

Homogeneous factorisations of graph products*

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Abstract

A homogeneous factorisation of a digraph Γ consists of a partition $\mathcal{P} = \{P_1, \dots, P_k\}$ of the arc set $A\Gamma$ and two vertex-transitive subgroups $M \leq G \leq \text{Aut}(\Gamma)$ such that M fixes each P_i setwise while G leaves \mathcal{P} invariant and permutes its parts transitively. Given two graphs Γ_1 and Γ_2 we consider several ways of taking a product of Γ_1 and Γ_2 to form a larger graph, namely the direct product, cartesian product and lexicographic product. We provide many constructions which enable us to lift homogeneous factorisations or certain arc partitions of Γ_1 and Γ_2 , to homogeneous factorisations of the various products.

1 Introduction

Let Γ be a digraph with vertex set $V\Gamma$ and arc set $A\Gamma$. Let $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ be a partition of $A\Gamma$ into k parts. Suppose that $M \leq G \leq \text{Aut}(\Gamma)$ such that M fixes each P_i setwise and acts transitively on the set of vertices of Γ , and G leaves \mathcal{P} invariant and acts transitively on the set \mathcal{P} . Then we call $(M, G, \Gamma, \mathcal{P})$ a *homogeneous factorisation of index k* . We refer to the parts P_i , and the subdigraphs of Γ induced by the P_i , as the *factors* of the homogeneous factorisation.

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Clearly, every vertex-transitive digraph Γ admits a homogeneous factorisation of index 1. We shall refer to such a homogeneous factorisation as *trivial*. The permutation group induced by the action of G on the set \mathcal{P} is denoted by $G^{\mathcal{P}}$.

An *isomorphic factorisation* of a graph is a partition of its arcs into spanning subgraphs which are pairwise isomorphic. Such factorisations have been extensively studied, see for example [4, 5]. Homogeneous factorisations form an interesting subclass of isomorphic factorisations. They were first introduced for complete graphs by the second and fourth authors in [8] to provide a framework for studying vertex-transitive, self-complementary graphs. Note that a vertex-transitive digraph is self-complementary if and only if it is a factor of an index two homogeneous factorisation of a complete graph. The concept of homogeneous factorisations was extended to arbitrary graphs and digraphs by the authors in [2]. Our early investigations uncovered several decompositions of homogeneous factorisations involving homogeneous factorisations, or arc partitions, of certain smaller related graphs. These suggested that there may be constructions of homogeneous factorisations of graph products that reflect the product structure. The purpose of this paper is to develop and study homogeneous factorisations for various common graph products focusing on general construction methods and their properties. Given two digraphs Γ_1 and Γ_2 , these are the *direct product* $\Gamma_1 \times \Gamma_2$, *Cartesian product* $\Gamma_1 \square \Gamma_2$ and *lexicographic product* $\Gamma_1[\Gamma_2]$. Recall that these three product graphs each have vertex set $V\Gamma_1 \times V\Gamma_2$ while adjacency for each of the graphs is defined as follows:

$\Gamma_1 \times \Gamma_2$: $((u_1, u_2), (v_1, v_2))$ is an arc if and only if (u_i, v_i) is an arc of Γ_i for $i = 1, 2$.

$\Gamma_1 \square \Gamma_2$: $((u_1, u_2), (v_1, v_2))$ is an arc if and only if either (u_1, v_1) is an arc of Γ_1 and $u_2 = v_2$, or $u_1 = v_1$ and (u_2, v_2) is an arc of Γ_2 .

$\Gamma_1[\Gamma_2]$: $((u_1, u_2), (v_1, v_2))$ is an arc if and only if either (u_1, v_1) is an arc of Γ_1 , or $u_1 = v_1$ and (u_2, v_2) is an arc of Γ_2 .

Since the direct and the Cartesian products are commutative and associative (up to isomorphism of digraphs), the *direct power* $\Gamma^{\times t}$ and the *Cartesian power* $\Gamma^{\square t}$ can be defined for every digraph Γ and integer $t \geq 2$ by letting $\Gamma^{\times t} = \Gamma_1 \times \dots \times \Gamma_t$ and $\Gamma^{\square t} = \Gamma_1 \square \dots \square \Gamma_t$ where $\Gamma_i = \Gamma$ for all $i \in \{1, \dots, t\}$.

The following theorem will be proved by providing explicit constructions for each of the three products (Construction 3.2 for the direct product, Construction 4.1 for the Cartesian product, and Construction 6.1 for the lexicographic product).

Theorem 1.1 *For $i = 1, 2$, let $\mathcal{F}_i = (M_i, G_i, \Gamma_i, \mathcal{P}_i)$ be a homogeneous factorisation of index k_i , and let Δ_i be a factor of \mathcal{F}_i . Then the following statements hold:*

- (i) *The direct product $\Gamma_1 \times \Gamma_2$ has a homogeneous factorisation of index $k_1 k_2$ with factors isomorphic to $\Delta_1 \times \Delta_2$.*

If, in addition, $k_1 = k_2 = k$ and $G_1^{\mathcal{P}_1}$ is permutationally isomorphic to $G_2^{\mathcal{P}_2}$, then

- (ii) the Cartesian product $\Gamma_1 \square \Gamma_2$ has a homogeneous factorisation of index k with factors isomorphic to $\Delta_1 \square \Delta_2$; and
- (iii) the lexicographic product $\Gamma_1[\Gamma_2]$ has a homogeneous factorisation of index k with factors isomorphic to $\Delta_1[\Delta_2]$.

Not all vertex-transitive graphs have nontrivial homogeneous factorisations. For example, K_6 has no homogeneous factorisations [2, Lemma 2.15]. Theorem 1.1 gives the existence of a nontrivial homogeneous factorisation of $\Gamma_1 \times \Gamma_2$ if either Γ_1 or Γ_2 have nontrivial homogeneous factorisations. We pose the following question.

Question 1.2 Suppose that Γ_1 and Γ_2 have no nontrivial homogeneous factorisations. Does $\Gamma_1 \times \Gamma_2$ have a nontrivial homogeneous factorisation?

In the case of Cartesian or lexicographic products, Theorem 1.1 only asserts the existence of a homogeneous factorisation of $\Gamma_1 \square \Gamma_2$ or $\Gamma_1[\Gamma_2]$ when Γ_1 and Γ_2 both have homogeneous factorisations of the same index and the two actions on parts are permutationally isomorphic. We pose the following question.

Question 1.3 Let Γ_1 and Γ_2 be digraphs and suppose that there do not exist homogeneous factorisations $(M_i, G_i, \Gamma_i, \mathcal{P}_i)$ of index k_i , for $i = 1, 2$ such that $k_1 = k_2$ and $G^{\mathcal{P}_1} \cong G^{\mathcal{P}_2}$. Do $\Gamma_1 \square \Gamma_2$ and $\Gamma_1[\Gamma_2]$ have homogeneous factorisations?

Given a digraph Γ , we are often interested in the set $\text{HomPartn}(\Gamma)$ of all partitions \mathcal{P} of $A\Gamma$ for which there exists $M, G \leq \text{Aut}(\Gamma)$ such that $(M, G, \Gamma, \mathcal{P})$ is a non-trivial homogeneous factorisation. Also, two homogeneous factorisations $\mathcal{F}_i = (M_i, G_i, \Gamma, \mathcal{P}_i)$, for $i = 1, 2$, will be called *parts-equal* if $\mathcal{P}_1 = \mathcal{P}_2$; we shall write $\mathcal{F}_1 \approx \mathcal{F}_2$ in this case.

We obtain a partial answer to Question 1.3 in the case of Cartesian products and $\Gamma_1 \cong \Gamma_2$. In particular, the next theorem shows that if Γ is a vertex transitive digraph and $t \geq 2$, then $\text{HomPartn}(\Gamma^{\square t})$ is non-empty even if $\text{HomPartn}(\Gamma)$ is empty. Given $A_1, A_2 \subseteq A\Gamma$, we can regard A_1 and A_2 as subdigraphs of Γ , each with vertex set $V\Gamma$, and we use $A_1 \square A_2$ to denote the digraph formed by the Cartesian product of these two subdigraphs.

Theorem 1.4 Let Γ be an N -vertex-transitive digraph, and let O_1, O_2, \dots, O_t be a partition of the arc set $A\Gamma$, where $t \geq 2$ and some O_i may be empty, such that each O_i is N -invariant. Then $\Gamma^{\square t}$ has a homogeneous factorisation of index t with factors isomorphic to $O_1 \square O_2 \square \dots \square O_t$.

Theorem 1.4 is proved by an explicit construction in Construction 4.8. This leads to the following question.

Question 1.5 Let Γ be a vertex-transitive digraph such that $\text{HomPartn}(\Gamma) = \emptyset$, and let $t \geq 2$. Does every partition $\mathcal{P} \in \text{HomPartn}(\Gamma^{\square t})$ arise from Construction 4.8?

Note that Question 1.3 remains open for lexicographic products, and for cartesian products when $\Gamma_1 \not\cong \Gamma_2$.

Theorems 1.1 and 1.4 have many interesting corollaries for certain families of graphs which we will now define. For a positive integer n , denote by K_n the complete graph with n vertices, and by $K_{n,n}$ the complete bipartite graph with n vertices in each part. Given two positive integers m and n we denote the complete multipartite graph with m parts of size n by $K_{m \times n}$. For a digraph Γ , denote by $n\Gamma$ the digraph consisting of n disjoint copies of Γ , and let $\bar{\Gamma}$ denote the complement of Γ . In particular, let \bar{K}_n denote the graph with n vertices and no arcs. Note that $K_m[\bar{K}_n] = K_{m \times n}$. For an integer $l \geq 2$, denote by \bar{C}_l the directed cycle of length l , and by C_l the undirected cycle of length l . If $l = 2$, then $\bar{C}_l = C_l = K_2$ is undirected. Note that $C_3 = K_3$. A *Hamming graph* $H(d, n)$, $d \geq 1$, $n \geq 2$, is the Cartesian product $K_n^{\square d}$. The graph $H(d, 2)$ is called a *d-dimensional cube*, also denoted by Q_d . A *grid graph* $G(d, n)$ and a *directed grid graph* $\vec{G}(d, n)$, $d \geq 1$, $n \geq 2$, are the Cartesian products $C_n^{\square d}$ and $\bar{C}_n^{\square d}$, respectively. Clearly, $G(d, 2) = H(d, 2) = Q_d$, and $G(d, 3) = H(d, 3)$.

We shall use the symbol $O(M)$ to denote the set of orders of the non-identity elements of a group M , and we let $\phi_M(l)$ denote the number of elements in M whose order is l . Note that if $M \cong \mathbb{Z}_n$, then $O(M)$ consists of all divisors of n other than 1, and for every $l \in O(M)$ we have that $\phi_M(l) = \phi(l)$, the Euler totient of l .

Theorems 1.1 and 1.4 have the following interesting consequence regarding homogeneous factorisations of Hamming graphs, the proof of which will be given in Section 4.

