

# Homogeneous factorisations of complete multipartite graphs <sup>★</sup>

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## Abstract

A *homogeneous factorisation* of a graph is a partition of its arc set such that there exist vertex transitive subgroups  $M < G \leq \text{Aut}(\Gamma)$  with  $M$  fixing each part of the partition setwise and  $G$  preserving the partition and transitively permuting the parts. In this paper we study homogeneous factorisations of complete multipartite graphs such that  $M$  acts regularly on vertices. We provide a necessary and sufficient condition for the existence of such factorisations and produce many interesting examples. In particular we give a complete determination of the parameters for which homogeneous factorisations exist with  $M$  cyclic.

*Key words:* homogeneous factorisations, complete multipartite graphs

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## 1 Introduction

Let  $\Gamma$  be an undirected graph with vertex set  $V\Gamma$ , edge set  $E\Gamma$  and arc set  $A\Gamma$ . An *isomorphic factorisation* of  $\Gamma$  is a partition of  $E\Gamma$  into spanning subgraphs which are pairwise isomorphic. The concept of isomorphic factorisations has been well studied, see for example [12], [13] with emphasis placed on establishing necessary conditions for their existence. A natural avenue to pursue is to add symmetry conditions on the isomorphic factorisations. This theme was pursued in [18], where a class of isomorphic factorisations called homogeneous factorisations was introduced and studied in the case where  $\Gamma$  is a complete graph. This is a natural class to study when looking at vertex-transitive graphs as the factors have a common vertex-transitive subgroup of automorphisms and there is a group of automorphisms of the graph which transitively permutes the factors. Their study was also motivated by the fact that the existence of a homogeneous factorisation of a complete graph into two factors is equivalent to the existence of a vertex-transitive self-complementary graph. The work of [18] was extended in [8], where homogeneous factorisations of arbitrary vertex-transitive digraphs was begun. This paper concentrates on the case when  $\Gamma$  is a complete multipartite graph.

Let  $\mathcal{P} = \{P_1, \dots, P_k\}$  be a partition of the arc set  $A\Gamma$  of  $\Gamma$  with  $k \geq 2$ , and let  $M, G$  be subgroups of  $\text{Aut}(\Gamma)$  which are transitive on  $V\Gamma$ , such that  $M \leq G$ ,

- (a)  $M$  fixes each  $P_i$  setwise, and
- (b) the partition  $\mathcal{P}$  is  $G$ -invariant and the induced permutation group  $G^{\mathcal{P}}$  of  $G$  on  $\mathcal{P}$  is transitive.

Then we say that  $(M, G, \Gamma, \mathcal{P})$  is a *homogeneous factorisation* of the graph  $\Gamma$  of index  $k$ . We call the elements of  $\mathcal{P}$  *factors* and sometimes interpret them as subdigraphs of  $\Gamma$ . If each  $P_i$  is symmetric, that is  $(u, v) \in P_i$  if and only if  $(v, u) \in P_i$ , then the factorisation is called *symmetric* and can be viewed as a partition of  $E\Gamma$ .

Given a graph  $\Gamma$  with  $n$  vertices, its *complement*  $\bar{\Gamma}$  is the graph with vertex set  $V\Gamma$  and edge set  $\{\{u, v\} \mid \{u, v\} \notin E\Gamma\}$ . We say that  $\Gamma$  is *self complementary* if  $\bar{\Gamma}$  is isomorphic to  $\Gamma$ . If  $\Gamma$  is a self-complementary graph with vertex-transitive automorphism group  $M$ , and  $\sigma$  is an isomorphism between  $\Gamma$  and its complement  $\bar{\Gamma}$ , then  $(M, \langle M, \sigma \rangle, K_n, \{A\Gamma, A\bar{\Gamma}\})$  is a symmetric homogeneous factorisation of index 2. Conversely, if  $(M, G, K_n, \mathcal{P})$  is a symmetric homogeneous factorisation of index 2, then its factors are self-complementary,  $M$ -vertex-transitive graphs. The methods used in studying homogeneous factorisations of complete graphs also arose in the study of exceptionality of permutation representations and their relation to arithmetically exceptional functions, see [10], [11].

Let  $(M, G, \Gamma, \mathcal{P})$  be a homogeneous factorisation of index  $k$ . Then for any vertex-transitive subgroup  $M'$  of  $G$  which fixes each part of  $\mathcal{P}$  setwise, we have that  $(M', G, \Gamma, \mathcal{P})$  is also a homogeneous factorisation. Note that the definition of a homogeneous factorisation does not require the normality of  $M$  in  $G$ . If  $K$  is the kernel of the action of  $G$  on  $\mathcal{P}$ , then  $M \leq K \triangleleft G$  and  $(K, G, \Gamma, \mathcal{P})$  is also a homogeneous factorisation of index  $k$ . So by making the assumption that  $M \triangleleft G$  we are not excluding any partitions of  $A\Gamma$  which give rise to homogeneous factorisations. Moreover, if  $M \triangleleft G$  then  $G$  has an action on the set of  $M$ -orbits on  $A\Gamma$  and the elements of  $\mathcal{P}$  are unions of  $M$ -orbits permuted transitively under this action. *Hence for the rest of the paper we will assume that if  $(M, G, \Gamma, \mathcal{P})$  is a homogeneous factorisation then  $M \triangleleft G$ .*

A permutation group  $G$  on the set  $\Omega$  is called *regular* if for each pair  $\alpha, \beta \in \Omega$  there exists exactly one element  $g \in G$  such that  $\alpha^g = \beta$ . If a homogeneous factorisation  $(M, G, \Gamma, \mathcal{P})$  is such that  $M$  acts regularly on  $V\Gamma$  and  $M \triangleleft G$ , then we call it an  *$M$ -Cayley homogeneous factorisation*, as each subdigraph induced by a factor of  $\mathcal{P}$  is a Cayley digraph for  $M$ . (See Section 2 for the definition of a Cayley digraph.)

We denote the complete multipartite graph with  $s$  parts of size  $t$  by  $K_{s \times t}$ . In the case where  $s = 2$  we often denote  $K_{2 \times t}$  by  $K_{t,t}$ . Note that  $K_{s \times 2} = K_{2s} - sK_2$ , that is, the complete graph  $K_{2s}$  minus the edges of a perfect matching, while  $K_{s \times 1}$  is the complete graph  $K_s$ . This article is devoted to the study of homogeneous factorisations of complete multipartite graphs, and in particular, to those which are  $M$ -Cayley. As it is shown in Lemma 2.1, if  $(M, G, K_{s \times t}, \mathcal{P})$  is an  $M$ -Cayley homogeneous factorisation then there exists a subgroup  $L$  of index  $s$  in  $M$  and order  $t$  such that  $K_{s \times t} = \text{Cay}(M, M \setminus L)$ . We make the following definition.

**Definition 1.1** Let  $M$  be a group,  $H$  a subgroup of  $\text{Aut}(M)$ ,  $L$  an  $H$ -invariant subgroup of  $M$ , and  $k$  an integer,  $k \geq 2$ . We say that the condition

- (i)  $\mathcal{R}(M, H, L, k)$  holds if there exists an index  $k$  subgroup  $R$  of  $H$  and a set  $X$  of orbit representatives of the natural action of  $H$  on  $M \setminus L$  such that, for each  $x \in X$ ,  $H_x \leq R$ .
- (ii)  $\mathcal{R}^{sym}(M, H, L, k)$  holds if there exists an index  $k$  subgroup  $R$  of  $H$  and a set  $X$  of orbit representatives of  $H$  on  $M \setminus L$  such that for each  $x \in X$ ,  $H_x \leq R$  and if  $x^{-1} \in x^H$  then  $x^{-1} \in x^R$ .

These two conditions transpire to be necessary and sufficient conditions for the existence of Cayley homogeneous factorisations and symmetric Cayley homogeneous factorisations of  $\text{Cay}(M, M \setminus L)$ .

**Theorem 1.2** *Given  $M, H, L$ , and  $k$  as in Definition 1.1, set  $s = |M : L|$  and  $t = |L|$ .*

- (i) *There exists a partition  $\mathcal{P}$  of the arc set of  $K_{s \times t}$  such that  $(M, M \rtimes H, K_{s \times t}, \mathcal{P})$  is an  $M$ -Cayley homogeneous factorisation of index  $k$  if and only if  $\mathcal{R}(M, H, L, k)$  holds.*
- (ii) *There exists a partition  $\mathcal{P}$  of the arc set of  $K_{s \times t}$  such that  $(M, M \rtimes H, K_{s \times t}, \mathcal{P})$  is a symmetric  $M$ -Cayley homogeneous factorisation of index  $k$  if and only if  $\mathcal{R}^{\text{sym}}(M, H, L, k)$  holds.*

Theorem 1.2 will be proved in Section 3. The heart of the proof is Construction 3.1 which allows us to construct an  $M$ -Cayley homogeneous factorisation of index  $k$  whenever  $\mathcal{R}(M, H, L, k)$  holds (see Proposition 3.2). Moreover, we show that any  $M$ -Cayley homogeneous factorisation arises from this construction (see Corollary 3.6).

If  $H = \langle \sigma \rangle$  is cyclic of prime power order  $p^a$  and  $L = C_M(\sigma)$  then  $\mathcal{R}(M, H, L, p)$  holds with  $R = \langle \sigma^p \rangle$ . Thus we have the following corollary.

**Corollary 1.3** *Let  $M$  be a group,  $|M| \geq 3$ , and let  $\sigma$  be an automorphism of  $M$  such that the order of  $\sigma$  is a power of some prime  $p$ . Let  $G = M \rtimes \langle \sigma \rangle$ ,  $t = |C_M(\sigma)|$  and  $s = |M|/t$ . Then there exists a partition  $\mathcal{P}$  of the arcs of  $K_{s \times t}$  such that  $(M, G, K_{s \times t}, \mathcal{P})$  is an  $M$ -Cayley homogeneous factorisation of index  $p$ . Furthermore, if  $p$  is odd, then  $\mathcal{P}$  can be chosen so that  $(M, G, K_{s \times t}, \mathcal{P})$  is symmetric.*

Theorem 1.2 motivates the following problem.

**Problem 1.4** *Characterise the triples  $(M, H, L)$ , where  $M$  is a group,  $H \leq \text{Aut}(M)$  and  $L$  is an  $H$ -invariant subgroup of  $M$ , such that there exists an  $H$ -invariant partition  $\mathcal{P}$  of  $M \setminus L$  with  $|\mathcal{P}| > 1$  and  $H$  transitive on  $\mathcal{P}$ .*

In Section 4 we give two constructions of  $M$ -Cayley homogeneous factorisations of complete bipartite graphs. The first relies on the fact that inversion  $x \mapsto x^{-1}$  is an automorphism of an abelian group while the second uses the two inequivalent actions of  $\text{GL}(3, 2)$  on eight points. In Section 5 we examine the case where  $M$  is an almost simple group with socle  $A_n$  and  $H$  is the groups of inner automorphisms induced by the socle, while in Section 6 we provide examples where either  $G$  or  $M$  is a Frobenius group.

