive and edge-primitive.*

Proof. Since G is edge-primitive and $T \triangleleft G$ it follows that T acts transitively on the set of edges with edge stabiliser $E \cap T$. Hence T is edge-primitive. Since Γ is G -arc-transitive, Γ is not a star and so by Lemma 3.4, Γ is also T -arc-transitive. \square

We have the following proposition.

Proposition 8.4. *Let $G = \text{PSL}(2, q)$, where $q = p^f$, E be a maximal subgroup of G and H be a subgroup of G such that $A = H \cap E$ is an index two subgroup of E and a proper subgroup of H . Suppose G is not 2-transitive on the set of cosets of H . Then one of the following holds.*

1. $q = p \equiv \pm 1, \pm 9 \pmod{40}$, $E = S_4$, $H = A_5$ and $A = A_4$.
2. $q = 17$, $E = D_{16}$, $H = S_4$ and $A = D_8$.
3. $q = 19$, $E = D_{20}$, $H = A_5$ and $A = D_{10}$.
4. $q = 25$, $E = D_{24}$, $H = \text{PGL}(2, 5)$ and $A = D_{12}$.

In the first case, given E there are two choices for H and these are conjugate in T . In the last three cases, given E there are four choices for H and these come in conjugate pairs.

Proof. We work our way through the list of maximal subgroups of G given in Theorem 8.1. We note first that E cannot be A_5 , A_4 , $\text{PSL}(2, p^m)$ for $p^m \neq 2$, or $[q] \rtimes C_{q-1}$ for q even, as these groups do not have an index 2 subgroup. Furthermore, $E \neq [q] \rtimes C_{(q-1)/2}$ for q odd as the only possible index 2 subgroup is $[q] \rtimes C_{(q-1)/4}$ which is only contained in E .

Suppose next that $E = D_{2(q-1)/(2, q-1)}$ and note that $q \notin \{5, 7, 9, 11\}$. Then $A = C_{(q-1)/(2, q-1)}$, the stabiliser of two points of the projective line, is an index two subgroup of E . The only possibility for H is a subgroup isomorphic to $[q] \rtimes C_{(q-1)/(2, q-1)}$, but in this case the action of G is 2-transitive. If $(q-1)/(2, q-1)$ is even then E also contains two subgroups isomorphic to $D_{(q-1)/2}$ which are conjugate in $\text{PGL}(2, q)$ but not $\text{PSL}(2, q)$. The restrictions on q imply that $(q-1)/(2, q-1) \geq 6$ and so if $A \cong D_{(q-1)/2}$ then A is not contained in a D_{q+1} . Furthermore, A is not contained in an A_4 . If A is contained in an A_5 then $(q-1)/2 = 6$ or 10 . The first implies that $q = 13$, but $\text{PSL}(2, 13)$ does not contain an A_5 while the second implies that $q = 21$, a contradiction. Thus A is not contained in an A_5 . If A is contained in an S_4 then $(q-1)/2 = 6$ or 8 . Again the first is not possible as $\text{PSL}(2, 13)$ does not contain an S_4 and so $q = 17$. Since D_8 is maximal in S_4 , it follows that in this case we have $H \cong S_4$. Counting again shows that given A there are two choices for H and these are conjugate in T . The two nonconjugate choices for A give us two nonconjugate pairs of choices for H . Thus we are in case (2). If $A \leq \text{PGL}(2, p^{f/2})$ then $(q-1)/2$ divides either $2(p^{f/2} - 1)$ or $2(p^{f/2} + 1)$. Since $q - 1 = (p^{f/2} - 1)(p^{f/2} + 1)$ either $p^{f/2} - 1$ or $p^{f/2} + 1$ divides 4. Thus $p^{f/2} = 3$ or 5 . Since $q \neq 9$ this give us $D_{12} \leq \text{PGL}(2, 5) \leq \text{PSL}(2, 25)$. Counting again gives that there are two choices for H and these are conjugate in T . Again the two nonconjugate choices for A give nonconjugate pairs of choices for H and we have case (4). If $A \leq \text{PSL}(2, p^{f/r})$ for $r \geq 3$ then $(q-1)/2$ divides either $p^{f/r} - 1$ or $p^{f/r} + 1$. Since $r \geq 3$ we have $p^f - 1 > 2(p^{f/r} \pm 1)$ and so this is not possible.

Next let $E = D_{2(q+1)/(2, q-1)}$ with $q \notin \{7, 9\}$. One choice for A is $C_{(q+1)/(2, q-1)}$. If $q = 5$ then $A = C_3$ and so $H \cong A_4$. However, in this case G is 2-transitive on the cosets of H . Thus $(q+1)/(2, q-1) \geq 6$ and so there is no possibility for H . If $(q+1)/2$ is even then A can also be one of the two choices of $D_{(q+1)/(2, q-1)}$ which are conjugate in $\text{PGL}(2, q)$ but not $\text{PSL}(2, q)$. Note then that $q \geq 11$ and so $(q+1)/(2, q-1) \geq 6$.