Theorem 1.6 *For positive integers $t, n \geq 2$, the following hold:*

- (i) *If $n = p^d$ for some prime p and positive integer d , then for every positive divisor s of t , the Hamming graph $H(t, n)$ admits a homogeneous factorisation of index $\gcd(2, p)^{\frac{(p^d-1)t}{2s}}$ with factors isomorphic to $p^{d(t-s)} G(s, p)$.*
- (ii) *If $n = rs$ for some $r, s \geq 2$, then the Hamming graph $H(t, n)$ admits a homogeneous factorisation of index t with factors isomorphic to $r^{t-1} s^{t-2} (K_s \square K_{r \times s})$.*
- (iii) *If M is an arbitrary group of order n and if $t \geq \frac{n-1+c}{2}$ where c is the number of elements of order 2 in M , then the Hamming graph $H(t, n)$ admits a homogeneous factorisation with factors isomorphic to*

$$\frac{n}{l_1^{d_1} \dots l_r^{d_r}} (G(d_1, l_1) \square \dots \square G(d_r, l_r)),$$

where $O(M) = \{l_1, \dots, l_r\}$, and $d_i = \gcd(2, l_i) \frac{\phi_M(l_i)}{2}$ for each $i \in \{1, \dots, r\}$.

We have the following corollary for grid graphs, the proof of which will be given in Example 4.7.

Theorem 1.7 *For any positive integers $d \geq 2$, $n \geq 3$, and a positive divisor s of d , the grid graph $G(d, n)$ admits a homogeneous factorisation of index $\frac{2d}{s}$ with*

factors isomorphic to $n^{d-s}\vec{G}(s, n)$. Moreover, if n is even, then the following hold:

- (i) $G(d, n)$ admits a homogeneous factorisation of index $\frac{2d}{s}$ with factors isomorphic to $\frac{n^d}{2^s}\mathbf{Q}_s$.
- (ii) If i, j, s_1, s_2 and m are positive integers such that $d = i + j$, $i = s_1m$ and $j = s_2m$, then there exists a homogeneous factorisation of $G(d, n)$ of index $\frac{2d}{s_1+s_2} = 2m$ with factors isomorphic to $\frac{n^{d-s_1}}{2^{s_2}}(\vec{G}(s_1, n)\square\mathbf{Q}_{s_2})$.

These examples suggest the following problem.

Problem 1.8 For a given vertex-transitive digraph Γ and $t \geq 2$, determine $\text{HomPartn}(\Gamma^{\square t})$. In particular, find $\text{HomPartn}(H(t, n))$ and $\text{HomPartn}(G(t, n))$.

In Section 5, we resolve this problem for $t = 2$ and $\Gamma = \mathbf{C}_n$, n odd. That is, we determine $\text{HomPartn}(G(2, 2m + 1))$ for any $m \geq 1$.

The article is organised as follows. In the remainder of this section, we recall a few graph- and group-theoretical notions that will be used throughout the paper. In Section 2, we provide a collection of factorisations of the graphs \mathbf{C}_n and \mathbf{K}_n which will be used in later sections to illustrate our results, and we determine all homogeneous factorisations of the cycles \mathbf{C}_n . In Sections 3, 4, and 6, we consider homogeneous factorisations of direct, Cartesian, and lexicographic products of graphs and digraphs, respectively, and prove the results stated in the Introduction. In Section 5 we examine homogeneous factorisations of grid graphs.

By a *digraph* Γ we shall mean an ordered pair $(V\Gamma, A\Gamma)$ of finite sets such that the *arc set* $A\Gamma$ is a subset of the set $V \times V \setminus \{(u, u) \mid u \in V\}$. If $A\Gamma$ is symmetric, that is, $(u, v) \in A\Gamma$ if and only if $(v, u) \in A\Gamma$, then Γ may be regarded as an undirected *graph* with *edge-set* $E\Gamma = \{\{u, v\} \mid (u, v) \in A\Gamma\}$. Similarly, if Σ is an undirected graph with edge set $E\Sigma$, then it may be regarded as a digraph with symmetric arc set $\{(u, v) \mid \{u, v\} \in E\Sigma\}$. If $\mathcal{F} = (M, G, \Sigma, \mathcal{P})$ is a homogeneous factorisation such that each factor $P_i \in \mathcal{P}$ is symmetric we call \mathcal{F} *symmetric* and \mathcal{P} may be regarded as a partition of the edge set of Σ .

The group of all permutations on a set V is denoted by $\text{Sym}(V)$. In particular, if $|V| = t$ for some integer t , then we write $S_t = \text{Sym}(V)$. If α is a permutation of V and $v \in V$, then we denote the image of v under α by v^α . Two permutation groups $G_1 \leq \text{Sym}(V_1)$, $G_2 \leq \text{Sym}(V_2)$ are *permutationally isomorphic* if there exists a bijection $f: V_1 \rightarrow V_2$ and a group isomorphism $\varphi: G_1 \rightarrow G_2$ such that for any $v \in V_1$ and $g \in G_1$ we have $f(v^g) = f(v)^{\varphi(g)}$.

If $G \leq \text{Sym}(V)$ is transitive (that is, for any $v \in V$, the *orbit* $v^G = \{v^g \mid g \in G\}$ is the full set V), then a partition \mathcal{P} of V is called G -invariant provided that $P^g = \{v^g \mid v \in P\}$ is an element of \mathcal{P} for every $P \in \mathcal{P}$ and $g \in G$. Clearly, the partitions of V into singletons and into V itself are always G -invariant, and are called *trivial*. If there exists a non-trivial G -invariant partition \mathcal{P} , then the permutation group G is said to be *imprimitive*, elements of \mathcal{P} are *blocks of imprimitivity*, and \mathcal{P} is called an *imprimitivity system* for G .

Transitive permutation groups which are not imprimitive, are called *primitive*. A permutation group $G \leq \text{Sym}(V)$ is *regular* if it is transitive and the *stabiliser* $G_v = \{g \mid g \in G, v^g = v\}$ of v in G is trivial for every $v \in V$.

In the forthcoming sections, we shall often refer to wreath products of permutation groups. Let V be a finite set of size $r \geq 2$, let $K \leq \text{Sym}(V)$, and let $H \leq S_t$ for some $t \geq 2$. Then the *wreath product* $K \wr H$ is the permutation group on the set V^t whose elements are $(t+1)$ -tuples $g = (g_1, \dots, g_t, h)$, $g_i \in K$, $h \in H$, acting on V^t according to the rule $(v_1, \dots, v_t)^g = (v_1^{g_1}, \dots, v_t^{g_t})$. The group $K \wr H$ (as an abstract group) is isomorphic to a semidirect product $K^t \rtimes H$, and acts primitively on V^t provided that K is primitive but not regular on V and that H is transitive (see for example [1, Lemma 2.7A]). Similarly, the *imprimitive wreath product* $K \text{ wr } H$ is the permutation group on the set $tV = V \times \{1, \dots, t\}$ whose elements are $(t+1)$ -tuples $g = (g_1, \dots, g_t, h)$, $g_i \in K$, $h \in H$, acting on tV according to the rule $(v, i)^g = (v^{g_i}, i^h)$ for each $(v, i) \in tV$. This group is clearly imprimitive with the imprimitivity system $\{V \times \{i\} \mid i \in \{1, \dots, t\}\}$. Note that $K \wr H$ and $K \text{ wr } H$ are isomorphic as abstract groups.

2 Examples of homogeneous factorisations

In this section, we present some homogeneous factorisations of cycles and complete graphs. These will then be used as inputs for the constructions described in the forthcoming sections, and will produce several interesting homogeneous factorisations of grid graphs and Hamming graphs.

We begin by constructing some homogeneous factorisations of the cycle $\Gamma = C_n$. Label the vertices of Γ with the elements of \mathbb{Z}_n such that vertex i is adjacent to $i-1$ and $i+1$, where addition is done modulo n . Let ρ be the “rotation” of the cycle such that $\rho : i \mapsto i+1 \pmod{n}$ for each $i \in \mathbb{Z}_n$ and let σ be the “reflection” $\sigma : i \mapsto -i \pmod{n}$. Then $\text{Aut}(\Gamma) = \langle \rho, \sigma \rangle \cong D_{2n}$.

Let $M_1 = \langle \rho \rangle = \mathbb{Z}_n$ and let $\mathcal{P}_1 = \{P_1, P_2\}$ be the decomposition of AF into two directed cycles. Then $\mathcal{F}_1 = (M_1, \text{Aut}(\Gamma), \Gamma, \mathcal{P}_1)$ is a homogeneous factorisation of index 2. This factorisation is demonstrated in Figure 1 in the case where $n = 6$.

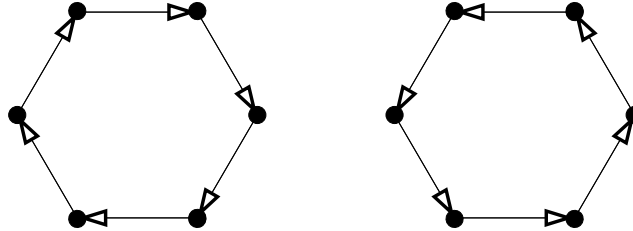


Figure 1: Homogeneous factorisation $(M_1, \text{Aut}(\Gamma), \Gamma, \mathcal{P}_1)$

If n is an even integer, then C_n admits another homogeneous factorisation: Let n be even and let $M_2 = \langle \rho^2, \sigma\rho \rangle \cong D_n$. Then $\sigma\rho$ fixes no vertices of Γ and M_2 has two symmetric orbits B_1 and B_2 on arcs as demonstrated in Figure 2 when $n = 6$. Each orbit consists of every second edge in the n -cycle. Letting $\mathcal{P}_2 = \{B_1, B_2\}$, we have that $\mathcal{F}_2 = (M_2, \text{Aut}(\Gamma), \Gamma, \mathcal{P}_2)$ is a homogeneous factorisation of index 2.

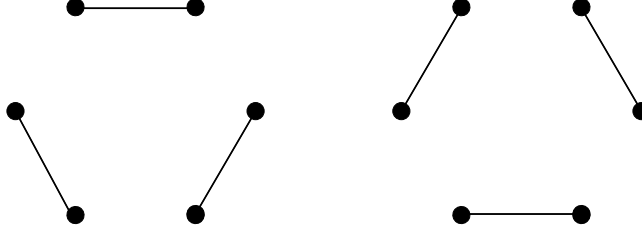


Figure 2: Homogeneous factorisation $(M_2, \text{Aut}(\Gamma), \Gamma, \mathcal{P}_2)$

We now prove that these are the only homogeneous factorisations of the cycle C_n .

Proposition 2.1 *Let $\Gamma = C_n$ be the cycle of length n and let $M_1, M_2, \mathcal{P}_1, \mathcal{P}_2$, and $\mathcal{F}_1, \mathcal{F}_2$ be as defined above.*

1. *If n is odd, then $\mathcal{F}_1 = (M_1, \text{Aut}(\Gamma), \Gamma, \mathcal{P}_1)$ is the only homogeneous factorisation of Γ .*
2. *If n is even, then the only homogeneous factorisations of Γ are \mathcal{F}_1 and $\mathcal{F}_2 = (M_2, \text{Aut}(\Gamma), \Gamma, \mathcal{P}_2)$.*

PROOF. Let $(M, G, \Gamma, \mathcal{P})$ be a homogeneous factorisation. Then M is a vertex-transitive subgroup of $\text{Aut}(\Gamma) \cong D_{2n}$, and since $M \neq G$ it follows that $G = \text{Aut}(\Gamma)$. If n is odd, then $M = M_1 = \mathbb{Z}_n$ is the only proper vertex-transitive subgroup of G , and hence $(M, G, \Gamma, \mathcal{P}) = (M_1, \text{Aut}(\Gamma), \Gamma, \mathcal{P}_1)$.