The attention of many authors has focused on the subclass of those vertex-transitive self-complementary graphs which admit a vertex-transitive action of a cyclic group of automorphisms (see [1,7,15,20,21,24]). These graphs are precisely the self-complementary Cayley graphs of cyclic groups and are therefore called *self-complementary circulants*. The notion of a self-complementary circulant generalises naturally to the notion of a *circulant homogeneous factorisation*, that is, a homogeneous factorisation  $(M, G, \Gamma, \mathcal{P})$  such that  $M$  contains a cyclic subgroup which is regular on vertices. If  $M$  itself is cyclic and  $M \triangleleft G$ , then we call the factorisation an  *$M$ -circulant homogeneous factorisation*.

tion. The first examples of circulant homogeneous factorisations of complete graphs of index  $k > 2$  were given in [18, Construction 6.3], where a circulant homogeneous factorisation of  $K_n$  of index  $k$  was constructed for each square free integer  $n$  such that each prime divisor of  $n$  was congruent to 1 modulo  $k$ . Symmetric circulant homogeneous factorisations of complete multipartite graphs  $K_{n \times 2}$  of index 2 were analysed in [6], and it was shown that such a factorisation exists if and only if each prime divisor of  $n$  is congruent to 1 modulo 4. In Section 8 we consider  $M$ -circulant homogeneous factorisations of an arbitrary complete multipartite graph  $K_{s \times t}$  and of an arbitrary index  $k$ . In particular, we prove the following. For a positive integer  $n$  and a prime  $p$ ,  $n_p$  denotes the highest power of  $p$  dividing  $n$ .

**Theorem 1.5** *Let  $s, t$  and  $k$  be integers such that  $t \geq 1$  and  $s, k > 1$ , let  $\Gamma = K_{s \times t}$ , and let  $M$  be a cyclic group of order  $st$  acting regularly on  $V\Gamma$ . Then:*

- (i) *There exist a partition  $\mathcal{P}$  of  $A\Gamma$  and  $G \leq N_{\text{Aut}(\Gamma)}(M)$  such that  $(M, G, \Gamma, \mathcal{P})$  is an  $M$ -circulant homogeneous factorisation of index  $k$  if and only if one of the following hold:*
  - (a) *both  $s$  and  $k$  are powers of 2 and  $k$  divides  $t$ ,*
  - (b)  *$s$  is odd and  $k$  divides  $(p-1)t_p$  for each prime divisor  $p$  of  $s$ .*
  - (c)  *$s$  is even and not a power of 2, and either (I)  $k = 2$  and  $t$  is even, or (II)  $k = 2^f \geq 4$ ,  $2k$  divides  $t$ , and  $k$  divides  $p-1$  for each odd prime divisor  $p$  of  $s$ .*
- (ii) *There exist a partition  $\mathcal{P}$  of  $A\Gamma$  and  $G \leq N_{\text{Aut}(\Gamma)}(M)$  such that  $(M, G, \Gamma, \mathcal{P})$  is a symmetric  $M$ -circulant homogeneous factorisation of index  $k$  if and only if, for each prime divisor  $p$  of  $s$ ,  $2k$  divides  $(p-1)t_p$ .*

An important ingredient of the proof of Theorem 1.5 is Construction 7.1, which allows us to lift homogeneous factorisations  $(M_1, G_1, K_{s_1 \times t_1}, \mathcal{P}_1)$  and  $(M_2, G_2, K_{s_2 \times t_2}, \mathcal{P}_2)$  of index  $k$  such that  $G_1^{\mathcal{P}_1}$  and  $G_2^{\mathcal{P}_2}$  are permutationally isomorphic, to a homogeneous factorisation of  $K_{s_1 s_2 \times t_1 t_2}$  of index  $k$ .

There are several natural areas for further work in the study of homogeneous factorisations of complete multipartite graphs. Theorem 1.5 provides necessary and sufficient conditions for the existence of homogeneous factorisations  $(M, G, K_{s \times t}, \mathcal{P})$  with  $M$  cyclic and  $M \triangleleft G$ . It would be interesting to remove the requirement  $M \triangleleft G$ .

**Problem 1.6** *Determine necessary and sufficient conditions for the existence of homogeneous factorisations  $(M, G, K_{s \times t}, \mathcal{P})$  such that  $M$  contains a transitive cyclic subgroup.*

Stringer [26] has determined necessary and sufficient conditions for the existence of homogeneous factorisations  $(M, G, K_n, \mathcal{P})$  where  $M$  contains a transitive cyclic subgroup.

Another line of investigation is to place more symmetry conditions on the factorisations. Peisert [22], following work of Zhang [27], has determined all arc-transitive self-complementary graphs and this work has been extended by Lim [19] who has determined all homogeneous factorisations  $(M, G, K_n, \mathcal{P})$  for which  $M$  is arc-transitive on each factor. This motivates the following.

**Problem 1.7** *Determine all homogeneous factorisations  $(M, G, K_{s \times t}, \mathcal{P})$  for which  $M$  is arc-transitive on each factor.*

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## 2 Preliminaries

A *digraph*  $\Gamma$  consists of a set of vertices  $V\Gamma$  and an arc set  $A\Gamma$  which is a subset of the set  $V\Gamma^{(2)}$  of ordered pairs of distinct elements of  $V\Gamma$ . Given a vertex  $v$ , the vertex  $w$  is an *out-neighbour* of  $v$  if  $(v, w) \in A\Gamma$ , while it is an *in-neighbour* if  $(w, v) \in A\Gamma$ . For any subset  $A$  of  $V\Gamma^{(2)}$ , we let  $A^* = \{(v, w) \mid (w, v) \in A\}$  and call  $A$  *self paired* if  $A = A^*$ . If  $A\Gamma$  is self paired, then we can replace each pair of arcs  $(v, w)$  and  $(w, v)$  with the edge  $\{v, w\}$  and so  $\Gamma$  can be regarded as an undirected graph.

Let  $M$  be a group and  $S$  be a subset of  $M \setminus \{1\}$ . Then the *Cayley digraph*  $\text{Cay}(M, S)$  of  $M$  with respect to  $S$ , is the digraph with vertex set  $M$  and arc set  $\{(m, n) \mid nm^{-1} \in S\}$ . Note that  $S$  is the set of out-neighbours of 1. Let  $S^{-1} = \{s^{-1} \mid s \in S\}$ . We say that  $S$  is *self-inverse* if  $S = S^{-1}$ . The Cayley digraph  $\text{Cay}(M, S)$  gives rise to an undirected graph if and only if  $S$  is self-inverse.

Given a Cayley digraph  $\Gamma = \text{Cay}(M, S)$ , the group  $M$  acts naturally on the vertices of  $\Gamma = \text{Cay}(M, S)$  by right multiplication and maps arcs to arcs. Furthermore, given any  $H \leq \text{Aut}(M)$  such that  $S^H = S$ ,  $H$  naturally acts on  $M$  and preserves arcs. Thus  $M \rtimes H \leq \text{Aut}(\Gamma)$ . In particular,  $M$  is a subgroup of  $\text{Aut}(\Gamma)$  acting regularly on vertices. Moreover, it is well known that a digraph is a Cayley digraph if and only if  $\text{Aut}(\Gamma)$  contains a subgroup which acts regularly on vertices.

For a group  $G$  acting on a set  $\Omega$  and for  $\alpha \in \Omega$ ,  $\alpha^G$  denotes the *G-orbit*  $\{\alpha^g \mid g \in G\}$ , and  $G_\alpha$  denotes the *vertex-stabiliser* of  $\alpha$  in  $G$ . Further, for a subset  $B \subseteq \Omega$  and  $g \in G$ ,  $B^g$  denotes the set  $\{b^g \mid b \in B\}$ , and  $G_B$  denotes the *setwise stabiliser*  $\{g \mid B^g = B\}$  of  $B$  in  $G$ .

Suppose that  $\Gamma$  is a digraph and  $M \triangleleft G \leq \text{Aut}(\Gamma)$ , where  $M$  acts regularly on  $V\Gamma$ . By standard permutation group theory (see, for example, [4, Section 1.7]), the vertices of  $\Gamma$  can be identified with the elements of  $M$  in such a way that the action of  $M$  on  $V\Gamma$  coincides with the action of  $M$  on itself by right multiplication. Moreover, if  $v$  is the vertex of  $\Gamma$  corresponding to the identity element  $1 \in M$ , then  $G = M \rtimes G_v$  and the action of  $G_v$  on  $V\Gamma$  coincides with the action of  $G_v$  on  $M$  by conjugation. Hence we can view  $G_v$  as a subgroup of  $\text{Aut}(M)$ . The digraph  $\Gamma$  is then isomorphic to the Cayley digraph  $\text{Cay}(M, S)$ , where  $S$  is the set of those elements of  $M$  which correspond to the out-neighbours of  $v$  in  $\Gamma$ . The following lemma characterises pairs  $M$  and  $S$  for which the Cayley digraph  $\text{Cay}(M, S)$  is isomorphic to a complete multipartite graph.

**Lemma 2.1** *If  $M$  is a group and  $L \leq M$  a subgroup of index  $s$  and order  $t$ , then  $\text{Cay}(M, M \setminus L)$  is isomorphic to the complete multipartite graph  $K_{s \times t}$ . Conversely, if  $\Gamma = K_{s \times t}$  is a complete multipartite graph and  $M$  a regular group of automorphisms of  $\Gamma$ , then there exists a subgroup  $L$  of order  $t$  and index  $s$  in  $M$  such that  $\Gamma$  is isomorphic to  $\text{Cay}(M, M \setminus L)$ .*

PROOF. See, for example, [16, Proposition 2.2]. ■

A partition  $\mathcal{B}$  of a set  $\Omega$  is called *trivial* if either all its members are singletons, or if its only member is  $\Omega$ . Let  $G$  be a group which acts transitively on a set  $\Omega$  and let  $\mathcal{B}$  be a partition of  $\Omega$ . If for all  $B \in \mathcal{B}$  and  $g \in G$  we have  $B^g \in \mathcal{B}$ , then we say that  $\mathcal{B}$  is a *system of imprimitivity* for  $G$ , and each  $B \in \mathcal{B}$  is called a *block*. Note that trivial partitions of  $\Omega$  are systems of imprimitivity for any transitive group  $G$ , and are hence called *trivial systems of imprimitivity*. We call  $G$  *primitive* if  $G$  has no nontrivial systems of imprimitivity. We recall some of the basic properties of systems of imprimitivity. For a more detailed account see [4] of [5].

Suppose that  $G$  acts transitively on a set  $\Omega$  and that it has a system of imprimitivity  $\mathcal{B}$ . For each  $\alpha \in \Omega$  there is a unique  $B \in \mathcal{B}$  containing  $\alpha$ . If  $g \in G$  fixes  $\alpha$ , then  $g$  fixes  $B$  setwise. Thus  $G_\alpha \leq G_B \leq G$ . Since  $G$  is transitive,  $G_B = G$  implies that  $B = \Omega$  and so  $\mathcal{B}$  is a trivial partition. Furthermore, given  $\beta \in B \setminus \{\alpha\}$  there exists  $g \in G$  such that  $\alpha^g = \beta$  and so  $g \in G_B \setminus G_\alpha$ . Therefore, if  $\mathcal{B}$  is a nontrivial system of imprimitivity then  $G_\alpha \neq G_B$  and  $G_B \neq G$ . The converse is also true.

**Lemma 2.2** *Let  $G$  be a transitive permutation group on a set  $\Omega$ , let  $\alpha \in \Omega$  and let  $R$  be a subgroup of  $G$  containing  $G_\alpha$ . If  $B = \alpha^R$  then  $\mathcal{B} = \{B^g \mid g \in G\}$  is a system of imprimitivity and  $G_B = R$ . Moreover, if  $|G : R| = k$  and  $\{h_1, \dots, h_k\}$  is a set of (right) coset representatives of  $R$  in  $G$ , then  $\mathcal{B} = \{B^{h_i} \mid i \in \{1, \dots, k\}\}$  and the size of  $\mathcal{B}$  is  $k$ .*

PROOF. The first assertion is standard permutation group theory. For  $B, B' \in \mathcal{B}$ , the subset of  $G$  of elements  $g$  that map the block  $B$  to  $B'$  forms a right coset of  $R = G_B$ , say  $G_B h_i$ , and  $B' = B^{h_i}$ . The second assertion follows. ■

Suppose that a group  $G$  acts transitively on two sets  $\Omega_1$  and  $\Omega_2$ . We say that the two actions are *equivalent* if for each  $\alpha \in \Omega_1$  and  $\beta \in \Omega_2$ , the point stabilisers  $G_\alpha$  and  $G_\beta$  are conjugate in  $G$ . Observe that this implies that  $G_\alpha$  is also the stabiliser of some point  $\gamma \in \Omega_2$ . Note that the equivalence of transitive actions as defined above is equivalent to the existence of a bijection  $f: \Omega_1 \rightarrow \Omega_2$ , such that  $f(\alpha)^g = f(\alpha^g)$  for all  $\alpha \in \Omega_1$  and  $g \in G$ .