Thus A is not contained in a A_4 . Since $(q+1)/2 \geq 6$ does not divide $q-1$ it follows that A is not contained in D_{q-1} . Now $A \leq A_5$, if and only if $(q+1)/2 = 10$ or 6 . For A_5 to be a subgroup of G we require that $q = 11$ or 19 . We do not have the first case as this yields a 2-transitive group. There are then two choices for H and these are conjugate in T . Moreover the two nonconjugate choices for A gives nonconjugate pairs of choices for H and we have case (3). To have $A \leq S_4$ we require $(q+1)/2 = 8$ or 6 . The first is not possible while the second has $q = 11$ in which case there is no S_4 . To have $A \leq \text{PGL}(2, p^{f/2})$ we require that $(q+1)/2$ divides either $2(p^{f/2} - 1)$ or $2(p^{f/2} + 1)$. Hence $p^f + 1$ divides either $4(p^{f/2} - 1)$ or $4(p^{f/2} + 1)$ and so $p^{f/2} - 1 \leq 4$. This implies that $p^{f/2} = 3$ or 5 . However, we then have $q = 9$ or 25 , and in both cases $(q+1)/2$ is odd. Hence A is not contained in $\text{PGL}(2, p^{f/2})$. For $A \leq \text{PSL}(2, p^m)$, for some $m < f/2$, we need $(q+1)/2$ to divide either $p^m - 1$ or $p^m + 1$. Neither of these are possible and so A and H are one of the groups listed.

Suppose next that $E = S_4$ and $q = p \equiv \pm 1 \pmod{8}$. Then $A = A_4$. Since $q = p$, the only other subgroup of G containing A is $H \cong A_5$ when $q \equiv \pm 1 \pmod{10}$. Since each A_5 contains 5 copies of A_4 and the normaliser in G of A_4 is S_4 it follows that there are two choices for H . This gives case (1).

Finally, if $E = \text{PGL}(2, p^{f/2})$ with p odd then $A = \text{PSL}(2, p^{f/2})$. The only way that A can be contained in another maximal subgroup of G is if A is soluble. Hence $q = 9$, $A = \text{PSL}(2, 3) \cong A_4$. Looking at the maximal subgroups of G it follows that $H \cong A_5$. However, in this case G is 2-transitive on the cosets of H , a contradiction. \square

We also need the following proposition concerning the exceptional cases in Theorem 8.2.

Proposition 8.5. *Let $T = \text{PSL}(2, q) \triangleleft G \leq \text{PGL}(2, q)$ and suppose that E is a maximal subgroup of G not containing $T = \text{PSL}(2, q)$ such that $E \cap T$ is not maximal in T . Suppose that G has a subgroup H such that $A = H \cap E$ is a proper subgroup of H and has index two in E , and that G is not 2-transitive on the set of cosets of H . Then one of the following holds.*

1. $G = \text{PGL}(2, 7)$, $E = D_{12}$, $H = S_4$ and $A = E \cap T = D_6$.
2. $G = \text{PGL}(2, 7)$, $E = D_{16}$, $H = S_4$ and $A = E \cap T = D_8$.
3. $G = \text{PGL}(2, 9)$, M_{10} , or $\text{P}\Gamma\text{L}(2, 9)$, $E = N_G(D_8)$, $H = N_G(\text{PGL}(2, 3))$ and $A = E \cap \text{P}\Sigma\text{L}(2, 9)$.
4. $G = \text{PGL}(2, 9)$, M_{10} , $\text{P}\Gamma\text{L}(2, 9)$, $E = N_G(D_{10})$, $H = N_G(A_5)$ and $A = E \cap \text{P}\Sigma\text{L}(2, 9)$.
5. $G = \text{PGL}(2, 11)$, $E = D_{20}$, $H = C_{11} \rtimes C_{10}$ and $A = C_{10}$.
6. $G = \text{PGL}(2, 11)$, $E = D_{20}$, $H = A_5$ and $A = E \cap T = D_{10}$.
7. $G = \text{PGL}(2, q)$, $q = p \equiv \pm 11, \pm 19 \pmod{40}$, $E = S_4$, $H = A_5$ and $A = A_4$.

In each case there are two conjugate choices for H .

Proof. Note that G and E are given by Theorem 8.2. The first 6 cases can all be dealt with by looking at the list of maximal subgroups in [3]. If $G = \text{PGL}(2, q)$ for $q = p \equiv 11, 19, 21, 29 \pmod{40}$ and $E = S_4$ then the only possibility for A is A_4 . There are then two choices for H being A_5 and these are the only possibilities. \square

We can now determine all G -edge-primitive graphs with $\text{soc}(G) = \text{PSL}(2, q)$.