If n is even, then there are two possibilities for M , these being M_1 and M_2 . Each of these possibilities has two orbits on arcs and hence gives rise to one of the two homogeneous factorisations stated in the theorem. ■

We shall now construct various homogeneous factorisations of the complete graph K_n . Some of them are best described in terms of *Cayley digraphs*, the definition of which we now recall: For a group G and a subset $S \subseteq G$ not containing 1, the Cayley digraph $\text{Cay}(G, S)$ is the digraph whose vertices are the elements of G and (x, y) is an arc of $\text{Cay}(G, S)$ if and only if $xy^{-1} \in S$. Note that $\text{Cay}(G, S)$ is connected if and only if $\langle S \rangle = G$, and is an undirected graph if and only if $S = S^{-1} = \{s^{-1} \mid s \in S\}$. Since G acts as a vertex-transitive automorphism group of $\text{Cay}(G, S)$ by right multiplication, all Cayley digraphs are vertex-transitive.

Clearly, if G is a group of order n , then $K_n \cong \text{Cay}(G, G \setminus \{1\})$. This observation is used in the following simple lemma, which provides various partitions of AK_n , some of which yield homogeneous factorisations of K_n .

Lemma 2.2 *Let n be a positive integer, let M be a group of order n , let $\Gamma = \text{Cay}(M, M \setminus \{1\}) \cong K_n$, and for $x \in M \setminus \{1\}$, let $\Delta_x = \text{Cay}(M, \{x, x^{-1}\})$. If $\mathcal{P} = \{A\Delta_x \mid x \in M \setminus \{1\}\}$, then \mathcal{P} is a partition of $A\Gamma$, M fixes each element of \mathcal{P} setwise, and $|\mathcal{P}| = \frac{n-1+c}{2}$ where c is the number of elements of order 2 in M . Moreover, each member of \mathcal{P} is isomorphic to $\frac{n}{l}C_l$ for some $l \in O(M)$, and for each $l \in O(M)$, there exists precisely $\gcd(2, l) \frac{\phi_M(l)}{2}$ members of \mathcal{P} isomorphic to $\frac{n}{l}C_l$.*

PROOF. Let $x \in M \setminus \{1\}$, and let C be the connected component of Δ_x containing 1. Clearly, $C \cong C_{|VC|}$ (where we let $\vec{C}_2 = C_2 = K_2$). Note that VC consists of all the powers of x , and so $|VC| = l$ where l is the order of x in M . Also, since Δ_x is vertex-transitive, the connected components of Δ_x are isomorphic, implying that $\Delta_x \cong mC_l$ where $m = \frac{|V\Delta_x|}{|VC|} = \frac{n}{l}$. Of course, $\Delta_x = \Delta_y$ if and only if $y \in \{x, x^{-1}\}$. Hence, if $l \neq 2$, then there are precisely $\frac{\phi_M(l)}{2}$ elements of \mathcal{P} that are isomorphic to $\frac{n}{l}C_l$. On the other hand, if $l = 2$, then $x = x^{-1}$ for every element of order l , and so there are $\phi_M(l)$ elements of \mathcal{P} isomorphic to $\frac{n}{l}C_l$. The assertion about the size of \mathcal{P} now follows easily by a simple counting argument. ■

Of course, the partition \mathcal{P} in Lemma 2.2 above can only yield a homogeneous factorisation of K_n if all the graphs Δ_x are isomorphic, that is, if all the non-trivial elements of M have the same order. In the case where M is abelian, as the following lemma shows, this condition is also sufficient.

Lemma 2.3 *For a prime p and a positive integer d , let $M = \mathbb{Z}_p^d$ be the additive group of a finite field \mathbb{F} of order p^d , let $\Gamma = \text{Cay}(M, M \setminus \{0\})$, and let $G = \text{AGL}(1, p^d) \cong M \rtimes C_{p^d-1}$ be the group of affine transformations of \mathbb{F} , acting on the set M in the natural way. For each $x \in M \setminus \{0\}$ let $\Delta_x = \text{Cay}(M, \{x, -x\})$. Let $\mathcal{P} = \{A\Delta_x \mid x \in M \setminus \{0\}\}$. Then $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index $\gcd(2, p) \frac{p^d-1}{2}$ with factors isomorphic to $p^{d-1}C_p$.*

PROOF. By Lemma 2.2, \mathcal{P} is a partition of the arc set of Γ , each part of \mathcal{P} is isomorphic to $p^{d-1}C_p$, M is a vertex-transitive subgroup of $\text{Aut}(\Gamma)$, and acts on \mathcal{P} trivially. Also, since every element of $M \setminus \{0\}$ has order p , the size of \mathcal{P} is $\frac{p^d-1}{2}$ if p is odd, and is $p^d - 1$ if $p = 2$. Since $G = \text{AGL}(1, p^d)$ clearly preserves the partition \mathcal{P} , the result follows. ■

In the next example the following lemma from [2] will be needed.

Lemma 2.4 [2, Lemma 2.18] *Suppose $(M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation such that $G^{\mathcal{P}}$ is imprimitive with system of imprimitivity $\{B_1, \dots, B_t\}$ where $t \geq 2$. For each $i = 1, 2, \dots, t$, let $P'_i = \cup_{A \in B_i} A$, and let $\mathcal{P}' = \{P'_1, \dots, P'_t\}$. Then $(M, G, \Gamma, \mathcal{P}')$ is a homogeneous factorisation of index t .*

Let $(M, G, K_{p^d}, \mathcal{P})$ and k be as in Lemma 2.3. Since G/M is a cyclic group of order $p^d - 1$, the permutation group $G^{\mathcal{P}}$ is cyclic of order k . If $z \in G$ is a generator of $G^{\mathcal{P}}$ and m is a divisor of k , then the orbits of $\langle z^m \rangle$ form a $G^{\mathcal{P}}$ -invariant partition $\mathcal{Q} = \{P^{\langle z^m \rangle} \mid P \in \mathcal{P}\} = \{Q_1, \dots, Q_m\}$. Hence Lemma 2.4 can be used to construct further examples of homogeneous factorisations of complete graphs of prime power order. We use \mathbb{Q}_4^+ to denote the graph obtained from the 4-dimensional cube \mathbb{Q}_4 by adding an edge between each pair of antipodal vertices of \mathbb{Q}_4 .

Lemma 2.5 *The complete graph K_{16} admits a homogeneous factorisation of index 3 with factors isomorphic to the graph \mathbb{Q}_4^+ .*

PROOF. Let \mathbb{F} be the finite field of order 16, let $M \cong \mathbb{Z}_2^4$ be its additive group, and let $(M, G, K_{16}, \mathcal{P})$ be the homogeneous factorisation of index 15 obtained by Lemma 2.3 for $p = 2$ and $d = 4$. Then $G = M \rtimes \langle z \rangle$ where z is an element of order 15 corresponding to a generator of the multiplicative group of the field \mathbb{F} . Note that $\langle z^3 \rangle$ has three orbits in its action on $M \setminus \{0\}$, each of which has the form $S = a_1^{\langle z^3 \rangle} = \{a_1, a_2, a_3, a_4, a_1 + a_2 + a_3 + a_4\}$, where $\langle a_1, a_2, a_3, a_4 \rangle = M$. (Here $a_{i+1} = a_1 z^{3i}$ and $a_1 z^{12} = a_1 + a_1 z^3 + a_1 z^6 + a_1 z^9$.)

Since the kernel of the action of G on \mathcal{P} is M , the group $\langle z \rangle$ projects bijectively onto the group $G^{\mathcal{P}}$, and so we may identify $G^{\mathcal{P}}$ with $\langle z \rangle$. By the discussion preceding the lemma, we know that the orbits of $\langle z^3 \rangle$ on \mathcal{P} yield a homogeneous factorisation $(M, G, K_{16}, \mathcal{P}')$ of index 3. The factors of this factorisation correspond to the orbits of $\langle z^3 \rangle$ on $M \setminus \{0\}$ and are hence isomorphic to $\Delta = \text{Cay}(M, S)$. Since $S = \{a_1, a_2, a_3, a_4, a_1 + a_2 + a_3 + a_4\}$ where $\langle a_1, a_2, a_3, a_4 \rangle = M$ it follows that $\Delta \cong \mathbb{Q}_4^+$. ■

3 Direct products

For two digraphs Γ_1 and Γ_2 , let $\Sigma = \Gamma_1 \times \Gamma_2$ denote their *direct product*, that is, the digraph with vertex set $V\Gamma_1 \times V\Gamma_2$ such that (u_1, u_2) is adjacent to (v_1, v_2) if and only if (u_i, v_i) is an arc of Γ_i for $i = 1, 2$. Note that $A(\Gamma_1 \times \Gamma_2) = A\Gamma_1 \times A\Gamma_2$. By definition, if $G_i \leq \text{Aut}(\Gamma_i)$ for $i = 1, 2$, then $G = G_1 \times G_2$ acts “componentwise” on Σ as a subgroup of $\text{Aut}(\Sigma)$, that is, $G_1 \times G_2 \leq \text{Aut}(\Sigma)$. In particular, if both G_i are vertex-transitive, then so is G . If Γ_i has valency d_i , then $\Gamma_1 \times \Gamma_2$ has valency $d_1 d_2$. Note that the digraph $\Gamma_1 \times \Gamma_2$ might be disconnected even though both Γ_i are connected. For undirected connected graphs Γ_1, Γ_2 it is well known that $\Gamma_1 \times \Gamma_2$ is disconnected if and only if both Γ_1 and Γ_2 are bipartite. In this case, the graph $\Gamma_1 \times \Gamma_2$ decomposes into 2 connected components, and each component is bipartite (see for example [6, Theorem 5.29]). This result does not extend to arbitrary digraphs. For example, for the product of directed cycles, we have the following result.

Lemma 3.1 *For any positive integers $m, n \geq 3$, we have $\vec{C}_m \times \vec{C}_n \cong g\vec{C}_l$, where $l = \text{lcm}(m, n)$ and $g = mn/l$.*

PROOF. Note that $\vec{C}_m = \text{Cay}(\mathbb{Z}_m, \{1\})$ and $\vec{C}_n = \text{Cay}(\mathbb{Z}_n, \{1\})$. By the definition of the direct product, the vertex set of $\Gamma = \vec{C}_m \times \vec{C}_n$ is the set $\mathbb{Z}_m \times \mathbb{Z}_n$, and there is an arc from a vertex (u_1, v_1) to (u_2, v_2) whenever $u_2 - u_1 = 1 \in \mathbb{Z}_m$ and $v_2 - v_1 = 1 \in \mathbb{Z}_n$. In other words, $\Gamma = \text{Cay}(\mathbb{Z}_m \times \mathbb{Z}_n, \{(1, 1)\})$. Since the order of the element $(1, 1) \in \mathbb{Z}_m \times \mathbb{Z}_n$ is $\text{lcm}(m, n)$, it follows that a connected component of Γ is isomorphic to \vec{C}_l , and the result follows. \blacksquare

The following construction and lemma prove Part (i) of Theorem 1.1.