Let  $G$  act transitively on  $\Omega_1$  and  $H$  act transitively on  $\Omega_2$ . We say that the actions of  $G$  and  $H$  are *permutationally isomorphic* if there exists a bijection  $f: \Omega_1 \rightarrow \Omega_2$  and a homomorphism  $\phi: G \rightarrow H$  such that for all  $\alpha \in \Omega_1$  and  $g \in G$  we have

$$f(\alpha^g) = (f(\alpha))^{\phi(g)}.$$

Note that if  $G$  acts on  $\Omega_1$  and  $\Omega_2$  such that the two actions are equivalent then they are permutationally isomorphic, with  $f$  being as determined in the previous paragraph and  $\phi = 1$ .

### 3 Cayley homogeneous factorisations

Recall Definition 1.1 that  $\mathcal{R}(M, H, L, k)$  holds if and only if we can choose a system  $X$  of representatives of the  $H$ -orbits on  $M \setminus L$ , such that some subgroup  $R \leq H$  of index  $k$  contains  $H_x$  for each  $x \in X$ . This is equivalent to requiring that for each  $y \in M \setminus L$ ,  $H_y$  is conjugate in  $H$  to a subgroup of  $R$ . For such a pair  $(R, X)$  we say that  $\mathcal{R}(M, H, L, k)$  *holds relative to*  $(R, X)$ . Similarly, if for each  $x \in X$  such that  $x^{-1} \in x^H$  then  $x^{-1} \in x^R$ , then we say that  $\mathcal{R}^{sym}(M, H, L, k)$  *holds relative to*  $(R, X)$ . We note here that if  $H$  acts semiregularly on  $M \setminus L$  then  $\mathcal{R}(M, H, L, k)$  holds relative to  $(1, X)$  for any set  $X$  of orbit representatives. We now give a construction which shows that if  $\mathcal{R}(M, H, L, k)$  holds then we can construct an  $M$ -Cayley homogeneous factorisation of a complete multipartite graph.

**Construction 3.1** *Given  $M, H, L, k, R$  and  $X$ , such that  $\mathcal{R}(M, H, L, k)$  holds relative to  $(R, X)$ , we construct an  $M$ -Cayley homogeneous factorisation  $(M, G, K_{s \times t}, \mathcal{P})$  of index  $k$ , where  $s = |M : L|$ ,  $t = |L|$  and  $G \cong M \rtimes H$ .*

Let  $S = M \setminus L$  and observe that the Cayley graph  $\Gamma = \text{Cay}(M, S)$  is isomorphic to the complete multipartite graph  $K_{s \times t}$ . Define  $S_1 = \{x^h \mid x \in X, h \in R\}$ . Choose a set  $\{h_1, h_2, \dots, h_k\}$  of coset representatives of  $R$  in  $H$  such that  $h_1 = 1$  and for each  $i = 2, \dots, k$  let  $S_i = S_1^{h_i}$ . For each  $i = 1, 2, \dots, k$ , let

$\Gamma_i = \text{Cay}(M, S_i)$  and  $P_i$  be the arc set of  $\Gamma_i$ . Finally, let  $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ , and let  $G = M \rtimes H$  be the permutation group acting on  $M$  in such a way that  $M$  acts by right multiplication, and  $H$  acts as a subgroup of  $\text{Aut}M$ .

**Proposition 3.2** *The output  $(M, G, K_{s \times t}, \mathcal{P})$  of Construction 3.1 is an  $M$ -Cayley homogeneous factorisation of index  $k$ . Moreover, this factorisation is symmetric if and only if*

- (a) *for each  $x \in X$  such that  $x^H$  is self-inverse, also  $x^R$  is self-inverse, and*
- (b) *for each  $x \in X$  such that  $x^H$  is not self-inverse, there exists  $y \in X$ , such that  $x^{-1}$  is contained in  $y^R$ .*

PROOF. Let us first prove that  $\mathcal{S} = \{S_1, \dots, S_k\}$  is a partition of  $S$  of size  $k$ . Take  $x \in X$  and recall that  $R$  contains  $H_x$ . By Lemma 2.2, the set  $\mathcal{B}_x = \{x^{Rh_i} \mid i \in \{1, \dots, k\}\}$  is a system of imprimitivity for the action of  $H$  on  $x^H$ , and  $|\mathcal{B}_x| = k$ . Since  $S_i = \cup_{x \in X} x^{Rh_i}$ , this shows that  $\mathcal{S}$  is a partition of  $S$  of size  $k$ . Moreover,  $\mathcal{S}$  is  $H$ -invariant and  $H$  acts on  $\mathcal{S}$  transitively. Since  $P_i$  is the arc set of the Cayley digraph  $\text{Cay}(M, S_i)$ , this implies that  $\mathcal{P}$  is an  $H$ -invariant partition of the arc set of  $\text{Cay}(M, S)$  of size  $k$ , upon which  $H$  acts transitively. On the other hand,  $M \leq \text{Aut}(\text{Cay}(M, S_i))$  for each  $i \in \{1, \dots, k\}$ , so  $M$  acts on  $\mathcal{P}$  trivially and acts regularly on  $V\Gamma_i = M$  for each  $i$ . But since  $G$  is generated by  $M$  and  $H$ , this implies that  $\mathcal{P}$  is  $G$ -invariant and  $G$  acts on  $\mathcal{P}$  transitively. This proves the assertion that  $(M, G, K_{s \times t}, \mathcal{P})$  is an  $M$ -Cayley homogeneous factorisation of index  $k$ .

By definition,  $(M, G, K_{s \times t}, \mathcal{P})$  is symmetric if and only if each  $\Gamma_i = \text{Cay}(M, S_i)$  is undirected, or equivalently, if and only if each  $S_i$  is self-inverse. The result now follows directly from the definition of the  $S_i$ . ■

It follows from Proposition 3.2 that if  $\mathcal{R}(M, H, L, k)$  holds for some  $H, M, L$  and  $k$ , then there exists an  $M$ -Cayley homogeneous factorisation of  $K_{|M:L| \times |L|}$  of index  $k$ . Moreover, it follows that a sufficient condition for the existence of a symmetric factorisation is that  $\mathcal{R}(M, H, L, k)$  holds relative to a pair  $(R, X)$  for which conditions (a) and (b) in Proposition 3.2 hold. The following lemma shows that condition (b) may be omitted, that is, if  $\mathcal{R}^{sym}(M, H, L, k)$  holds then we can construct a symmetric homogeneous factorisation.

**Lemma 3.3** *Let  $M, H, L, k, R$  and  $X$  be such that  $\mathcal{R}(M, H, L, k)$  holds relative to  $(R, X)$ . If, in addition,  $R$  and  $X$  satisfy condition (a) in Proposition 3.2, then there exists  $\hat{X}$ , such that  $\mathcal{R}(M, H, L, k)$  holds relative to  $(R, \hat{X})$ , and  $R$  and  $\hat{X}$  satisfy conditions (a) and (b) in Proposition 3.2.*

PROOF. Let  $X = \{x_1, \dots, x_r\}$ , let  $\mathcal{J}$  be the set of indices  $i \in \{1, \dots, r\}$ , such that  $x_i^H$  is self-inverse, and let  $\mathcal{J}' = \{1, \dots, r\} \setminus \mathcal{J}$ . Observe that elements of  $\mathcal{J}'$  come in pairs  $\{i, i'\}$ , such that  $x_{i'}^H = (x_i^H)^{-1}$ . For each such pair  $\{i, i'\}$

choose a representative, say  $i$ , and define  $y_i = x_i$ ,  $y_{i'} = x_i^{-1}$ . Further, for each  $i \in \mathcal{J}$ , let  $y_i = x_i$ . Finally, set  $\hat{X} = \{y_1, \dots, y_r\}$ . Since for any  $x \in M$ , we have  $H_x = H_{x^{-1}}$ , the group  $R$  contains  $H_{y_i}$  for each  $i \in \{1, \dots, r\}$ . Thus  $\mathcal{R}(M, H, L, k)$  holds relative to  $(R, \hat{X})$ . Further, for any  $y_i \in \hat{X}$ , such that  $y_i^H$  is self-inverse, we have  $y_i = x_i$ , and by assumption also  $y_i^R$  is self-inverse. Hence  $R$  and  $X$  satisfy condition (a). Finally, for each  $y_i \in \hat{X}$  such that  $y_i^H$  is not self-inverse, there exists  $i'$  (the one, for which  $x_{i'}^H = (x_i^H)^{-1}$ ), such that  $y_i^{-1} = y_{i'}$ , and thus  $y_i^{-1} \in y_{i'}^R$ . Therefore,  $R$  and  $\hat{X}$  also satisfy condition (b).  $\blacksquare$

**Corollary 3.4** *Let  $M, H, L$  and  $k$  be as in Definition 1.1. If the condition  $\mathcal{R}^{sym}(M, H, L, k)$  holds then there exists a symmetric  $M$ -Cayley homogeneous factorisation  $(M, M \rtimes H, K_{|M:L| \times |L|}, \mathcal{P})$  of index  $k$ .*

PROOF. If  $\mathcal{R}^{sym}(M, H, L, k)$  holds then by Definition 1.1, there exists a pair  $(R, X)$  for which condition (a) in Proposition 3.2 is satisfied and  $\mathcal{R}(M, H, L, k)$  holds relative to  $(R, X)$ . Lemma 3.3 then implies the existence of  $\hat{X}$  such that  $\mathcal{R}(M, H, L, k)$  holds relative to  $(R, \hat{X})$  and such that the pair  $(R, \hat{X})$  satisfies both conditions (a) and (b) in Proposition 3.2. The result now follows from Proposition 3.2.  $\blacksquare$

Let  $(M, G, K_{s \times t}, \mathcal{P})$  be an  $M$ -Cayley homogeneous factorisation. Recall that  $K_{s \times t}$  can be identified with the Cayley graph  $\text{Cay}(M, M \setminus L)$ , for some  $L \leq M$  such that  $|M : L| = s$ , and  $G = M \rtimes H$  for some  $H \leq \text{Aut}M$  such that  $L^H = L$ . Furthermore, each  $P_i \in \mathcal{P}$  is the arc set of a Cayley graph  $\text{Cay}(M, S_i)$  for some  $S_i \subseteq M \setminus L$ . To complete the proof of Theorem 1.2 it remains to show that there exists an index  $k$  subgroup  $R$  of  $H$  and a set  $X$  of representatives for the orbits of  $H$  on  $M \setminus L$  such that  $\mathcal{R}(M, H, L, k)$  holds relative to  $(R, X)$ . We also intend to show that  $(M, G, K_{s \times t}, \mathcal{P})$  is the same homogeneous factorisation as obtained from Construction 3.1 using  $M, H, L, k, R$  and  $X$ . Thus if  $(M, G, K_{s \times t}, \mathcal{P})$  is symmetric then Proposition 3.2 implies that  $\mathcal{R}^{sym}(M, H, L, k)$  holds. The following lemma is crucial to this endeavour.