Proof. (of Theorem 1.3) Let Γ be a G -edge-primitive graph such that $T = \text{soc}(G) = \text{PSL}(2, q)$ with $q > 3$. Then by Proposition 2.5 there exists a maximal subgroup E of G with an index 2 subgroup A also contained in a proper corefree subgroup H of G such that $\Gamma \cong \text{Cos}(G, H, HgH)$ for some $g \in E \setminus A$. If G is 2-transitive on the set of cosets of H then Γ is a complete graph and G is primitive on 2-subsets. By Theorem 2.1, G appears in Table 2. Thus we can assume that G is not 2-transitive on vertices. Then by Proposition 8.5 either Γ is T -edge-primitive with $E \cap T$, $A \cap T$ and $H \cap T$ given by Proposition 8.4, or G, E, A and H are given by Proposition 8.5.

Next let $q = p \equiv \pm 1, \pm 9 \pmod{40}$, $E \cap T = S_4$, $A \cap T = A_4$ and $H \cap T = A_5$. Since there are two conjugacy classes of A_5 subgroups in $\text{PSL}(2, q)$ and these are fused in $\text{PGL}(2, q)$ it follows that $\text{PGL}(2, q)$ is not an automorphism group of this graph and so we have row 9 of Table 1.

The remaining cases from Proposition 8.4 are

1. $q = 17$, $E \cap T = D_{16}$, $A \cap T = D_8$ and $H \cap T = S_4$
2. $q = 19$, $E \cap T = D_{20}$, $A \cap T = D_{10}$ and $H \cap T = A_5$
3. $q = 25$, $E \cap T = D_{24}$, $A \cap T = D_{12}$ and $H \cap T = \text{PGL}(2, 5)$.

In all cases there are two T -conjugacy classes of subgroups $H \cap T$, and these are fused in $\text{PGL}(2, q)$. Hence we get isomorphic graphs. Also the only possibilities for G are then $\text{PSL}(2, 17)$, $\text{PSL}(2, 19)$, $\text{PSL}(2, 25)$ and $\text{P}\Sigma\text{L}(2, 25)$. These give us rows 6–8 of Table 1.

It remains to deal with the groups left from Proposition 8.5.

If $G = \text{PGL}(2, 7)$, $E = D_{12}$, $A = D_6$ and $H = S_4$ then since $H \leq \text{PSL}(2, 7)$ it follows that Γ is bipartite. Note that $G \cong \text{Aut}(\text{PSL}(3, 2))$, H is the stabiliser in $\text{PSL}(3, 2)$ of a 1-space U and A is the stabiliser in H of a 2-space which is complementary to U . Thus we have row 2.

Next let $G = \text{PGL}(2, 7)$, $E = D_{16}$, $A = D_8$ and $H = S_4$. Again we have that Γ is bipartite, and H is the stabiliser in $\text{PSL}(3, 2)$ of a 1-space U . However, this time A is the stabiliser in H of a 2-space containing U and so Γ is the Heawood graph, so we have row 1.

Now let $G = \text{PGL}(2, 9)$, M_{10} , or $\text{P}\Gamma\text{L}(2, 9)$, $E = N_G(D_8)$, $A = E \cap \text{P}\Sigma\text{L}(2, 9)$ and $H = N_G(\text{PGL}(2, 3))$. Note that $\text{P}\Gamma\text{L}(2, 9) \cong \langle \text{PSp}(4, 2), \tau \rangle$ where τ is a duality of the associated polar space. Moreover, H is the stabiliser of a totally isotropic 1-space and A is the stabiliser in H of an incident totally isotropic 2-space. Thus Γ is the Tutte–Coxeter graph and we have row 4.

When $G = \text{PGL}(2, 9)$, M_{10} or $\text{P}\Gamma\text{L}(2, 9)$, $E = N_G(D_{10})$, $A = E \cap \text{P}\Sigma\text{L}(2, 9)$ and $H = N_G(A_5)$, we have that $H \leq (G \cap \text{P}\Sigma\text{L}(2, 9))$ and $G \cap \text{P}\Sigma\text{L}(2, 9)$ is an index two subgroup of G . Thus Γ is bipartite. The vertices of Γ are two sets of size 6 with $\text{PSL}(2, 9) \cong A_6$ acting on each with two different actions. Since the stabiliser in A_6 of a

point in one action is still transitive in the other action it follows that $\Gamma \cong K_{6,6}$ and we have row 3 of Table 1.

When $G = \text{PGL}(2, 11)$, $E = D_{20}$, $A = E \cap T = D_{10}$ and $H = A_5$ we have that $H \leq \text{PSL}(2, 11)$ and so we get a bipartite graph on 22 vertices with valency 6. Thus we have row 5.

Finally, let $G = \text{PGL}(2, q)$, $q = p \equiv 11, 19, 21, 29 \pmod{40}$, $E = S_4$, $A = A_4$ and $H = A_5$. Then we get the bipartite graph in row 10 of Table 1. \square

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