Construction 3.2 For $i = 1, 2$, let $\mathcal{F}_i = (M_i, G_i, \Gamma_i, \mathcal{P}_i)$ be a homogeneous factorisation of index k_i (k_i may equal 1 so \mathcal{F}_i may be trivial), and let $\mathcal{P}_i = \{P_{i1}, \dots, P_{ik_i}\}$. Let $\mathcal{F}_1 \times \mathcal{F}_2 = (M, G, \Gamma, \mathcal{P})$ where $M = M_1 \times M_2$, $G = G_1 \times G_2$, $\Gamma = \Gamma_1 \times \Gamma_2$, and $\mathcal{P} = \{P_{1j_1} \times P_{2j_2} \mid 1 \leq j_1 \leq k_1, 1 \leq j_2 \leq k_2\}$.

Lemma 3.3 For $i = 1, 2$, let $\mathcal{F}_i = (M_i, G_i, \Gamma_i, \mathcal{P}_i)$ be a homogeneous factorisation of index k_i and factors isomorphic to Δ_i . Then the 4-tuple $\mathcal{F}_1 \times \mathcal{F}_2 = (M, G, \Gamma, \mathcal{P})$ of Construction 3.2 is a homogeneous factorisation of index $k_1 k_2$ with factors isomorphic to $\Delta_1 \times \Delta_2$. Moreover, $G^{\mathcal{P}} \cong G_1^{\mathcal{P}_1} \times G_2^{\mathcal{P}_2}$, and $\mathcal{F}_1 \times \mathcal{F}_2$ is symmetric if and only if each \mathcal{F}_i is symmetric.

PROOF. Clearly, M fixes each $P \in \mathcal{P}$ setwise and acts transitively on VT . Also G acts transitively on \mathcal{P} and so $\mathcal{F}_1 \times \mathcal{F}_2$ is a homogeneous factorisation of index $k_1 k_2$. Moreover, each $P \in \mathcal{P}$ is the arc set of a digraph $P_{j_1} \times P_{j_2} \cong \Delta_1 \times \Delta_2$, where $1 \leq j_1 \leq k_1$ and $1 \leq j_2 \leq k_2$. Hence P is symmetric if and only if each P_{j_i} is symmetric. \blacksquare

We illustrate the above construction by proving the following result about homogeneous factorisations of the direct product of two cycles.

Proposition 3.4 Let $m, n \geq 3$ be integers and $\Gamma = C_m \times C_n$. Then the following hold:

- (i) Γ admits a homogeneous factorisation of index 4 with factors isomorphic to $g\vec{C}_l$ where $g = \text{gcd}(m, n)$ and $l = \text{lcm}(m, n)$.
- (ii) If n is even, then Γ also admits a homogeneous factorisation of index 4 with factors isomorphic to $s\vec{C}_t$ where $s = \text{gcd}(m, 2)$ and $t = \text{lcm}(m, 2)$.
- (iii) If both m and n are even, then Γ also admits a homogeneous factorisation of index 4 with factors isomorphic to $\frac{mn}{2}K_2$.

PROOF. Let $r \geq 3$ be an integer. All homogeneous factorisations of the cycle C_r were determined in Proposition 2.1. When r is odd it was seen that there is only one homogeneous factorisation $\mathcal{F}_1(r) = (C_r, D_{2r}, C_r, \mathcal{P}_1)$ and the factors are isomorphic to \vec{C}_r , while when r is even it was seen that there exists an additional homogeneous factorisations $\mathcal{F}_2(r) = (D_r, D_{2r}, C_r, \mathcal{P}_2)$, where the factors of \mathcal{P}_2 are isomorphic to $\frac{r}{2}K_2$. Each of these homogeneous factorisations has index 2. We can then use Construction 3.2 to obtain the following homogeneous factorisations of index 4:

- (i) For all $m, n \geq 3$, there is the factorisation $\mathcal{F}_1(m) \times \mathcal{F}_1(n)$, whose factors are isomorphic to $\vec{C}_m \times \vec{C}_n$, which is, by Lemma 3.1, isomorphic to $g\vec{C}_l$ where $l = \text{lcm}(m, n)$ and $g = \text{gcd}(m, n)$;
- (ii) If n is even, then there is the factorisation $\mathcal{F}_1(m) \times \mathcal{F}_2(n)$, whose factors are isomorphic to $\vec{C}_m \times (\frac{n}{2}\mathbf{K}_2)$. By Lemma 3.1, $\vec{C}_m \times \mathbf{K}_2 \cong \frac{2m}{t}\vec{C}_t$ where $t = \text{lcm}(m, 2)$, and so $\vec{C}_m \times (\frac{n}{2}\mathbf{K}_2) \cong \frac{mn}{t}\vec{C}_t$. Finally, observe that $\frac{mn}{t} = \frac{n \text{gcd}(m, 2)}{2}$;
- (iii) If m and n are both even, then there is the factorisation $\mathcal{F}_2(m) \times \mathcal{F}_2(n)$, whose factors are isomorphic to $(\frac{m}{2}\mathbf{K}_2) \times (\frac{n}{2}\mathbf{K}_2)$. Since $\mathbf{K}_2 \times \mathbf{K}_2 \cong 2\mathbf{K}_2$ we have that $(\frac{m}{2}\mathbf{K}_2) \times (\frac{n}{2}\mathbf{K}_2) \cong \frac{mn}{2}\mathbf{K}_2$. ■

3.1 Direct powers of a homogeneous factorisation

If $\mathcal{F} = (M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of index k , then a recursive application of Construction 3.2 yields a factorisation $(M^t, G^t, \Gamma^{\times t}, \mathcal{P}^{\times t})$ of index k^t . However, it is not this factorisation of the graph $\Gamma^{\times t}$ that we define as the t -th direct power of \mathcal{F} . The automorphism group of the digraph $\Gamma^{\times t}$ contains a subgroup $\vec{G} = G \wr S_t$, properly containing the group G^t and still preserving the partition $\mathcal{P}^{\times t}$. Hence, the group G^t in the above factorisation of $\Gamma^{\times t}$ can be substituted by \vec{G} .

Definition 3.5 Let $\mathcal{F} = (M, G, \Gamma, \mathcal{P})$ be a homogeneous factorisation and t a positive integer. Then the homogeneous factorisation $(M^t, G^t \wr S_t, \Gamma^{\times t}, \mathcal{P}^{\times t})$ described above is called the t -th direct power of \mathcal{F} and denoted by $\mathcal{F}^{\times t}$.

Among all vertex-transitive graphs, those admitting a primitive group of automorphisms often play an important role. Similarly, homogeneous factorisations $(M, G, \Gamma, \mathcal{P})$ for which G acts primitively on $V\Gamma$ deserve special attention (see [2, Question 1.4]). Such homogeneous factorisations will be called *vertex-primitive*. An interesting feature of the direct power of a homogeneous factorisation is that it preserves vertex-primitivity, and thus gives rise to infinite families of vertex-primitive homogeneous factorisations.

Proposition 3.6 Let $\mathcal{F} = (M, G, \Gamma, \mathcal{P})$ be a vertex-primitive homogeneous factorisation of index k and let t be a positive integer. Then $\mathcal{F}^{\times t} = (M^t, G \wr S_t, \Gamma^{\times t}, \mathcal{P}^{\times t})$ is a vertex-primitive homogeneous factorisation of index k^t .

PROOF. Since the product action of $K \wr H$ of two permutation groups K and H is primitive whenever K is primitive and nonregular, and H is transitive (see for example [1, Lemma 2.7A]), the statements of the proposition then follow easily. ■

Example 3.7 Let p be an odd prime, d a positive integer, and let $\mathcal{F} = (M, G, \mathbf{K}_{p^d}, \mathcal{P})$ be the homogeneous factorisation from Lemma 2.3. Since $G = \text{AGL}(1, p^d)$ acts

2-transitively on the vertices of K_{p^d} , \mathcal{F} is vertex-primitive. Hence by Proposition 3.6, for any $t \geq 2$, the graph $K_{p^d}^{\times t}$ admits a vertex-primitive factorisation $\mathcal{F}^{\times t} = (M^t, G \wr S_t, K_{p^d}^{\times t}, \mathcal{P}^{\times t})$ with factors isomorphic to the graph $p^{t(d-1)}C_p^{\times t}$.

3.2 Bipartite products

As mentioned before, if both Γ_1 and Γ_2 are bipartite undirected connected graphs, then $\Gamma_1 \times \Gamma_2$ is disconnected with two connected components, say Σ' and Σ'' . If both Γ_1, Γ_2 are vertex-transitive, then so is $\Gamma_1 \times \Gamma_2$, and therefore Σ' and Σ'' are isomorphic vertex-transitive bipartite graphs. This allows us to introduce the notion of a *bipartite product* of two connected vertex-transitive bipartite graphs:

Definition 3.8 For $i = 1, 2$, let Γ_i be a connected bipartite vertex-transitive graph, and let Σ', Σ'' be the connected components of the graph $\Gamma_1 \times \Gamma_2$. Since Σ' and Σ'' are isomorphic, we call a graph isomorphic to $\Sigma' \cong \Sigma''$ the *bipartite product* of Γ_1 and Γ_2 and denote it by $\Gamma_1 \times_{\text{bip}} \Gamma_2$.

Example 3.9 It is easy to see that the direct product of $K_{m,m}$ and $K_{n,n}$ is isomorphic to the graph $2K_{mn,mn}$. Hence $K_{m,m} \times_{\text{bip}} K_{n,n} = K_{mn,mn}$.

Suppose now that $\mathcal{F}_i = (M_i, G_i, \Gamma_i, \mathcal{P}_i)$, $i = 1, 2$, is a homogeneous factorisation of a connected undirected bipartite graph Γ_i of index k_i . Let Σ' and Σ'' be the connected components of $\Gamma_1 \times \Gamma_2$, and let $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 = (M, G, \Gamma, \mathcal{P})$. Then $\{V\Sigma', V\Sigma''\}$ is a G -invariant partition of $V\Gamma$. By [2, Lemma 2.16], the restriction $\mathcal{F}' = (M_{V\Sigma'}^V, G_{V\Sigma'}^V, \Sigma', \mathcal{P}')$ is a homogeneous factorisation of index $k_1 k_2$, where $\mathcal{P}' = \{P \cap A\Sigma' \mid P \in \mathcal{P}\}$. Since $\Sigma' \cong \Gamma_1 \times_{\text{bip}} \Gamma_2$, this enables us to define the following:

Definition 3.10 Let $\mathcal{F}_1, \mathcal{F}_2$ and \mathcal{F}' be as in the paragraph above. Then the homogeneous factorisation \mathcal{F}' of the graph $\Gamma_1 \times_{\text{bip}} \Gamma_2$ is called the *bipartite product* of \mathcal{F}_1 and \mathcal{F}_2 , and is denoted by $\mathcal{F}_1 \times_{\text{bip}} \mathcal{F}_2$.

In view of Example 3.9 and the above construction, some homogeneous factorisations of $K_{mn,mn}$ can be produced as the bipartite product of homogeneous factorisations of $K_{m,m}$ and $K_{n,n}$. Various homogeneous factorisations of the complete bipartite graphs are given in [3]. With them as inputs into the above construction, we can construct many interesting examples.