**Lemma 3.5** *Let  $M$  be a group, let  $L$  be a subgroup of order  $t$  and index  $s$  in  $M$ , and let  $\Gamma = \text{Cay}(M, M \setminus L) \cong K_{s \times t}$ . Suppose that  $\mathcal{P} = \{P_1, \dots, P_k\}$  is a partition of  $A\Gamma$  and  $H$  a subgroup of  $\text{Aut}(M)$  such that  $(M, M \rtimes H, \Gamma, \mathcal{P})$  is an  $M$ -Cayley homogeneous factorisation of index  $k$ . For each  $i = 1, \dots, k$ , let  $S_i \subseteq M \setminus L$  be such that  $P_i$  is the arc set of  $\text{Cay}(M, S_i)$ . Further, let  $\mathcal{O} = \{O_1, \dots, O_k\}$  denote the set of  $H$  orbits on  $M \setminus L$ . Then*

- (i)  $k$  divides the size of each  $O_i$ . In particular,  $H$  acts without fixed points on  $M \setminus L$ ,
- (ii) for each  $O_j \in \mathcal{O}$ ,  $\mathcal{B}_j = \{O_j \cap S_i \mid i = 1, \dots, k\}$  is a system of imprimitivity for  $H$  on  $O_j$  of size  $k$  such that for each  $j, l \in \{1, \dots, k\}$ , the actions of  $H$  on  $\mathcal{B}_j$  and  $\mathcal{B}_l$  are equivalent and transitive.

PROOF. First observe that part (i) is a direct consequence of part (ii). Thus it suffices to prove part (ii). Let  $G = M \rtimes H$  and note that for each  $i \in \{1, \dots, k\}$ ,  $S_i = \{x \mid (1, x) \in P_i\}$ . Observe that  $S_i^h = \{x^h \mid (1, x) \in P_i\} = \{x \mid (1, x) \in P_i^h\}$ , for any  $h \in H$ . Since  $\mathcal{P}$  is a  $G$ -invariant partition of the arc set of  $\Gamma$ , the above shows that  $\mathcal{S} = \{S_1, \dots, S_k\}$  is an  $H$ -invariant partition of  $S$ . Moreover, since  $G$  is transitive on  $\mathcal{P}$ ,  $M$  acts trivially on  $\mathcal{P}$  and  $G = MH$ , we see that  $H$  acts transitively on  $\mathcal{S}$ . Furthermore, for each  $j = 1, \dots, r$ ,  $\mathcal{B}_j$  is an  $H$ -invariant partition of  $O_j$  upon which  $H$  acts transitively. Moreover, for any  $j, l \in \{1, \dots, r\}$  and  $h \in H$ ,  $(O_j \cap S_1)^h = O_j \cap S_1$  if and only if  $S_1^h = S_1$  which holds if and only if  $(O_l \cap S_1)^h = O_l \cap S_1$ . Hence the  $H$ -stabilisers of  $O_j \cap S_1$ ,  $O_l \cap S_1$  and  $S_1$  coincide. Thus the actions of  $H$  on  $\mathcal{B}_j$ ,  $\mathcal{B}_l$  and  $\mathcal{S}$  are equivalent, and in particular are transitive. ■

**Corollary 3.6** *Let  $M, H, L, k, \Gamma$  and  $\mathcal{P}$  be as in Lemma 3.5. Then there exists an index  $k$  subgroup  $R$  of  $H$  and a set  $X$  of representatives of the orbits of  $H$  on  $M \setminus L$  such that for each  $x \in X$ ,  $R$  contains  $H_x$ . Furthermore,  $(M, M \rtimes H, \Gamma, \mathcal{P})$  is the same as that obtained from Construction 3.1 using  $M, H, L, k, R$  and  $X$ . In particular, if  $(M, M \rtimes H, \Gamma, \mathcal{P})$  is symmetric, then  $\mathcal{R}^{sym}(M, H, L, k)$  holds.*

PROOF. Let  $R$  be the stabiliser in  $H$  of  $B_1 = O_1 \cap S_1$ , that is,  $R$  is the stabiliser of a block of the system of imprimitivity  $\mathcal{B}_1$ . Then since the actions of  $H$  on each  $\mathcal{B}_i$  are equivalent, it follows that  $R$  fixes a block of each  $\mathcal{B}_i$ . In fact  $R$  fixes each  $B_i = O_i \cap S_1$ . For each  $i$  choose  $x_i \in B_i$  and let  $X = \{x_1, \dots, x_k\}$ . Then  $R$  contains each  $H_{x_i}$ . Furthermore, each  $B_i = x_i^R$  and  $S_1 = B_1 \cup \dots \cup B_k = \{x^h \mid x \in X, h \in R\}$ . Choose a set  $\{h_1, \dots, h_k\}$  of coset representatives for  $R$  in  $H$  such that  $h_1 = 1$ . Then for each  $i = 1, \dots, k$ ,  $S_i = S_1^{h_i}$ . Since for each  $i$ ,  $P_i$  is the arc set of  $\text{Cay}(M, S_i)$ , it follows that the homogeneous factorisation obtained from Construction 3.1 using  $M, H, L, k, R$  and  $X$  is  $(M, M \rtimes H, \Gamma, \mathcal{P})$ . Finally, suppose that  $(M, M \rtimes H, \Gamma, \mathcal{P})$  is symmetric. We have just shown that this factorisation may be obtained by Construction 3.1. Thus Proposition 3.2 implies that  $\mathcal{R}^{sym}(M, H, L, k)$  holds. ■

Finally in this section we draw together the components of the proof of Theorem 1.2. Let  $M, H, L$  and  $k$  be as in Definition 1.1, and set  $s = |M : L|$  and  $t = |L|$ . Then Theorem 1.2(i) follows from Proposition 3.2, Lemma 3.5 and Corollary 3.6; and Theorem 1.2(ii) follows from Corollary 3.4, Lemma 3.5 and Corollary 3.6. Corollary 1.3 then follows by letting  $H = \langle \sigma \rangle$ , a cyclic group of prime power order  $p^a$ , and  $R = \langle \sigma^p \rangle$ .

## 4 Some constructions in the bipartite case

In this section we use Theorem 1.2 to give two explicit constructions of families of homogeneous factorisations of complete bipartite graphs.

**Example 4.1** Let  $B$  be an abelian group of order  $t$  and let  $G = (B \times B) \rtimes \langle \tau \rangle$  where  $\tau$  is the automorphism of  $B \times B$  defined by  $(a, b)^\tau = (b^{-1}, a^{-1})$ . Let  $L = \{(b, b) \mid b \in B\}$  and  $M = L \rtimes \langle \tau \rangle$ . Then  $M \triangleleft G$  and  $\Gamma = \text{Cay}(M, M \setminus L) \cong K_{t,t}$ . Let  $H = \{(1, b) \mid b \in B\}$ . Then  $G = M \rtimes H$ ,  $H \lesssim \text{Aut}(M)$  and  $L$  is  $H$ -invariant. Now  $H$  acts regularly on  $M \setminus L$  by conjugation, (explicitly  $((a, a)\tau)^{(1,b)} = (b^{-1}a, b^{-1}a)\tau$ ), so let  $X = \{\tau\}$  and  $R = H_\tau = 1$ . Then  $\mathcal{R}(M, H, L, t)$  holds relative to  $(R, X)$ . Furthermore, as  $\tau$  has order two,  $\mathcal{R}^{sym}(M, H, L, t)$  holds. Construction 3.1 applied to  $(R, X) = (1, \{\tau\})$  defines a partition  $\mathcal{P}_0 = \{P_b \mid b \in \mathcal{B}\}$  of  $A\Gamma$  where  $P_1$  is the arc set of  $\text{Cay}(M, \{\tau\})$  and in general  $P_b$  is the arc set of  $\text{Cay}(M, \{(b^{-1}, b^{-1})\tau\})$ . Thus we obtain a symmetric  $M$ -Cayley homogeneous factorisation  $(M, G, \Gamma, \mathcal{P}_0)$  of index  $t$ . Note further, that if  $R'$  is an index  $k$  subgroup of  $H$  then  $\mathcal{R}^{sym}(M, H, L, k)$  holds relative to  $(R', X)$  and so we can also construct symmetric  $M$ -Cayley homogeneous factorisations of index  $k$ .

A digraph for which every vertex has a unique out-neighbour is called a *1-factor* and a factorisation of a graph into 1-factors is called a *1-factorisation*. The homogeneous factorisation  $(M, G, \Gamma, \mathcal{P}_0)$  in Example 4.1 is a homogeneous 1-factorisation and we wish to know if these examples are typical of homogeneous 1-factorisations of  $K_{t,t}$ . The following proposition provides a necessary condition on  $M$  for the existence of a homogeneous 1-factorisation based on  $M$ . First of all we recall a lemma from [8].

**Lemma 4.2** *If  $(M, G, \Gamma, \mathcal{P})$  is a homogeneous 1-factorisation of a connected graph  $\Gamma$ , then  $M$  acts regularly on  $V\Gamma$  and coincides with the kernel of the action of  $G$  on  $\mathcal{P}$ .*

**Proposition 4.3** *Let  $(M, G, \Gamma, \mathcal{P})$  be a symmetric homogeneous 1-factorisation of a complete bipartite graph  $\Gamma$  and let  $M^+$  denote the index 2 subgroup of  $M$  which preserves the bipartition of  $\Gamma$ . Then any element  $\tau \in M \setminus M^+$  has order 2, and  $\tau^{-1}g\tau = g^{-1}$  for any  $g \in M^+$ . In particular,  $M^+$  is abelian.*

**PROOF.** For each  $A \in \mathcal{P}$  and vertex  $v$ , let  $A(v)$  be the (uniquely determined) vertex of  $\Gamma$  such that  $(v, A(v)) \in A$ . Since the factorisation is symmetric,  $A(A(v)) = v$ . Similarly,  $A(v)^g = A(v^g)$  for each  $g \in M$ . In particular, if  $\tau \in M \setminus M^+$ , and  $A \in \mathcal{P}$  is the part containing  $(v, v^\tau)$ , then  $A(v) = v^\tau$ ,  $A(v^\tau) = v$ , and thus  $v^{\tau^2} = A(v)^\tau = A(v^\tau) = v$ . But since  $M$  acts regularly on  $V\Gamma$  (by Lemma 4.2), this implies that  $\tau^2 = 1$ .

Now, let  $g$  be an arbitrary element of  $M^+$ . Then  $g\tau \in M \setminus M^+$  and so by the previous paragraph  $g\tau g\tau = 1$ , and thus  $g^\tau = g^{-1}$ . Finally, if  $g, h$  are arbitrary elements of  $M^+$ , then  $g^{-1}h^{-1} = g^\tau h^\tau = (gh)^\tau = (gh)^{-1} = h^{-1}g^{-1}$ , showing that  $g$  and  $h$  commute.  $\blacksquare$

Our next construction uses the two inequivalent actions of  $\text{AGL}(3, 2)$  on eight points. It involves the product action of a wreath product. Let  $G$  be a permutation group on a set  $V$  and let  $l \geq 2$ . The wreath product  $G \text{ wr } S_l = G^l \rtimes S_l$  has a natural product action on  $V^l$  defined as follows. For  $g = (g_1, \dots, g_l) \in G^l$ ,  $\sigma \in S_l$  and  $x = (x_1, \dots, x_l) \in V^l$ ,  $x^g = (x_1^{g_1}, \dots, x_l^{g_l})$  and  $x^\sigma = (x_{1\sigma^{-1}}, \dots, x_{l\sigma^{-1}})$ .

**Example 4.4** Let  $M$  be an elementary abelian group of order  $2^4$ ,  $L$  an index two subgroup and  $\Gamma = \text{Cay}(M, M \setminus L) \cong K_{8,8}$ . Let  $H = \text{GL}(3, 2) \cong \text{PSL}(3, 2)$ . Then  $H$  has an action on  $M$  as a group of automorphisms such that the action on  $L$  is its natural action which fixes a point and its action on  $M \setminus L$  is the 2-transitive action of  $\text{PSL}(2, 7)$  on 8 points (see for example [23]). Let  $G = M \rtimes H$ ,  $X = \{x\}$  for some  $x \in M \setminus L$  and  $R = H_x$ . Then since  $H$  is transitive on  $M \setminus L$  it follows that  $\mathcal{R}(M, H, L, 8)$  holds. Furthermore,  $x = x^{-1}$  and so  $\mathcal{R}^{\text{sym}}(M, H, L, 8)$  holds. Thus we can use Construction 3.1 to find a partition  $\mathcal{P}$  of the arc set of  $K_{8,8}$  such that  $(M, G, \Gamma, \mathcal{P})$  is a symmetric  $M$ -Cayley homogeneous factorisation of index 8.

For each  $l \geq 2$  let  $M'$  be an elementary abelian group of order  $2^{3l+1}$  and  $L'$  be an index two subgroup. Then  $\Gamma' = \text{Cay}(M', M' \setminus L') \cong K_{2^{3l}, 2^{3l}}$ . Let  $K$  be a subgroup of the symmetric group  $S_l$  and  $H' = \text{GL}(3, 2) \text{ wr } K$ . Then  $H'$  acts on  $M'$  as a group of automorphisms such that  $H'$  acts on  $L'$  as the point stabiliser of  $\text{AGL}(3, 2) \text{ wr } K$  in its natural product action on  $8^l$  points; and  $H'$  acts on  $M' \setminus L'$  as the natural product action of  $\text{PSL}(3, 2) \text{ wr } K$  on  $8^l$  points. Let  $G' = M' \rtimes H'$ ,  $X' = \{x'\}$  for some  $x' \in M' \setminus L'$  and  $R' = H'_{x'}$ . Then as  $x'$  has order two,  $\mathcal{R}^{\text{sym}}(M', H', L', 8^l)$  holds relative to  $(R', X')$ . Hence we can use Construction 3.1 to find a partition  $\mathcal{P}'$  of the arc set of  $K_{8^l, 8^l}$  such that  $(M', G', \Gamma', \mathcal{P}')$  is a symmetric  $M'$ -Cayley homogeneous factorisation of index  $8^l$ . Note that if  $K$  is not a transitive subgroup of  $S_l$  then  $R'$  is not a maximal subgroup of  $H'$ . In this case, there are then overgroups of  $R'$  in  $H'$  and any such overgroup will give rise to a new homogeneous factorisation of  $\Gamma'$  using Construction 3.1.

The very special nature of these examples is demonstrated by the following proposition. For a bipartite graph with a group  $G$  of automorphisms, we denote by  $G^+$  the index two subgroup of  $G$  which fixes setwise both parts of the bipartition.

**Proposition 4.5** *Let  $(M, G, K_{t,t}, \mathcal{P})$  be a homogeneous factorisation such that  $M \triangleleft G$ ,  $G$  is arc-transitive on  $K_{t,t}$  and  $G^+$  is primitive on the two parts  $U$  and  $W$  of the bipartition. Furthermore, suppose that  $G^+$  acts faithfully on both*

$U$  and  $W$ . Then  $G^+ \cong \text{AGL}(3, 2) \text{ wr } K$  for some transitive subgroup  $K$  of  $S_k$ ,  $M^+ \cong (C_2^3)^k$  and  $t = 2^{3k}$ .

PROOF. Let  $\Gamma = K_{t,t}$ . Since  $G^+$  acts faithfully on  $U$ ,  $G^+ \cong (G^+)^U$  and is a primitive permutation group. Let  $\alpha \in U$ . Then  $(G^+)_\alpha = G_\alpha$  acts transitively on  $\Gamma(\alpha) = W$ . Let  $\beta \in W$  and  $\tau \in G \setminus G^+$  such that  $\alpha^\tau = \beta$ . Then  $G_\beta = G_\alpha^\tau$  and so  $(G^+) = G_\alpha G_\alpha^\tau$  and  $\tau$  induces an automorphism of  $G^+$ . All such factorisations of primitive groups were determined in [2] and the examples are  $\text{AGL}(3, 2)$  acting on 8 points, various almost simple groups, or a wreath product of one of these two. Now  $G^+$  has a normal subgroup  $M^+$  such that  $(M^+)_\alpha$  is not transitive on  $W$ . This rules out all the almost simple groups as a possibility for  $G^+$  as in all cases, the socle also factorises. Similarly,  $G^+$  is not a wreath product of an almost simple group and so  $G^+ \cong \text{AGL}(3, 2) \text{ wr } K$  for some transitive subgroup  $K$  of  $S_k$  and  $t = 2^{3k}$ . Now  $M^+$  is a nontrivial normal subgroup of  $G^+$  which is not edge transitive and so  $M^+ \cong (C_2^3)^k$ . ■

## 5 Examples using almost simple groups

In this section we construct several examples of  $M$ -Cayley homogeneous factorisations for which  $M$  is an almost simple group. One method is to use Corollary 1.3. Let  $M$  be an almost simple group and  $\sigma$  be an automorphism of  $M$  of prime power order. Let  $L = C_M(\sigma)$ ,  $s = |M : L|$  and  $t = |L|$ . Then by Corollary 1.3, there exists a partition  $\mathcal{P}$  of the arc set of  $K_{s \times t}$  such that  $(M, M \rtimes \langle \sigma \rangle, K_{s \times t}, \mathcal{P})$  is an  $M$ -Cayley homogeneous factorisation. We also have the following examples.

**Example 5.1** Let  $M$  be an almost simple group,  $L = \text{soc}(M)$  and let  $\sigma$  be an automorphism of  $M$  of prime power order  $p^a$  which fixes no element of  $M \setminus L$ . Set  $s = |M : L|$  and  $t = |L|$ . Then each orbit of  $H = \langle \sigma \rangle$  on  $M \setminus L$  has size a power of  $p$ . Thus letting  $R = \langle \sigma^{p^{a-1}} \rangle$  we see that  $\mathcal{R}(M, H, L, p)$  holds and so we can use Construction 3.1, to build a partition  $\mathcal{P}$  of the arc set of  $K_{s \times t}$  such that  $(M, M \rtimes H, K_{s \times t}, \mathcal{P})$  is an  $M$ -Cayley homogeneous factorisation of index  $p$ . The simplest example of such an  $M$  and  $\sigma$  is  $M = S_n$  for  $n = p^a \geq 5$  and  $\sigma$  the automorphism of  $M$  given by conjugation by an  $n$ -cycle.

Now let  $M$  be an almost simple group with socle  $L$  and let  $H = \text{Inn}_L(M) \cong L$ , be the group of automorphisms of  $M$  induced by conjugation by the elements of  $L$ . We wish to determine when there exists an integer  $k$  such that  $\mathcal{R}(M, H, L, k)$  holds. Note that given an element  $x \in M \setminus L$ , then in the natural action of  $H$  on  $M \setminus L$ ,  $H_x$  corresponds to the centraliser in  $L$  of  $x$ . First we give a couple of examples.

**Example 5.2**  $K_{60,60}$  has a homogeneous factorisation of index 10. Let  $M =$

$S_5$ ,  $L = A_5$  and  $H = \text{Inn}_L(M)$ . Then  $H$  has three orbits on the elements of  $M \setminus L$ , and  $X = \{(1, 2), (1, 2)(3, 4, 5), (1, 4, 2, 3)\}$  is a set of representatives for these orbits. Now  $C_L((1, 2)) = \langle (1, 2)(3, 4), (3, 4, 5) \rangle \cong S_3$ ,  $C_L((1, 2)(3, 4, 5)) = \langle (3, 4, 5) \rangle \leq C_L((1, 2))$  and  $C_L((1, 4, 2, 3)) = \langle (1, 2)(3, 4) \rangle \leq C_L((1, 2))$ . Hence letting  $R$  be the subgroup of  $H = \text{Inn}_L(M)$  corresponding to  $C_L((1, 2))$ , we see that  $\mathcal{R}(M, H, L, 10)$  holds relative to  $(R, X)$  and so we can use Construction 3.1 to build an  $M$ -Cayley homogeneous factorisation  $(M, M \rtimes H, K_{60,60}, \mathcal{P})$  of index 10. By Proposition 3.2, this factorisation is not symmetric since  $(1, 4, 2, 3)^R$  does not contain  $(1, 3, 2, 4)$ .

**Example 5.3**  $K_{360,360}$  has symmetric homogeneous factorisations of indices 10, 15, 45 and 90. Let  $M = M_{10}$ ,  $L = A_6$  and  $H = \text{Inn}_L(M)$ . Then  $H$  has three orbits  $\mathcal{O}_1, \mathcal{O}_2$  and  $\mathcal{O}_3$  on  $M \setminus L$ :  $\mathcal{O}_1$  contains elements of order 4 whose centraliser in  $L$  is isomorphic to  $C_2$  while  $\mathcal{O}_2$  and  $\mathcal{O}_3$  contain elements of order 8 whose centraliser in  $L$  is isomorphic to  $C_4$ . There is only one class of  $C_4$ 's and only one class of  $C_2$ 's in  $L$ . Hence there exists  $x \in \mathcal{O}_1$ ,  $y \in \mathcal{O}_2$  and  $z \in \mathcal{O}_3$  such that  $C_L(x) \leq C_L(y) = C_L(z)$ . Thus if we let  $R$  be the subgroup of  $H$  corresponding to  $C_L(z)$  and let  $X = \{x, y, z\}$ , it follows that  $\mathcal{R}(M, H, L, 90)$  holds with respect to  $(R, X)$ . Now  $\mathcal{O}_2$  and  $\mathcal{O}_3$  are not self-inverse and we may have chosen  $z = y^{-1}$ . Furthermore,  $x^R$  contains  $x^{-1}$  and so  $\mathcal{R}^{sym}(M, H, L, 90)$  holds relative to  $(R, \{x, y, y^{-1}\})$ . So we can use Construction 3.1, to obtain a symmetric  $M$ -Cayley homogeneous factorisation  $(M, M \rtimes H, K_{360,360}, \mathcal{P})$  of index 90. Furthermore,  $C_L(z)$  is not maximal in  $L$  and so if we let  $R'$  be any overgroup of  $R$  in  $H$  and  $k = |H : R'|$  we see that  $\mathcal{R}(M, H, L, k)$  holds relative to  $(R', X)$ . Thus we can also construct symmetric homogeneous factorisations  $(M, M \rtimes H, K_{360,360}, \mathcal{P}')$  of indices 45, 15 and 10.

We now show that these two examples are the only examples which arise when  $M$  is an almost simple group with socle  $L = A_n$  and  $H = \text{Inn}_L(M)$ .

**Theorem 5.4** *Let  $M$  be an almost simple group with socle  $L = A_n$ ,  $n \geq 5$ ,  $s = |M : L|$ ,  $t = |L| = n!/2$  and let  $H = \text{Inn}_L(M)$ . Then there is a partition  $\mathcal{P}$  of the arc set of  $K_{s \times t}$  such that  $(M, M \rtimes H, K_{s \times t}, \mathcal{P})$  is a homogeneous factorisation of index  $k$  if and only if  $(M, L, k) = (S_5, A_5, 10)$ , or  $(M_{10}, A_6, k)$ , with  $k \in \{10, 15, 45, 90\}$ .*

PROOF. By Theorem 1.2, there is a partition  $\mathcal{P}$  of the arc set of  $K_{s \times t}$  such that  $(M, M \rtimes H, K_{s \times t}, \mathcal{P})$  is a homogeneous factorisation of index  $k$  if and only if  $\mathcal{R}(M, H, L, k)$  holds. We have already seen in Example 5.2 that when  $n = 5$ ,  $\mathcal{R}(M, H, L, k)$  holds if and only if  $k = 10$ . We also saw in Example 5.3 that if  $M = M_{10}$  and  $L = A_6$  then  $\mathcal{R}(M, H, L, k)$  holds if and only if  $k = 90, 45, 15$  or  $10$ . Thus it remains to show that if  $n \geq 6$  and  $\mathcal{R}(M, H, L, k)$  holds then  $n = 6$  and  $M = M_{10}$ .