Example 3.11 By [3, Examples 5.2-5.3], $K_{60,60}$ has a homogeneous factorisation $\mathcal{F}_1 = (S_5, S_5 \rtimes A_5, K_{60,60}, \mathcal{P}_1)$ of index 10 and $K_{360,360}$ has a homogeneous factorisation $\mathcal{F}_2 = (M_{10}, M_{10} \rtimes A_6, K_{360,360}, \mathcal{P}_2)$ of index 10. Thus, the graph $K_{2^5 3^3 5^2, 2^5 3^3 5^2}$ has a homogeneous factorisation $\mathcal{F}_1 \times_{\text{bip}} \mathcal{F}_2$ of index 100.

In the rest of the section, we shall concentrate on the bipartite powers of a bipartite graph, which we now define: Let Γ be a connected vertex-transitive

bipartite graph, and let t be a positive integer. Then the t -th bipartite power $\Gamma^{\times_{\text{bip}} t}$ of Γ is defined recursively by

$$\Gamma^{\times_{\text{bip}} 1} = \Gamma \quad \text{and} \quad \Gamma^{\times_{\text{bip}} (i+1)} = \Gamma^{\times_{\text{bip}} i} \times_{\text{bip}} \Gamma \quad \text{for } 1 \leq i \leq t-1.$$

Note that $\Gamma^{\times_{\text{bip}} t}$ is isomorphic to a connected component of $\Gamma^{\times t}$.

If $\mathcal{F} = (M, G, \Gamma, \mathcal{P})$ is a homogeneous factorisation of a connected vertex-transitive bipartite graph Γ , t a positive integer, and $\Sigma \cong \Gamma^{\times_{\text{bip}} t}$ a connected component of the graph $\Gamma^{\times t}$, then the factorisation $\mathcal{F}^{\times t} = (M^t, G \wr S_t, \Gamma^{\times t}, \mathcal{P}^{\times t})$ induces a homogeneous factorisation $\mathcal{F}' = (M', G', \Sigma, \mathcal{P}')$ where M' and G' are restrictions of the setwise stabilisers of $V\Sigma$ in M^t and $G \wr S_t$, respectively, and $\mathcal{P}' = \{P \cap A\Sigma \mid P \in \mathcal{P}^{\times t}\}$. This homogeneous factorisation \mathcal{F}' of the graph $\Gamma^{\times_{\text{bip}} t}$ will be called the t -th bipartite power of \mathcal{F} and denoted by $\mathcal{F}^{\times_{\text{bip}} t}$. Note that according to this definition, we have that $\mathcal{F} \times_{\text{bip}} \mathcal{F} \approx \mathcal{F}^{\times_{\text{bip}} 2}$, as defined after Theorem 1.1, but $\mathcal{F} \times_{\text{bip}} \mathcal{F} \neq \mathcal{F}^{\times_{\text{bip}} 2}$.

The same role that vertex-primitive homogeneous factorisations play for non-bipartite graphs, *vertex-biprimitive* homogeneous factorisations play in the case of bipartite graphs. (A homogeneous factorisation $(M, G, \Gamma, \mathcal{P})$ of a bipartite digraph Γ is said to be *vertex-biprimitive* if G is imprimitive and all nontrivial block systems have precisely two blocks.) We leave it to the reader to prove the following proposition.

Proposition 3.12 *If $\mathcal{F} = (M, G, \Gamma, \mathcal{P})$ is a vertex-biprimitive homogeneous factorisation of a connected bipartite digraph Γ with index k , then $\mathcal{F}^{\times_{\text{bip}} t}$ is a vertex-biprimitive homogeneous factorisation with index k^t .*

This provides a method for constructing vertex-biprimitive homogeneous factorisations (refer to [2, Question 1.4]).

3.3 Covers

In this section we lift homogeneous factorisations of a digraph Γ to homogeneous factorisations of covers of Γ . Construction 3.2 enabled us to lift homogeneous factorisations of Γ to homogeneous factorisations of the standard double cover of Γ . We now move on to studying general covers.

For a digraph Γ and $v \in V\Gamma$, $\Gamma^+(v) = \{u \in V\Gamma \mid (v, u) \in A\Gamma\}$ and $\Gamma^-(v) = \{u \in V\Gamma \mid (u, v) \in A\Gamma\}$. A *covering projection* from a digraph $\tilde{\Gamma}$ to another digraph Γ is a digraph epimorphism (that is, a surjective function $\varphi: V\tilde{\Gamma} \rightarrow V\Gamma$ mapping $A\tilde{\Gamma}$ onto $A\Gamma$) such that, for each $v \in V\Gamma$ and each $\tilde{v} \in \varphi^{-1}(v)$, φ maps $\tilde{\Gamma}^+(\tilde{v})$ and $\tilde{\Gamma}^-(\tilde{v})$ bijectively onto $\Gamma^+(v)$ and $\Gamma^-(v)$, respectively. Further, a group $G \leq \text{Aut}(\Gamma)$ is said to *lift along* φ if for every $g \in G$ there exists $\tilde{g} \in \text{Aut}(\tilde{\Gamma})$ such that $\varphi \circ \tilde{g} = g \circ \varphi$. If \tilde{G} is the set of all \tilde{g} that satisfy $\varphi \circ \tilde{g} = g \circ \varphi$ for some $g \in G$, then we say that G *lifts (along φ) to \tilde{G}* . A *regular covering projection* is a covering projection φ of connected digraphs where the trivial group $1 \leq \text{Aut}(\Gamma)$ lifts along φ to a group which is transitive on the fibre $\varphi^{-1}(u)$ for some vertex u . In the case of regular covering projections of

graphs, there is a rather nice theory which deals with the question of which automorphisms do lift, see for example [9].

We now give the following construction. Its proof is straightforward and is omitted.

Construction 3.13 *Let $(M, G, \Gamma, \mathcal{P})$ be a symmetric homogeneous factorisation of index k and $\wp : \tilde{\Gamma} \rightarrow \Gamma$ a covering projection such that G lifts along \wp . Further, let \tilde{G} and \tilde{M} denote the lifts of G and M , respectively. For $P \in \mathcal{P}$ let*

$$\tilde{P} = \{(\tilde{u}, \tilde{v}) \mid (\tilde{u}, \tilde{v}) \in A\tilde{\Gamma}, (\wp(\tilde{u}), \wp(\tilde{v})) \in P\},$$

and let $\tilde{\mathcal{P}} = \{\tilde{P} \mid P \in \mathcal{P}\}$. Then $(\tilde{M}, \tilde{G}, \tilde{\Gamma}, \tilde{\mathcal{P}})$ is a symmetric homogeneous factorisation of index k . Moreover, $\tilde{G}^{\tilde{\mathcal{P}}} \cong G^{\mathcal{P}}$.

4 Cartesian products

For sets V_1, V_2 , if $A_i \subseteq V_i \times V_i$ for $i = 1, 2$, we denote by $A_1 \square A_2$ the set of all pairs $((u_1, v_1), (u_2, v_2))$ such that either $u_1 = v_1$ and $(u_2, v_2) \in A_2$, or $u_2 = v_2$ and $(u_1, v_1) \in A_1$. The *Cartesian product* of digraphs Γ_1, Γ_2 , denoted by $\Gamma_1 \square \Gamma_2$, is the digraph with vertex set $V\Gamma_1 \times V\Gamma_2$ and arc set $A\Gamma_1 \square A\Gamma_2$.

Suppose that Γ_i with $i = 1, 2$ has a homogeneous factorisation $(M_i, G_i, \Gamma_i, \mathcal{P}_i)$ of index k with factors isomorphic to Δ_i , such that $G_1^{\mathcal{P}_1}$ is permutationally isomorphic to $G_2^{\mathcal{P}_2}$. Then the following construction produces a homogeneous factorisation of $\Gamma_1 \square \Gamma_2$ of index k whose factors are isomorphic to $\Delta_1 \times \Delta_2$. Note that this proves Part (ii) of Theorem 1.1.

Construction 4.1 For $i = 1, 2$, let $\mathcal{F}_i = (M_i, G_i, \Gamma_i, \mathcal{P}_i)$ be a homogeneous factorisation of index k . Let $\mathcal{P}_i = \{P_{i1}, \dots, P_{ik}\}$, where $i = 1, 2$. Assume that $G_1^{\mathcal{P}_1}$ is permutationally isomorphic to $G_2^{\mathcal{P}_2}$. Let $\varphi_i : G_i \rightarrow H$ be a homomorphism, where $H \leq S_k$ such that $H \cong G_1^{\mathcal{P}_1} \cong G_2^{\mathcal{P}_2}$. Let $\mathcal{F}_1 \square \mathcal{F}_2 = (M, G, \Gamma, \mathcal{P})$, where

$$\begin{aligned} M &= M_1 \times M_2, \\ G &= \{(g_1, g_2) \in G_1 \times G_2 \mid \varphi_1(g_1) = \varphi_2(g_2)\}, \\ \Gamma &= \Gamma_1 \square \Gamma_2, \\ \mathcal{P} &= \{P_{1j} \square P_{2j} \mid 1 \leq j \leq k\}. \end{aligned}$$

Lemma 4.2 *The 4-tuple $\mathcal{F}_1 \square \mathcal{F}_2 = (M, G, \Gamma, \mathcal{P})$ given by Construction 4.1 is a homogeneous factorisation of index k with factors $P_{1j} \square P_{2j}$ for $1 \leq j \leq k$. Furthermore, $\mathcal{F}_1 \square \mathcal{F}_2$ is symmetric if and only if both \mathcal{F}_1 and \mathcal{F}_2 are symmetric.*

PROOF. Since, for each i , M_i is contained in the kernel of φ_i , we have that $M \leq G$. Thus, M fixes setwise each member of \mathcal{P} . By definition, G is transitive on \mathcal{P} . Hence $\mathcal{F}_1 \square \mathcal{F}_2 = (M, G, \Sigma, \mathcal{P})$ is a homogeneous factorisation of index k . ■

The homogeneous factorisation $\mathcal{F}_1 \square \mathcal{F}_2$, constructed in Construction 4.1, is called the *Cartesian product* of \mathcal{F}_1 and \mathcal{F}_2 . Since the Cartesian product of

graphs is associative, for homogeneous factorisations $\mathcal{F}_i = (M_i, G_i, \Gamma_i, \mathcal{P}_i)$ where $1 \leq i \leq t$, we can recursively define the Cartesian product of the \mathcal{F}_i :

$$\mathcal{F}_1 \square \dots \square \mathcal{F}_{t-1} \square \mathcal{F}_t = (\mathcal{F}_1 \square \dots \square \mathcal{F}_{t-1}) \square \mathcal{F}_t,$$

which is a homogeneous factorisation of $\Gamma_1 \square \dots \square \Gamma_t$. If $\mathcal{F}_1 = \dots = \mathcal{F}_t = \mathcal{F}$, then we have the t -th Cartesian power

$$\mathcal{F}^{\square t} = \mathcal{F}_1 \square \dots \square \mathcal{F}_t.$$

Example 4.3 Construction 4.1 can be used to produce many interesting homogeneous factorisations of graphs which are isomorphic to Cartesian products of smaller graphs. Here are a few examples:

- (i) *Given vertex-transitive self-complementary graphs $\Delta_1, \dots, \Delta_t$ of respective orders n_1, \dots, n_t , we can construct a symmetric index-2 homogeneous factorisation of the graph $\mathsf{K}_{n_1} \square \dots \square \mathsf{K}_{n_t}$, with factors isomorphic to $\Delta_1 \square \dots \square \Delta_t$.*

Indeed, for each $i = 1, \dots, t$, let $M_i = \text{Aut}(\Delta_i)$, σ_i be an isomorphism between Δ_i and its complement $\overline{\Delta}_i$, let $G_i = \langle M_i, \sigma_i \rangle$ and $\mathcal{P}_i = \{A\Delta_i, A\overline{\Delta}_i\}$. Then $\mathcal{F}_i = (M_i, G_i, \mathsf{K}_{n_i}, \mathcal{P}_i)$ is a homogeneous factorisation of index 2 with $G_i^{\mathcal{P}_i} = S_2$ (the symmetric group of degree 2). By Lemma 4.2 and the discussion following it, $\mathcal{F}_1 \square \dots \square \mathcal{F}_t$ is a symmetric homogeneous factorisation of index 2 with each factor isomorphic to $\Delta_1 \square \dots \square \Delta_t$.