Suppose that  $M = S_n$ ,  $n \geq 6$  and  $\mathcal{R}(M, H, L, k)$  holds. Then there exists an index  $k$  subgroup  $R$  of  $L$  such that every centraliser of an element of  $M \setminus L$  is conjugate to a subgroup of  $R$ . Let  $x = (1, 2) \in M \setminus L$ . Then  $C_L(x) = (\text{Sym}\{1, 2\} \times \text{Sym}\{3, \dots, n\}) \cap L \cong S_{n-2}$ , the stabiliser in  $L$  of a 2-subset and  $C_L(x)$  is a maximal subgroup of  $L$ . Thus  $R$  is the centraliser of some transposition  $(a, b)$ . Next let  $y = (1, 2)(3, 4)(5, 6) \in M \setminus L$ . Then  $C_L(y) = (C_{S_6}((1, 2)(3, 4)(5, 6)) \times \text{Sym}\{7, \dots, n\}) \cap L$ . However,  $C_L(y)$  contains the element  $(1, 3, 5)(2, 4, 6)$  which does not centralise a transposition  $(a, b)$  where either  $a$  or  $b$  belongs to  $\{1, 2, \dots, 6\}$ . Hence  $n \geq 8$  and  $a, b > 6$ . Now  $C_L(y)$  contains  $\text{Alt}\{7, 8, \dots, n\}$  which only centralises a transposition if  $n = 8$ . Thus  $n = 8$  and the only transposition which can possibly be centralised by  $C_L(y)$  is  $(7, 8)$ . Now let  $z = (5, 6, 7, 8)$ . Then  $C_L(z)$  contains  $(1, 2, 3, 4)(5, 6, 7, 8)$  which does not centralise any transposition in  $S_8$ . Hence no suitable  $R$  exists.

When  $n > 6$ , the only almost simple groups with socle  $A_n$  are  $S_n$  and  $A_n$ . For  $n = 6$  there are four overgroups of  $L = A_6$ , these being  $S_6, \text{PGL}(2, 9), M_{10}$  and  $\text{Aut}(A_6)$ . Thus it remains to eliminate  $M = \text{PGL}(2, 9)$  and  $\text{Aut}(A_6)$ . If  $M = \text{Aut}(A_6)$ , then  $S_6 \leq M$  and the argument in the previous paragraph shows that this is not possible. Thus  $M = \text{PGL}(2, 9)$ . A quick computer check using MAGMA [3] shows that this does not give rise to any suitable groups  $R$  and so we are done. ■

## 6 Examples using Frobenius groups

We begin with an example that starts with a cyclic group.

**Example 6.1** Let  $p$  be a prime, and  $d$  and  $s$  be a positive integers such that  $s$  divides  $p^{d-1}(p-1)$  if  $p$  is odd or  $s$  divides  $2^{d-2}$  if  $p = 2$ . Let  $L = \langle a \rangle \cong C_{p^d}$  and  $M = L \rtimes C_s \leq L \rtimes \text{Aut}(L)$ . Let  $\sigma$  be the automorphism of  $M$  given by conjugation by  $a$ . Then  $C_M(\sigma) = L$ . Let  $t = p^d$ . Then by Corollary 1.3, there exists a partition  $\mathcal{P}$  of the arc set of  $K_{s \times t}$  such that  $(M, M \rtimes \langle \sigma \rangle, K_{s \times t}, \mathcal{P})$  is an  $M$ -Cayley homogeneous factorisation of index  $p$ .

We can also construct several interesting examples of homogeneous factorisations using Frobenius groups. A transitive permutation group  $G$  on a set  $\Omega$  is a *Frobenius group* if  $G$  is not regular and every element of  $G$  which is not the identity fixes at most one point. The set of fixed point free elements of  $G$  together with the identity form a normal subgroup of  $G$  called the *Frobenius kernel*. See for example [5, Section 3.4]. We now have the following construction using an arbitrary Frobenius group.

**Example 6.2** Let  $M$  be a Frobenius group on a set  $\Omega$  with Frobenius kernel

$L$ . Then we can identify  $\Omega$  with  $L$  such that  $L$  acts by right multiplication and  $M = L \rtimes M_1$  such that  $M_1$ , the stabiliser of  $1 \in L$ , acts on  $\Omega = L$  by conjugation. The nontrivial elements of  $L$  are fixed point free on  $\Omega$  while all the elements of  $M \setminus L$  fix exactly one point. Let  $\Gamma = \text{Cay}(M, M \setminus L)$ . Then  $\Gamma \cong K_{|M_1| \times |L|}$ .

Let  $H$  be the group of inner automorphisms of  $M$  given by conjugation by the elements of  $L$ . Let  $x \in M \setminus L$ . Since  $M$  is a Frobenius group with kernel  $L$  it follows that  $x$  fixes a unique element of  $\Omega$ , say  $\omega$ . Now suppose that there exists  $l \in L$  such that  $l^{-1}xl = x$ . Then  $x$  also fixes  $\omega^l$ , and so  $\omega^l = \omega$ . Since the nontrivial elements of  $L$  fix no points of  $\Omega$ , this implies that  $l = 1$ . Thus no nontrivial element of  $H$  fixes a point of  $M \setminus L$ . Hence  $H \cong L$  and  $H$  acts semiregularly on  $M \setminus L$ . It follows that  $\mathcal{R}(M, H, L, k)$  holds relative to  $(R, X)$  for any index  $k$  subgroup  $R$  of  $H$  and set  $X$  of orbit representatives for  $H$  on  $M \setminus L$ .

We also have the following construction using a Frobenius kernel.

**Example 6.3** Let  $G$  be a Frobenius group with Frobenius kernel  $M$  such that  $M$  is not a minimal normal subgroup of  $G$ . Let  $L$  be a nontrivial normal subgroup of  $G$  properly contained in  $M$ . Then  $G = M \rtimes G_1$  and  $G_1$  acts by conjugation on  $M$  such that each element of  $G_1$  fixes only the identity element. Thus  $G_1$  acts semiregularly on  $M \setminus L$ . It follows that  $\mathcal{R}(M, G_1, L, k)$  holds relative to  $(R, X)$  for any index  $k$  subgroup  $R$  of  $G_1$  and set  $X$  of orbit representatives for  $G_1$  on  $M \setminus L$ .

## 7 A product construction

Suppose that  $(M_1, G_1, K_{s_1 \times t_1}, \mathcal{P}_1)$  and  $(M_2, G_2, K_{s_1 \times t_2}, \mathcal{P}_2)$  are homogeneous factorisations of index  $k$  such that  $G_1^{\mathcal{P}_1}$  and  $G_2^{\mathcal{P}_2}$  are permutationally isomorphic. We would like to be able to use these two homogeneous factorisations to construct a homogeneous factorisation of index  $k$  of  $K_{s_1 s_2 \times t_1 t_2}$ . We do this in the following construction. This construction is similar to the product constructions given in [9].

**Construction 7.1** Let  $l, k \geq 2$  be positive integers and let  $H$  be a transitive subgroup of the symmetric group  $\text{Sym}(k)$ . For each  $i = 1, \dots, l$ , let  $(M_i, G_i, K_{s_i \times t_i}, \mathcal{P}_i)$  be a homogeneous factorisation of index  $k$  such that  $G_i^{\mathcal{P}_i}$  is permutationally isomorphic to  $H$ . We construct a homogeneous factorisation  $(M, G, K_{s \times t}, \mathcal{P})$  of index  $k$  where  $s = s_1 s_2 \dots s_l$ ,  $t = t_1 t_1 \dots t_l$  and  $M = M_1 \times \dots \times M_l$ .

First note that the vertex set of each graph  $\Gamma_i = K_{s_i \times t_i}$  can be viewed as

the set  $S_i \times T_i$  where  $|S_i| = s_i$  and  $|T_i| = t_i$ . Two vertices  $(u_{i1}, w_{i1})$  and  $(u_{i2}, w_{i2})$  such that  $u_{i1}, u_{i2} \in S_i$  and  $w_{i1}, w_{i2} \in T_i$  are adjacent if and only if  $u_{i1} \neq u_{i2}$ . Hence the  $s_i$  parts of size  $t_i$  of pairwise nonadjacent vertices are the sets of all vertices with a given first coordinate. Let  $S = S_1 \times \cdots \times S_l$  and  $T = T_1 \times \cdots \times T_l$ . Define  $\Gamma$  to be the graph with vertex set  $S \times T$  such that two vertices  $(\mathbf{u}_1, \mathbf{w}_1)$  and  $(\mathbf{u}_2, \mathbf{w}_2)$  are adjacent if and only if  $\mathbf{u}_1 \neq \mathbf{u}_2$ . Then  $\Gamma \cong K_{s \times t}$ . Viewing  $S \times T$  as  $(S_1 \times T_1) \times \cdots \times (S_l \times T_l)$ , we see that the group  $G_1 \times \cdots \times G_l$  acts on  $V\Gamma$  in its natural product action and preserves adjacency. Hence  $G_1 \times \cdots \times G_l \leq \text{Aut}(\Gamma)$ .

For each  $i = 1, \dots, l$ ,  $G_i^{\mathcal{P}_i}$  is permutationally isomorphic to  $H$ . Hence there exist a homomorphism  $\phi_i : G_i \rightarrow H$  and a labelling  $\mathcal{P}_i = \{(P_i)_1, \dots, (P_i)_k\}$  such that for each  $j = 1, \dots, k$  and  $g \in G_i$ ,  $((P_i)_j)^g = (P_i)_{j\phi_i(g)}$ . Note that  $\phi_i(M_i) = 1$  for each  $i$ . Now let  $G$  be the subgroup of  $G_1 \times \cdots \times G_l$  given by

$$\{(g_1, g_2, \dots, g_l) \mid \phi_1(g_1) = \phi_2(g_2) = \cdots = \phi_l(g_l)\}.$$

Then  $M \leq G \leq \text{Aut}(\Gamma)$ .

We now define a partition  $\mathcal{P} = \{P_1, \dots, P_k\}$  of the arc set of  $\Gamma$ . We place the arc  $((\mathbf{u}_1, \mathbf{w}_1), (\mathbf{u}_2, \mathbf{w}_2))$  in  $P_j$  if and only if, for the first coordinate  $i$  such that  $u_{i1} \neq u_{i2}$ , the arc  $((u_{i1}, w_{i1}), (u_{i2}, w_{i2}))$  of  $\Gamma_i$  lies in  $(P_i)_j$ . (Here  $u_{ji}$  denotes the  $i^{\text{th}}$  coordinate of  $\mathbf{u}_j$ .) This defines a partition of  $A\Gamma$  since  $((\mathbf{u}_1, \mathbf{w}_1), (\mathbf{u}_2, \mathbf{w}_2))$  is an arc if and only if  $\mathbf{u}_1 \neq \mathbf{u}_2$ . For each  $(g_1, \dots, g_l) \in G$ , we have that for all  $i, m = 1, \dots, l$ , and  $j = 1, \dots, k$ ,  $((P_i)_j)^{g_i} = ((P_m)_j)^{g_m}$ . Hence  $M$  fixes each  $P_j$  setwise and  $G$  transitively permutes the set  $\mathcal{P}$ . Thus  $(M, G, \Gamma, \mathcal{P})$  is a homogeneous factorisation of index  $k$ . If each  $(M_i, G_i, \Gamma_i, \mathcal{P}_i)$  is symmetric then so is  $(M, G, \Gamma, \mathcal{P})$ .

## 8 Circulant homogeneous factorisations of $K_{s \times t}$

Observe that a graph  $\Gamma$  on  $n$  vertices, admitting an  $M$ -circulant homogeneous factorisation  $\mathcal{F} = (M, G, \Gamma, \mathcal{P})$ , may be identified with a Cayley graph  $\text{Cay}(M, S)$ , with  $M$  the additive group of the finite ring  $\mathbb{Z}_n$  and  $S$  a subset of  $M \setminus \{0\}$ . Furthermore, the group  $G$  is a semidirect product  $M \rtimes H$  with  $M$  acting on itself by addition, and  $H$  a subgroup of the group  $\mathbb{Z}_n^*$  of units of the ring  $\mathbb{Z}_n$ , acting on  $M$  by multiplication.

In Subsection 8.1 we shall characterise all  $M$ -circulant homogeneous factorisations of complete multipartite graphs in a special case, where  $G$  is the largest possible group, that is, the group  $\text{Hol}(M) = M \rtimes \mathbb{Z}_n^*$ . Combining these homogeneous factorisations with the product construction in Construction 7.1, we prove Theorem 1.5 in Subsection 8.2.

We start with the following useful observation, which is a straightforward consequence of the definition of condition  $\mathcal{R}$  and the fact that the action of  $H$  by conjugation on itself is trivial in the case of an abelian group  $H$ .

**Lemma 8.1** *Let  $M$  be a group, let  $H \leq \text{Aut}M$ , let  $L$  be an  $H$ -invariant subgroup of  $M$ , and let  $k \geq 2$  an integer. If  $H$  is abelian, then  $\mathcal{R}(M, H, L, k)$  holds if and only if there exists an index  $k$  subgroup  $R$  of  $H$  which contains  $H_x$  for all  $x \in M \setminus L$ . Moreover,  $\mathcal{R}^{\text{sym}}(M, H, L, k)$  holds if and only if there exists an index  $k$  subgroup  $R$  of  $H$  which contains  $H_x$  for each  $x \in M \setminus L$ , and such that for each  $x \in M \setminus L$  satisfying  $x^{-1} \in x^H$ , also  $x^{-1} \in x^R$ .*

A crucial role in our analysis will be played by the action of the group of units  $\mathbb{Z}_n^*$  on the elements of  $\mathbb{Z}_n$  by multiplication. For a subgroup  $H \leq \mathbb{Z}_n^*$  and an element  $x \in \mathbb{Z}_n$ , let  $H_x$  denote the stabiliser of  $x$  in  $H$  acting on  $\mathbb{Z}_n$  by multiplication. Note that  $H_x = \{g \in H, x(g-1) = 0\}$ . Further, for an element  $x \in \mathbb{Z}_n$ , let  $\langle x \rangle$  denote the ideal of  $\mathbb{Z}_n$  generated by  $x$ . Finally, elements of the set  $\{0, 1, \dots, n-1\} \subseteq \mathbb{Z}$  will conveniently be considered either as integers or as elements of  $\mathbb{Z}_n$  where the meaning is clear from the context. The following lemma reveals some properties of the action of  $\mathbb{Z}_n^*$  on  $\mathbb{Z}_n$ .

**Lemma 8.2** *Let  $n$  be a positive integer with prime power factorisation  $n = p_1^{d_1} p_2^{d_2} \dots p_r^{d_r}$ , let  $H$  be a subgroup of  $\mathbb{Z}_n^*$  acting on  $\mathbb{Z}_n$  by multiplication, and let  $x$  be an element of  $\mathbb{Z}_n$  of additive order  $m$ . Then  $H_x = (1 + \langle m \rangle) \cap H$ . Moreover, if  $m = p_1^{f_1} p_2^{f_2} \dots p_r^{f_r}$ , let  $I(m) = \{i \mid 1 \leq i \leq r, f_i > 0\}$  and  $\Pi = \prod_{i \in I(m)} (p_i - 1) p_i^{f_i - 1}$ . Then the following hold.*

- (i) *If  $H = \mathbb{Z}_n^*$ , then the index of  $H_x$  in  $H$  is  $\Pi$  and the quotient group  $H/H_x$  is isomorphic to the direct product  $Q_1 \times Q_2 \times \dots \times Q_r$ , where*
  - (a)  *$Q_i$  is a trivial group, if  $f_i = 0$ , or  $f_i = 1$  and  $p_i = 2$ ,*
  - (b)  *$Q_i$  is isomorphic to the cyclic group  $C_{(p_i-1)p_i^{f_i-1}}$ , if  $p_i$  is odd and  $f_i \geq 1$ , or  $p_i = 2$  and  $f_i = 2$ ,*
  - (c)  *$Q_i$  is isomorphic to a direct product  $C_2 \times C_{2^{f_i-2}}$ , if  $p_i = 2$  and  $f_i \geq 3$ .*
- (ii) *If  $H$  is a proper subgroup of  $\mathbb{Z}_n^*$ , then the index of  $H_x$  in  $H$  divides  $\Pi$  and the quotient group  $H/H_x$  is isomorphic to a subgroup of  $Q_1 \times Q_2 \times \dots \times Q_r$ , where  $Q_i$  are as in part (i), and all subgroups can occur.*

**PROOF.** Since the additive order of  $x \in \mathbb{Z}_n$  is  $m$ ,  $xl = 0$  for some  $l \in \mathbb{Z}$  if and only if  $m$  divides  $l$ . But then  $xg = x$  for some  $g \in \mathbb{Z}_n$  if and only if  $g-1 \in \langle m \rangle$  (where  $m$  is now viewed as an element of  $\mathbb{Z}_n$ ), implying that  $H_x = (1 + \langle m \rangle) \cap H$ .

Let us first assume that  $H = \mathbb{Z}_n^*$ . There exists a natural isomorphism

$$\phi: \mathbb{Z}_n \rightarrow \mathbb{Z}_{p_1^{d_1}} \oplus \mathbb{Z}_{p_2^{d_2}} \oplus \dots \oplus \mathbb{Z}_{p_r^{d_r}},$$

which maps  $\mathbb{Z}_n^*$  to  $\mathbb{Z}_{p_1^{d_1}}^* \times \mathbb{Z}_{p_2^{d_2}}^* \times \dots \times \mathbb{Z}_{p_r^{d_r}}^*$ . Note that all elements of the same

additive order in  $\mathbb{Z}_n$  belong to the same  $\mathbb{Z}_n^*$ -orbit, and since  $\mathbb{Z}_n^*$  is abelian, they have the same stabiliser in  $\mathbb{Z}_n^*$ . Thus we may assume that

$$\phi(x) = (p_1^{d_1-f_1}, p_2^{d_2-f_2}, \dots, p_r^{d_r-f_r}) \in \mathbb{Z}_{p_1^{d_1}} \oplus \mathbb{Z}_{p_2^{d_2}} \oplus \dots \oplus \mathbb{Z}_{p_r^{d_r}}.$$

Then  $\phi(H_x) = F_1 \times F_2 \times \dots \times F_r$ , where  $F_i \leq \mathbb{Z}_{p_i^{d_i}}^*$  is the stabiliser of  $p_i^{d_i-f_i} \in \mathbb{Z}_{p_i^{d_i}}^*$ . Clearly, if  $Q_i$  denotes the quotient group  $\mathbb{Z}_{p_i^{d_i}}^*/F_i$ , then  $H/H_x$  is isomorphic to  $Q_1 \times Q_2 \times \dots \times Q_r$ . We have shown above that  $F_i = (1 + \langle p_i^{f_i} \rangle) \cap \mathbb{Z}_{p_i^{d_i}}^*$ . Hence, if  $f_i = 0$  then  $F_i = \mathbb{Z}_{p_i^{d_i}}^*$ , and so  $Q_i$  is trivial. If  $f_i > 0$ , then each element in  $1 + \langle p_i^{f_i} \rangle$  is coprime with  $p$  and so  $F_i = 1 + \langle p_i^{f_i} \rangle$ . In this case the index of  $F_i$  in  $\mathbb{Z}_{p_i^{d_i}}^*$  is  $|\mathbb{Z}_{p_i^{d_i}}^*|/p_i^{d_i-f_i} = (p_i-1)p_i^{f_i-1}$ . If  $p_i = 2$  and  $f_i = 1$ , then the index of  $F_i$  in  $\mathbb{Z}_{p_i^{d_i}}^*$  is 1, so  $Q_i$  is trivial. If  $p_i = 2$  and  $f_i = 2$ , then the index of  $F_i$  in  $\mathbb{Z}_{p_i^{d_i}}^*$  is 2, so  $Q_i$  is cyclic. If  $p_i$  is odd, then  $Q_i$  is cyclic because  $\mathbb{Z}_{p_i^{d_i}}^*$  is cyclic. Finally, if  $p_i = 2$  and  $f_i \geq 3$ , then  $\mathbb{Z}_{p_i^{d_i}}^*$  is isomorphic to a direct product of the group  $C_2 = \langle -1 \rangle$  with the group  $C_{2^{d_i-2}} = 1 + \langle 4 \rangle$ . Recall that  $F_i = 1 + \langle 2^{f_i} \rangle$ . Since  $f_i \geq 3$ ,  $F_i$  is contained in  $1 + \langle 4 \rangle = C_{2^{d_i-2}}$ . Hence  $Q_i \cong C_2 \times C_{2^{f_i-2}}$ . This completes the proof of part (i). To prove part (ii) observe that for any subgroup  $H$  of  $\mathbb{Z}_n^*$ , the quotient group  $H/H_x = H/(H \cap (\mathbb{Z}_n^*)_x)$  is isomorphic to  $(\mathbb{Z}_n^*)_x H / (\mathbb{Z}_n^*)_x$ , and can thus be viewed as a subgroup of  $\mathbb{Z}_n^*/(\mathbb{Z}_n^*)_x$ . ■

### 8.1 $M$ -Circulant homogeneous factorisations with $G = \text{Hol}(M)$

In this subsection we analyse homogeneous factorisations  $(M, M \rtimes H, K_{s \times t}, \mathcal{P})$  where  $M$  is a cyclic group and  $H$  its full automorphism group. We begin with a construction of subgroups  $L \leq M$  and  $R \leq H$ , such that  $\mathcal{R}(M, H, L, k)$  holds relative to  $(R, X)$  for some  $k$  and  $X$ .

Let  $p$  be a prime,  $m$  a positive integer not divisible by  $p$  and  $n = p^d m$  for some integer  $d \geq 1$ . Let  $M$  denote the additive group of the finite ring  $\mathbb{Z}_n$ , let  $H$  be its automorphism group viewed as the group of units  $\mathbb{Z}_n^*$  acting on  $\mathbb{Z}_n$  by right multiplication, and let  $L = \langle p^e \rangle$  for some integer  $e$  such that  $1 \leq e \leq d$ . Further, let  $K = (1 + \langle p^{d-e+1} \rangle) \cap H$ , and let  $R$  be any subgroup of index  $k \geq 2$  in  $H$  containing  $K$ . Note that by Lemma 8.2,  $|H : K| = (p-1)p^{d-e}$  and there exists such an  $R$  of index  $k$  for any  $k$  dividing  $(p-1)p^{d-e}$ . Finally, let  $X$  be any system of representatives of  $H$ -orbits on  $M \setminus L$ .

**Proposition 8.3** *With the notation above,  $\mathcal{R}(M, H, L, k)$  holds relative to  $(R, X)$ . Moreover,  $\mathcal{R}^{\text{sym}}(M, H, L, k)$  holds relative to  $(R, X)$  if and only if  $R$  contains  $-1 \in \mathbb{Z}_n^*$ . In particular, if  $k$  is odd, then  $\mathcal{R}^{\text{sym}}(M, H, L, k)$  holds relative to  $(R, X)$ .*

**PROOF.** By Lemma 8.2,  $K = (1 + \langle p^{d-e+1} \rangle) \cap H$  is the vertex stabiliser of the element  $mp^{e-1} \in \mathbb{Z}_n$  in the group  $H$ , and the index of  $K$  in  $H$  is  $(p-1)p^{d-e}$ .