- (ii) *For every positive integer t , a homogeneous factorisation $\mathcal{F} = (M, G, \mathsf{K}_n, \mathcal{P})$ of a complete graph K_n with index k and factors isomorphic to Δ induces a homogeneous factorisation $\mathcal{F}^{\square t}$ of the Hamming graph $H(t, n) = \mathsf{K}_n^{\square t}$ with index k and factors isomorphic to $\Delta^{\square t}$.*

For instance, by Lemma 2.5, there exists a homogeneous factorisation of the complete graph K_{16} of index 3 and factors isomorphic to Q_4^+ . Hence, for every positive integer t , the Hamming graph $H(t, 16)$ admits an index-3 homogeneous factorisation whose factors are isomorphic to the graph $(\mathsf{Q}_4^+)^{\square t}$.

- (iii) *For all primes p and positive integers d and t , the Hamming graph $H(t, p^d)$ admits a homogeneous factorisation of index $\gcd(2, p) \frac{p^d - 1}{2}$ with factors isomorphic to $p^{t(d-1)} G(t, p)$.*

This follows by applying Part (ii) of this example with the homogeneous factorisations described in Lemma 2.3 as input.

- (iv) *For any positive integers t, n , with $n \geq 3$, the grid graph $G(t, n) = \mathsf{C}_n^{\square t}$ admits an index-2 homogeneous factorisation with factors isomorphic to the directed grid graph $\vec{G}(t, n)$. Moreover, if n is even, then for any integers $i, j \geq 0$ such that $i + j = t$, there exists an index-2 homogeneous factorisation of $G(t, n)$ with factors isomorphic to $\frac{n^j}{2^j} (\vec{G}(i, n) \square \mathsf{Q}_j)$.*

Indeed, the first factorisation is obtained by applying Construction 4.1 to the factorisation $\mathcal{F}_1 = (M_1, D_{2n}, C_n, \mathcal{P}_1)$ from Proposition 2.1. It was also seen there that if n is even, there exists another index-2 factorisation $\mathcal{F}_2 = (M_2, D_{2n}, C_n, \mathcal{P}_2)$ with factors isomorphic to $\frac{n}{2}K_2$. Hence the graph $G(t, n)$ has a factorisation $\mathcal{F} = \mathcal{F}_1^{\square i} \square \mathcal{F}_2^{\square j}$ of index 2 with factors isomorphic to the graph $\vec{C}_n^{\square i} \square (\frac{n}{2}K_2)^{\square j} \cong (\frac{n}{2})^j (\vec{G}(i, n) \square Q_j)$.

In the Cartesian product $\mathcal{F}_1 \square \mathcal{F}_2$, each factor is isomorphic to $\Delta_1 \square \Delta_2$. If $\mathcal{F}_1 = \mathcal{F}_2$, then $\Delta_1 \cong \Delta_2$, and $\Delta_1 \square \Delta_2$ can be further homogeneously factorised as two factors: ‘horizontal’ copies of Δ_1 and ‘vertical’ copies of Δ_2 . This observation is made explicit in the following construction.

Construction 4.4 Let $\mathcal{F} = (M, G, \Gamma, \mathcal{P})$, where $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$, be a homogeneous factorisation of index k and let t be a positive integer. For $i \in \{1, \dots, k\}$, let

$$\begin{aligned} Q_{1,i} &= P_i \square \bar{K}_n \square \dots \square \bar{K}_n \leq \Gamma^{\square t}, \\ Q_{2,i} &= \bar{K}_n \square P_i \square \dots \square \bar{K}_n \leq \Gamma^{\square t}, \\ &\vdots \\ Q_{t,i} &= \bar{K}_n \square \bar{K}_n \square \dots \square P_i \leq \Gamma^{\square t}, \end{aligned}$$

let $\mathcal{P}^{\parallel t}$ be the collection of all the $Q_{j,i}$, and let $\bar{M} = M^t \leq \text{Aut}(\Gamma^{\square t})$. Finally, let $\bar{G} = G \wr S_t$ and let $\mathcal{F}^{\parallel t} = (\bar{M}, \bar{G}, \Gamma^{\square t}, \mathcal{P}^{\parallel t})$.

Lemma 4.5 *The 4-tuple $\mathcal{F}^{\parallel t}$ defined in Construction 4.4 is a homogeneous factorisation of index kt . The factors of $\mathcal{F}^{\parallel t}$ are isomorphic to the digraph $n^{t-1}\Delta$ where Δ is a factor of \mathcal{F} . In particular, $\mathcal{F}^{\parallel t}$ is symmetric if and only if \mathcal{F} is symmetric.*

PROOF. Since $X \square \bar{K}_m \cong mX$ for every digraph X and integer m , the digraphs $Q_{j,i}$ are indeed all isomorphic to $n^{t-1}\Delta$. By definition, \bar{M} acts transitively on $V\Gamma$ and fixes each $Q_{j,i} \in \mathcal{P}^{\parallel t}$ setwise. Moreover, $\bar{G} = G \wr S_t$ preserves the partition $\mathcal{P}^{\parallel t}$ and so $\mathcal{F}^{\parallel t}$ is a homogeneous factorisation. The result then follows. \blacksquare

Remark 4.6 1. We can use the homogeneous factorisation $\mathcal{F}^{\parallel r} = (M^r, G \wr S_r, \Gamma^{\square r}, \mathcal{P}^{\parallel r})$ as input for Construction 4.4 to obtain a new homogeneous factorisation $(\mathcal{F}^{\parallel r})^{\parallel s} = (M^{rs}, (G \wr S_r) \wr S_s, \Gamma^{\square rs}, (\mathcal{P}^{\parallel r})^{\parallel s})$ of index kr with factors isomorphic to $n^{rs-1}\Delta$. Note also that $(\mathcal{F}^{\parallel r})^{\parallel s} \approx (\mathcal{F}^{\parallel s})^{\parallel r}$ but equality does not hold as the groups involved are different.

2. We can also use the homogeneous factorisation $\mathcal{F}^{\square s}$ obtained from Construction 4.1 as input in Construction 4.4 to obtain a homogeneous factorisation $(\mathcal{F}^{\square s})^{\parallel r} = (M^{rs}, G_0 \wr S_r, \Gamma^{\square rs}, (\mathcal{P}^{\square s})^{\parallel r})$ of index kr and with factors isomorphic to $n^{rs-s}\Delta^{\square s}$. Reversing the order gives a homogeneous factorisation $(\mathcal{F}^{\parallel r})^{\square s}$ with factors isomorphic to $n^{rs-s}\Delta^{\square s}$ however, $(\mathcal{P}^{\square s})^{\parallel r} \neq (\mathcal{P}^{\parallel r})^{\square s}$.

Example 4.7 Since the Hamming graphs and the grid graphs are all Cartesian powers, Constructions 4.1 and 4.4 provide further examples of homogeneous factorisations of these graphs.

- (i) If Δ is a factor of some homogeneous factorisation \mathcal{F} of K_n of index k and $t = rs$, then $(\mathcal{F}^{\square s})^{\parallel r}$ is a homogeneous factorisation of the Hamming graph $H(t, n)$ of index kr and with factors isomorphic to $n^{t-s}\Delta^{\square s}$. In particular, in view of Lemma 2.3, we have the following generalisation of Example 4.3(iii). Note that this proves Part (i) of Theorem 1.6.

For any prime p and positive integer d , the Hamming graph $H(rs, p^d)$ admits a homogeneous factorisation of index $\gcd(2, p) \frac{(p^d-1)r}{2}$ with factors isomorphic to $p^{d(t-s)}G(s, p)$. In particular, $H(r, p^d)$ admits a homogeneous factorisation with factors isomorphic to $p^{d(t-1)}C_p$.

- (ii) Each cycle C_n has a homogeneous factorisation of index 2 with factors isomorphic to \vec{C}_n (see Proposition 2.1). Hence for $d = rs$, applying Construction 4.1 and then Construction 4.4 yields a homogeneous factorisation of the grid graph $G(d, n)$ of index $2r$ with factors isomorphic to $n^{d-s}\vec{C}_n^{\square s} = n^{d-s}\vec{G}(s, n)$. If n is even, then C_n admits also a homogeneous factorisation with factors isomorphic to $\frac{n}{2}K_2$. Hence $G(d, n)$ admits a homogeneous factorisation of index $2r$ with factors isomorphic to $n^{d-s}((\frac{n}{2}K_2)^{\square s}) \cong \frac{n^d}{2^s}K_2^{\square s} = \frac{n^d}{2^s}Q_s$. Note that this proves the first claim and Part (i) of Theorem 1.7.

- (iii) Suppose now that n is even, that $d = i + j$ for some positive integers i, j , and suppose that $i = s_1m, j = s_2m$ for some positive integers s_1, s_2 and m . Let $\mathcal{F}_1 = (M_1, G_1, G(i, n), \mathcal{P}_1)$ and $\mathcal{F}_2 = (M_2, G_2, G(j, n), \mathcal{P}_2)$ be the two homogeneous factorisations of index $\frac{2i}{s_1} = \frac{2j}{s_2} = 2m$ obtained by Part (ii) of this example. Then the factors of \mathcal{F}_1 and \mathcal{F}_2 are isomorphic to $\Delta_1 = n^{i-s_1}\vec{G}(s_1, n)$ and $\Delta_2 = \frac{n^j}{2^{s_2}}Q_{s_2}$, respectively. Moreover, both $G_1^{\mathcal{P}_1}$ and $G_2^{\mathcal{P}_2}$ are isomorphic to the action of the group C_2 wr S_m on a set of cardinality $2m$. Hence we may apply Construction 4.1 to obtain the homogeneous factorisation $\mathcal{F}_1 \square \mathcal{F}_2$ of the graph $G(i, n) \square G(j, n) \cong G(d, n)$, of index $2m$ with factors isomorphic to $n^{i-s_1}\vec{G}(s_1, n) \square \frac{n^j}{2^{s_2}}Q_{s_2} \cong \frac{n^{d-s_1}}{2^{s_2}}(\vec{G}(s_1, n) \square Q_{s_2})$. Note that this proves Part (ii) of Theorem 1.7.

Constructions 4.1 and 4.4 produce homogeneous factorisations as Cartesian products of smaller homogeneous factorisations. The following construction shows that any Cartesian power of any vertex-transitive digraph has a homogeneous factorisation. Note that Lemma 4.9 below proves Theorem 1.4.

Construction 4.8 Let Γ be a digraph and let $N \leq \text{Aut}(\Gamma)$ be transitive on $V\Gamma$. Let $\{O_1, O_2, \dots, O_t\}$ be a partition of $A\Gamma$ (some O_i may be empty) such that each O_i is N -invariant. Let $\Sigma = \Gamma^{\square t}$, and let τ be the element of $\text{Aut}(\Sigma)$

such that $\tau : (v_1, v_2, \dots, v_t) \mapsto (v_t, v_1, v_2, \dots, v_{t-1})$. Let

$$\begin{aligned} M &= N^t, \\ G &= \langle M, \tau \rangle = N \wr \langle \tau \rangle = N^t \rtimes \langle \tau \rangle, \\ \mathcal{P} &= \{(O_1 \square O_2 \square \dots \square O_t)^{\tau^i} \mid 1 \leq i \leq t\}. \end{aligned}$$

Lemma 4.9 *The 4-tuple $(M, G, \Sigma, \mathcal{P})$ given by Construction 4.8 is a homogeneous factorisation of index t . Furthermore, $(M, G, \Sigma, \mathcal{P})$ is symmetric if and only if each O_i is symmetric.*

PROOF. By definition, M fixes each P_i setwise, \mathcal{P} is a G -invariant partition of the arcs of Σ and G acts transitively on \mathcal{P} . Hence $(M, G, \Sigma, \mathcal{P})$ is a homogeneous factorisation. Moreover, each P_i is symmetric if and only if each O_i is symmetric and so the result follows. \blacksquare

Example 4.10 Let Γ be an N -vertex-transitive graph of order n .

- (i) Let $\bar{\Gamma}$ be the complement of Γ in K_n . Then $\{A\Gamma, A\bar{\Gamma}\}$ is a partition of the arc set of K_n and each of $A\Gamma, A\bar{\Gamma}$ is N -invariant, so Construction 4.8 yields a homogeneous factorisation of $K_n \square K_n = H(2, n)$ of index 2 with factors isomorphic to $\Gamma \square \bar{\Gamma}$. For instance, if we let Γ be the Petersen graph P , then this yields a homogeneous factorisation of $H(2, 10)$ of index 2 with factors isomorphic to $P \square \bar{P}$.
- (ii) In Construction 4.8, if we let $O_1 = A\Gamma$ and $O_i = \emptyset$ for $i \geq 2$, we obtain a homogeneous factorisation of $\Gamma^{\square t}$ with index t and factors isomorphic to $n\Gamma$. This is the same partition of the arc set of $\Gamma^{\square t}$ given by the homogeneous factorisation $\mathcal{F}^{\square t}$ obtained by Construction 4.4 by starting with the trivial homogeneous factorisation \mathcal{F} of Γ (that is, where \mathcal{P} has only one part).

Equipped with Construction 4.8, we can now prove Theorem 1.6.

PROOF OF THEOREM 1.6. Part (i) follows directly from Example 4.7(i). To prove Part (ii), let $N = S_s^r \rtimes S_r \cong S_s \text{ wr } S_r$ be the imprimitive wreath product of the symmetric groups S_s and S_r in its natural imprimitive action on a set V of cardinality $sr = n$. Further, let $\Gamma \cong K_n$ be the complete graph on the set V . Then the arc set AK_n decomposes into two N -orbits O_1 and O_2 , with O_1 containing all the arcs having both end-points inside the same block of imprimitivity of N , and O_2 consisting of the arcs whose end-points belong to different blocks of N . Note that $O_1 \cong rK_s$ and $O_2 \cong K_{r \times s}$. Now, let $O_i = \bar{K}_n$ for $3 \leq i \leq t$, and apply Construction 4.8 to $\{O_1, \dots, O_t\}$. Part(ii) then follows by Lemma 4.9.

We now prove Part (iii). Let $N = M$, for each $x \in M \setminus \{1\}$ let $\Delta_x = \text{Cay}(M, \{x, x^{-1}\})$ and let $\mathcal{P} = \{A\Delta_x \mid x \in M \setminus \{1\}\}$. Then by Lemma 2.2, \mathcal{P} is a partition of the arc set of $\Gamma = \text{Cay}(M, M \setminus \{1\}) \cong K_n$ with each part N -invariant, and $|\mathcal{P}| = \frac{n-1+c}{2} = m$ where c is the number of elements of order 2 in M . Extend

\mathcal{P} to a partition $\mathcal{P}^* = \{O_1, \dots, O_d\}$ with $\mathcal{P} = \{O_1, \dots, O_m\}$ and $O_i = \emptyset$ for all $i, m+1 \leq i \leq d$. With this partition of $A\Gamma$ as the input, Construction 4.8 yields a homogeneous factorisation of $H(t, n)$ with factors isomorphic to $\Delta \cong O_1 \square \dots \square O_m \square \overline{K}_{n^{d-m}} \cong n^{d-m} (O_1 \square \dots \square O_m)$. By Lemma 2.2, each graph $O_j, j \leq m$, is isomorphic to $\frac{n}{l_i} C_{l_i}$ for some $l_i \in O(M) = \{l_1, \dots, l_r\}$, and each $\frac{n}{l_i} C_{l_i}$ occurs among the O_j exactly $\gcd(2, l_i) \frac{\phi_M(l_i)}{2} = d_i$ times. For $i \in \{1, \dots, r\}$, let Σ_i be the Cartesian product of all the d_i graphs O_j that are isomorphic to $\frac{n}{l_i} C_{l_i}$. Then $\Sigma_i \cong \frac{n^{d_i}}{l_i^{d_i}} G(d_i, l_i)$, and $\Delta \cong n^{d-m} \Sigma_1 \square \dots \square \Sigma_r$. Hence a connected component of Δ is isomorphic to $G(d_1, l_1) \square \dots \square G(d_r, l_r)$, and the result follows. \blacksquare

Example 4.11 To illustrate the construction from the proof of Part (ii) above, let us have a closer look at two homogeneous factorisations of the graph $H(2, 6) \cong K_6 \square K_6$. Since $6 = 2 \cdot 3 = 3 \cdot 2$, Part (ii) of Theorem 1.6 gives rise to two index-2 homogeneous factorisations of $H(2, 6)$, one with factors isomorphic to $\Delta_1 = 3(K_2 \square K_{3 \times 2})$, and the other with factors isomorphic to $\Delta_2 = 2(K_3 \square K_{2 \times 3})$. Observe that $K_{3 \times 2}$ is isomorphic to the skeleton of the Octahedron. Hence each connected component of Δ_1 is isomorphic to the graph $K_2 \square K_{3 \times 2}$, shown on the left-hand side of Figure 3. A connected component of Δ_2 , shown on the right-hand side of Figure 3, is isomorphic to $K_3 \square K_{3,3}$.

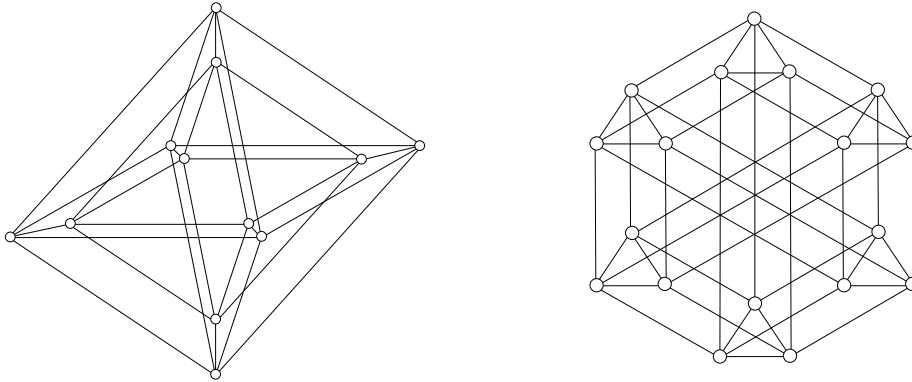


Figure 3: Connected components of factors of the two homogeneous factorisations of $H(2, 6)$ given in Example 4.11.

5 Homogeneous factorisations of grid graphs

In the previous section we presented three ways of constructing homogeneous factorisations of Cartesian powers $\Gamma^{\square t}$. Since $\Gamma^{\square mn} \cong (\Gamma^{\square m})^{\square n}$, any sequence of these constructions will produce a homogeneous factorisation of some Cartesian power of Γ . This leads us to ask the following question.

Question 5.1 For which digraphs Γ and positive integers t do all members of $\text{HomPartn}(\Gamma^{\square t})$ arise from applying some sequence of Constructions 4.1, 4.4 and 4.8?

We investigate this question in the case where $\Gamma = C_n$, the cycle of length n , and $t = 2$.

Proposition 5.2 Let $\Gamma = C_n$ for n odd and $\Sigma = \Gamma^{\square 2}$. Then all members of $\text{HomPartn}(\Sigma)$ arise from Constructions 4.1, 4.4 or 4.8.

PROOF. Let $\text{Aut}(\Gamma) = \langle \rho, \sigma \rangle \cong D_{2n}$, where $\rho : i \mapsto i + 1 \pmod{n}$ and $\sigma : i \mapsto -i \pmod{n}$. By Proposition 2.1, the only (non-trivial) homogeneous factorisation of Γ is $\mathcal{F} = (\langle \rho \rangle, \text{Aut}(\Gamma), \Gamma, \mathcal{P}_\Gamma)$ where \mathcal{P}_Γ is the partition of $A\Gamma$ into two directed cycles. We denote these two parts by P_1 and P_2 respectively and note that they are the two orbits of $\langle \rho \rangle$ on $A\Gamma$.

Suppose that $(M, G, \Sigma, \mathcal{P})$ is a homogeneous factorisation of Σ . Note that $\text{Aut}(\Sigma) \cong (D_{2n} \times D_{2n}) \rtimes \langle \tau \rangle = D_{2n} \wr \langle \tau \rangle$, where τ is the automorphism of Σ which swaps the coordinates of each $(v_1, v_2) \in V\Gamma \times V\Gamma$. Since M is vertex-transitive, we have that n^2 divides $|M|$. Then as $N = \langle \rho \rangle^2$ is the unique subgroup of $\text{Aut}(\Sigma)$ of order n^2 , and n^2 is coprime to $|\text{Aut}(\Sigma) : N|$ it follows that $N \leq M$. As N is vertex-transitive it follows that $(N, G, \Sigma, \mathcal{P})$ is a homogeneous factorisation and so each part of \mathcal{P} is a union of orbits of N on $A\Sigma$. Now N has four orbits on $A\Sigma$, these being

$$\begin{aligned} O_1 &= \{((u, w), (v, w)) \mid (u, v) \in P_1\}, \\ O_1^* &= \{((u, w), (v, w)) \mid (u, v) \in P_2\}, \\ O_2 &= \{((u, v), (u, w)) \mid (v, w) \in P_1\} \end{aligned}$$

and

$$O_2^* = \{((u, v), (u, w)) \mid (v, w) \in P_2\}.$$

Hence there are four possibilities for \mathcal{P} , these being $\mathcal{P}_1 = \{O_1 \cup O_2, O_1^* \cup O_2^*\}$, $\mathcal{P}_2 = \{O_1 \cup O_1^*, O_2 \cup O_2^*\}$, $\mathcal{P}_3 = \{O_1 \cup O_2^*, O_2 \cup O_1^*\}$, and $\mathcal{P}_4 = \{O_1, O_2, O_1^*, O_2^*\}$. If we let $G = \langle N, (\sigma, \sigma) \rangle$ then $\mathcal{F}_1 = (N, G, \Sigma, \mathcal{P}_1) = \mathcal{F}^{\square 2}$ can be obtained by Construction 4.1. Further, $\mathcal{F}_2 = (N, N \rtimes \langle \tau \rangle, \Sigma, \mathcal{P}_2)$ is a homogeneous factorisation that can be obtained from Construction 4.8 by letting $t = 2$ and taking $\{A\Gamma, \emptyset\}$ for the partition of $A\Gamma$. Also $\mathcal{F}_3 = (N, N \rtimes \langle \tau \rangle, \Sigma, \mathcal{P}_3)$ is obtained from Construction 4.8 applied with $t = 2$ and the partition $\{P_1, P_2\}$ of $A\Gamma$. Finally, $(N, \text{Aut}(\Sigma), \Sigma, \mathcal{P}_4) = \mathcal{F}^{\parallel 2}$, as obtained from Construction 4.4. \blacksquare

The situation is different when n is even as the following example provides a homogeneous factorisation of $C_n \square C_n$ for n even which does not arise from any sequence of Constructions 4.1, 4.4 and 4.8.

Example 5.3 Let n be even, $\Gamma = C_n$ and $\Sigma = \Gamma \square \Gamma$. Label the vertices of Γ be $0, 1, 2, \dots, n-1$. Let $\text{Aut}(\Gamma) = \langle \rho, \sigma \rangle \cong D_{2n}$, where $\rho : i \mapsto i + 1 \pmod{n}$ and $\sigma : i \mapsto -i \pmod{n}$. Then $\text{Aut}(\Sigma) = D_{2n} \wr \langle \tau \rangle$ where τ is as in the proof of Proposition 5.2.

Let $M = \langle (\rho^2, 1), (1, \rho^2), (\rho, \rho), (1, \rho)\tau \rangle$. Then $|M| = n^2$ and acts semiregularly on $V\Sigma$. Hence M acts regularly on $V\Sigma$. Now Σ has $4n^2$ arcs and so M has 4 orbits of length n^2 on $A\Sigma$. These are

$$P_1 = \{((i, j), (i, j + 1)), ((j + 1, i), (j + 2, i)) \mid |i - j| \text{ even}\}, \quad (5.1)$$

$$P_2 = \{((i, j), (i, j - 1)), ((j + 1, i), (j, i)) \mid |i - j| \text{ even}\}, \quad (5.2)$$

$$P_3 = \{((i, j), (i, j + 1)), ((j + 1, i), (j + 2, i)) \mid |i - j| \text{ odd}\}, \quad (5.3)$$

and

$$P_4 = \{((i, j), (i, j - 1)), ((j + 1, i), (j, i)) \mid |i - j| \text{ odd}\},$$

where addition is modulo n . Now $N_{\text{Aut}(\Sigma)}(M) = \langle (\rho, 1), (1, \rho), (\sigma, \sigma), \tau \rangle$ which acts transitively on $A\Sigma$. Hence if we let $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ and $G = N_{\text{Aut}(\Sigma)}(M)$, it follows that $\mathcal{F} = (M, G, \Sigma, \mathcal{P})$ is a homogeneous factorisation of index 4. Since by Proposition 2.1, the only homogeneous factorisations of C_n have index 2, the partition \mathcal{P} cannot be obtained from either Construction 4.8 or Construction 4.1. The only possibilities of index 4 are via Construction 4.4 using the partitions \mathcal{P}_1 and \mathcal{P}_2 from Proposition 2.1. In both the factorisations obtained, each part of the partition contains only horizontal arcs, that is, arcs of the form $((u, v), (w, v))$, or only vertical arcs, that is, arcs of the form $((v, u), (v, w))$. However, each $P_i \in \mathcal{P}$ contains both horizontal and vertical arcs and hence there exists a homogeneous factorisation of Σ which is not obtained from any sequence of applications of Constructions 4.1, 4.4 and 4.8.

6 Lexicographic products

In this section, we present a construction of a homogeneous factorisation of a lexicographic product of two graphs, and thus complete the proof of Theorem 1.1.

For sets V_1, V_2 , if $A_i \subseteq V_i \times V_i$ for $i = 1, 2$, we denote by $A_1[A_2]$ the set of all pairs $((u_1, v_1), (u_2, v_2))$ such that either $(u_1, v_1) \in A_1$, or $u_1 = v_1$ and $(u_2, v_2) \in A_2$. Then the lexicographic product of digraphs Γ_1, Γ_2 , has arc set $A\Gamma_1[A\Gamma_2]$. We are able to lift homogeneous factorisations of Γ_1 and Γ_2 to homogeneous factorisations of $\Gamma_1[\Gamma_2]$. This is a generalisation of the construction in [7] for homogeneous factorisations of complete graphs. Note that this proves Theorem 1.1(iii).

Construction 6.1 For $i = 1, 2$, let $\mathcal{F}_i = (M_i, G_i, \Gamma_i, \mathcal{P}_i)$ be a homogeneous factorisation of index k such that $G_1^{\mathcal{P}_1}$ and $G_2^{\mathcal{P}_2}$ are permutationally isomorphic to a transitive permutation group $H \leq S_k$, and let $\varphi_i : G_i \rightarrow H$ be the corresponding epimorphisms. Label $\mathcal{P}_i = \{P_{i1}, P_{i2}, \dots, P_{ik}\}$ such that for any $g \in G$, we have $P_{ij}^g = P_{i, j\varphi_i(g)}$. Let $\mathcal{F}_1[\mathcal{F}_2] = (M, G, \Gamma, \mathcal{P})$ where

$$\begin{aligned} M &= M_1 \times M_2, \\ G &= \{(g_1, g_2) \in G_1 \times G_2 \mid \varphi_1(g_1) = \varphi_2(g_2)\}, \\ \Gamma &= \Gamma_1[\Gamma_2], \\ \mathcal{P} &= \{P_{1j}[P_{2j}] \mid 1 \leq j \leq k\}. \end{aligned}$$

Lemma 6.2 *The 4-tuple $\mathcal{F}_1[\mathcal{F}_2]$ is a homogeneous factorisation of Γ of index k . Moreover, $\mathcal{F}_1[\mathcal{F}_2]$ is symmetric if and only if each \mathcal{F}_i is symmetric.*

PROOF. Since M_i lies in the kernel of φ_i for each $i = 1, 2$, we have that M is a vertex-transitive subgroup of G , and M fixes each $P_{1j}[P_{2j}]$ setwise. Furthermore, if $(g_1, g_2) \in G$ and $j \in \{1, 2, \dots, k\}$, such that $P_{1j}^{g_1} = P_{1j'}$ and $P_{2j}^{g_2} = P_{2j'}$, then

$$(P_{1j}[P_{2j}])^{(g_1, g_2)} = P_{1j'}[P_{2j'}].$$

Hence \mathcal{P} is G -invariant, and $G^{\mathcal{P}}$ is transitive. Therefore, $\mathcal{F}_1[\mathcal{F}_2]$ is a homogeneous factorisation of index k . Furthermore, $\mathcal{F}_1[\mathcal{F}_2]$ is symmetric if and only if both \mathcal{F}_1 and \mathcal{F}_2 are symmetric. ■

We give explicit examples of the use of Construction 6.1 using both self-complementary and almost self-complementary graphs as factors, where the latter are defined in the next paragraph.

Let $\Gamma_1 = K_m$. If $\Gamma_2 = K_n$, then $\Gamma_1[\Gamma_2] = K_{mn}$; if $\Gamma_2 = K_n[\overline{K}_2] = K_{n \times 2}$, the complete multipartite graph with n parts of size 2, then $\Gamma_1[\Gamma_2] = K_{mn \times 2}$. We say that a graph Γ with n vertices is *almost self-complementary* if Γ is isomorphic to its complement in $K_{\frac{n}{2} \times 2}$. Clearly, factors of an index-2 homogeneous factorisation of $K_{\frac{n}{2} \times 2}$ are vertex-transitive almost self-complementary graphs. Note that not all vertex-transitive almost self-complementary graphs are factors of a homogeneous factorisation of some $K_{\frac{n}{2} \times 2}$ (for example, C_6 is such a vertex-transitive almost self-complementary graph). However, an almost self-complementary graph which arises as a factor of an index-2 homogeneous factorisation of $K_{n \times 2}$ is called *homogeneously almost self-complementary* (see [10]). The smallest homogeneously almost self-complementary graph is $2K_2$. Thus, by Construction 6.1, we have the following example.

Example 6.3 Let Δ_1 be a vertex-transitive self-complementary graph.

- (i) As shown in [7], if Δ_2 is a vertex-transitive self-complementary graph, then so is $\Delta_1[\Delta_2]$. For instance, $C_5[C_5]$ is a self-complementary graph.
- (ii) If Δ_2 is a graph which is homogeneously almost self-complementary, then so is $\Delta_1[\Delta_2]$. In particular, $\Delta_1[2K_2]$ is homogeneously almost self-complementary, and so $C_5[2K_2]$ is homogeneously almost self-complementary.

Similar to the direct product and Cartesian product, for homogeneous factorisations $(M_i, G_i, \Gamma_i, \mathcal{P}_i)$ where $1 \leq i \leq t$ of index k such that the $G_i^{\mathcal{P}_i}$ are permutationally isomorphic, we can recursively define the *t-th iterated lexicographic products*:

$$\mathcal{F}_1[\mathcal{F}_2[\dots[\mathcal{F}_{t-1}[\mathcal{F}_t]]\dots]] = (\mathcal{F}_1[\mathcal{F}_2[\dots[\mathcal{F}_{t-1}] \dots]])[\mathcal{F}_t].$$

Example 6.4

- (i) Let Δ be a vertex-transitive self-complementary graph. Then so is the iterated lexicographic product $\Delta[\Delta[\dots[\Delta]\dots]]$. If Σ is homogeneously almost self-complementary, then so is $\Delta[\Delta[\dots[\Delta[\Sigma]]\dots]]$. In particular, since C_5 is a vertex-transitive self-complementary graph, $C_5[C_5[\dots[C_5]\dots]]$ is also self-complementary; if further Δ is homogeneously almost self-complementary, then so is $C_5[C_5[\dots[C_5[\Delta]]\dots]]$.
- (ii) If Δ is a factor of a homogeneous factorisation of K_n of index k , then the t -th iterated lexicographic product $\Delta[\Delta[\dots[\Delta]\dots]]$ is a factor of a homogeneous factorisation of K_{n^t} of index k . In particular, by Lemma 2.5, Q_4^+ is a factor of an index-3 homogeneous factorisation of K_{16} . Hence $Q_4^+[Q_4^+[\dots[Q_4^+]\dots]]$ is a factor of a homogeneous factorisation of K_{16^t} of index 3.

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