Therefore it suffices to show that  $K$  contains the vertex stabiliser  $H_x$  for each  $x \in M \setminus L$ . Observe that  $x \in M \setminus L$  if and only if the (additive) order of  $x$  is divisible by  $p^{d-e+1}$ . Therefore, by Lemma 8.2,  $x \in M \setminus L$  if and only if  $H_x = (1 + \langle tp^{d-e+1} \rangle) \cap H$  for some  $t \in \mathbb{Z}_n$ . But since  $\langle tp^{d-e+1} \rangle \subseteq \langle p^{d-e+1} \rangle$ , this shows that  $H_x \subseteq K$  for each  $x \in M \setminus L$ . We now prove that  $\mathcal{R}^{sym}(M, H, L, k)$  holds relative to  $(R, X)$  if and only if  $R$  contains  $-1 \in \mathbb{Z}_n^*$ . Observe first that since  $H$  contains the element  $-1$  each  $H$ -orbit is self-inverse. In view of Lemma 8.1 we need to prove that for each  $x \in M \setminus L$  the  $R$ -orbit of  $x$  is self-inverse. If  $-1 \in R$ , then the latter clearly holds. Conversely, if for each  $x \in M \setminus L$  the  $R$ -orbit of  $x$  is self-inverse, then also the  $R$ -orbit of the element  $1 \in M \setminus L$  is self-inverse. But then  $-1 \in R$ . Observe that if  $k = [H : R]$  is odd, then  $R$  contains all involutions of  $H$ . In particular,  $-1 \in R$ .  $\blacksquare$

Recall that for an integer  $n$ ,  $n_p$  denotes the largest power of  $p$  which divides  $n$ .

**Corollary 8.4** *Let  $\Gamma = K_{s \times t}$  be a complete multipartite graph with  $s = p^e$  for some prime  $p$ , let  $M$  be a cyclic group of order  $n = st$ ,  $H = \text{Aut}(M)$  and  $G = M \rtimes H$ . Finally, let  $k \geq 2$  be an integer.*

(i) *If  $k$  divides  $(p-1)t_p$ , then there exists a partition  $\mathcal{P}$  of  $A\Gamma$  such that  $(M, G, \Gamma, \mathcal{P})$  is a homogeneous factorisation of index  $k$ . Furthermore, unless  $p = 2$  and  $k = t_p > 2$ , such a factorisation exists with  $G^{\mathcal{P}} \cong C_k$ .*

(ii) *If  $2k$  divides  $(p-1)t_p$ , then there exists a partition  $\mathcal{P}$  of  $A\Gamma$  such that  $(M, G, \Gamma, \mathcal{P})$  is a symmetric homogeneous factorisation of index  $k$ . Furthermore,  $G^{\mathcal{P}} \cong C_k$ .*

PROOF. Let  $L = \langle p^e \rangle$  and  $\Gamma = \text{Cay}(M, M \setminus L) \cong K_{s \times t}$ . Let  $K = (1 + \langle p^{d-e+1} \rangle) \cap H$  where  $n_p = p^d$ . Note that  $t_p = p^{d-e}$ . Suppose first that  $k$  divides  $(p-1)t_p$ . By Lemma 8.2, the index of  $K$  in  $H$  is equal to  $(p-1)p^{d-e}$  and  $H/K \cong C_{(p-1)p^{d-e}}$  unless  $p = 2$  and  $d-e \geq 2$ , in which case  $H/K \cong C_2 \times C_{2^{d-e-1}}$ . In both cases  $H/K$  is abelian and so we can take  $R$  to be an index  $k$  subgroup of  $H$  containing  $K$  and  $X$  to be any set of orbit representatives for  $H$  on  $M \setminus L$ . Then by Proposition 8.3,  $\mathcal{R}(M, H, L, k)$  holds relative to  $(R, X)$ . Hence Theorem 1.2 implies that there exists a partition  $\mathcal{P}$  of  $A\Gamma$  such that  $(M, G, \Gamma, \mathcal{P})$  is a homogeneous factorisation of index  $k$ . Since  $H$  is abelian,  $G^{\mathcal{P}} \cong H/R$ . If  $p$  is odd, or  $p = 2$  and  $d-e = 1$  (so that  $k = t_p = 2$ ), then  $H/K$  is cyclic and so  $G^{\mathcal{P}} \cong C_k$ . If  $p = 2$  and  $d-e \geq 2$ , then  $H/K \cong C_2 \times C_{2^{d-e-1}}$ . Not all sections of  $H/K$  in this case are cyclic. However, if  $k \neq t_2 = 2^{d-e}$  then we can choose  $R$  so that  $G^{\mathcal{P}} \cong H/R$  is cyclic. Thus part (ii) is proved.

Now assume that  $2k$  divides  $(p-1)t_p = (p-1)p^{d-e}$ . By Lemma 8.2,  $K$  is the stabiliser of the element  $p^{e-1} \in \mathbb{Z}_n$  in the group  $H$ . Hence  $-1 \in K$  if and only if  $n = 2^e$ . If  $n$  were equal to  $2^e$ , then  $s = n$  and  $k = 1$ ; a contradiction.

Thus  $-1 \notin K$ . By Proposition 8.3,  $\mathcal{R}(M, H, L, k)$  holds relative to  $(R, X)$  if and only if  $R$  contains  $-1$ . Hence  $R$  contains the group  $\langle K, -1 \rangle$  which has index  $\frac{1}{2}(p-1)p^{d-e}$  in  $H$ . Furthermore, since  $\mathbb{Z}_{2^d}^* \cong C_2 \times C_{2^{d-2}}$  where the  $C_2$  is generated by  $-1$ , it follows that  $G^{\mathcal{P}} \cong H/R$  is always cyclic. ■

The following proposition shows that all homogeneous factorisations  $(M, M \rtimes H, K_{s \times t}, \mathcal{P})$ , where  $M$  is a cyclic group and  $H$  its full automorphism group, arise in the context of Proposition 8.3.

**Proposition 8.5** *Let  $M$  be the additive group of  $\mathbb{Z}_n$ , let  $H = \text{Aut}M$ , and let  $L, k, R$ , and  $X$  be such that  $\mathcal{R}(M, H, L, k)$  holds relative to  $(R, X)$ . Then  $L = \langle p^e \rangle$  where  $p$  is a prime and  $p^e$  divides  $n$ , and  $R$  is a subgroup of  $H$  containing  $K = (1 + \langle p^{d-e+1} \rangle) \cap H$ . In particular,  $k$  divides  $(p-1)p^{d-e}$ , where  $d$  is the largest integer such that  $p^d$  divides  $n$ .*

PROOF. Since  $L$  is a subgroup of a cyclic group it is itself cyclic and thus  $L = \langle s \rangle$  for some divisor  $s$  of  $n$ , viewed as an element of  $\mathbb{Z}_n$ . Let  $p$  be any prime dividing  $s$  and let  $d$  be the largest integer such that  $p^d$  divides  $n$ . Set  $m = n/p^d$ . If  $s$  is not a power of  $p$ , then as all elements of  $L$  are multiples of  $s$ ,  $p^d$  and  $m$  (viewed as elements of  $M$ ) belong to  $M \setminus L$ . By Lemma 8.2, their stabilisers in  $H$  are  $H_{p^d} = (1 + \langle m \rangle) \cap H$  and  $H_m = (1 + \langle p^d \rangle) \cap H$ . Since the ideals  $\langle m \rangle$  and  $\langle p^d \rangle$  of  $\mathbb{Z}_n$  are coprime, we have  $(1 + \langle m \rangle)(1 + \langle p^d \rangle) = 1 + \langle m \rangle + \langle p^d \rangle = \mathbb{Z}_n$ . So, for each  $h \in \mathbb{Z}_n$  there exist  $a \in 1 + \langle m \rangle$  and  $b \in 1 + \langle p^d \rangle$ , such that  $h = ab$ . Moreover, if  $h \in \mathbb{Z}_n^*$ , also  $a, b \in H$ , which shows that the group  $H$  is generated by the stabilisers  $H_m$  and  $H_{p^d}$ . Lemma 8.1 then implies that  $R = H$ , which is a contradiction. Therefore,  $L = \langle p^e \rangle$  for some  $e \leq d$ . In this case it follows, as in the proof of Proposition 8.3, that  $K$  is the group generated by all vertex stabilisers  $H_x$ ,  $x \in M \setminus L$ , and by Lemma 8.1,  $R$  contains  $K$ . ■

We wrap up this subsection with an example of a symmetric homogeneous factorisation  $(M, M \rtimes H, K_{3 \times 6}, \mathcal{P})$  of index 3, where  $M$  is a cyclic group of order 18 and  $H = \text{Aut}(M)$ , exemplifying the general construction given in the proof above.

**Example 8.6** Let  $s = 3$ ,  $t = 6$ ,  $n = st = 18$ ,  $k = 3$ ,  $M = \mathbb{Z}_n$  and  $H = \mathbb{Z}_n^*$ . Further, let  $L = \langle 3 \rangle = \{0, 3, 6, 9, 12, 15\}$ , let  $K = (1 + \langle 9 \rangle) \cap H = \{1\}$ , and let  $R = \langle -1, K \rangle = \{1, 17\}$ . Choose a set  $X$  of representatives of  $H$ -orbits on  $M \setminus L$  to be  $X = \{1, 10\}$ . As in Construction 3.1 let  $S_1 = \{xg \mid x \in X, g \in R\} = \{1, 8, 10, 17\}$ , and  $S_i = S_1^{h_i}$ , where  $\{h_1, h_2, h_3\} = \{1, 5, 7\}$  is a system of coset representatives of  $R$  in  $H$ , that is  $S_2 = \{4, 5, 13, 14\}$ ,  $S_3 = \{2, 7, 11, 16\}$ . Observe that  $\{S_1, S_2, S_3\}$  is a partition of  $S = M \setminus L$ . Finally, let  $\Gamma = \text{Cay}(M, S)$ , and let  $P_i$  be the arc set of  $\Gamma_i = \text{Cay}(M, S_i)$ , for  $i \in \{1, 2, 3\}$ . In view of Proposition 3.2,  $(M, M \rtimes H, \Gamma, \{P_1, P_2, P_3\})$  is a symmetric homogeneous factorisation of index 3. The three factors of the homogeneous

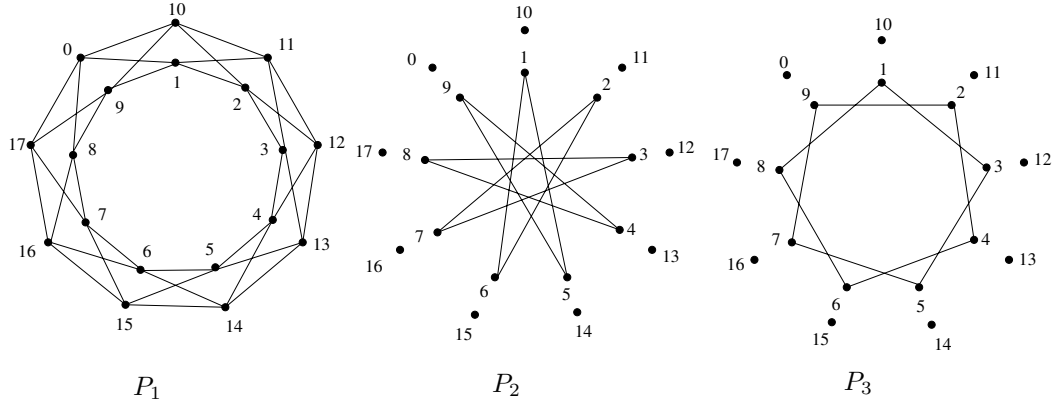


Fig. 1. The homogeneous factorisation resulting from Example 8.6 using  $X = \{1, 10\}$

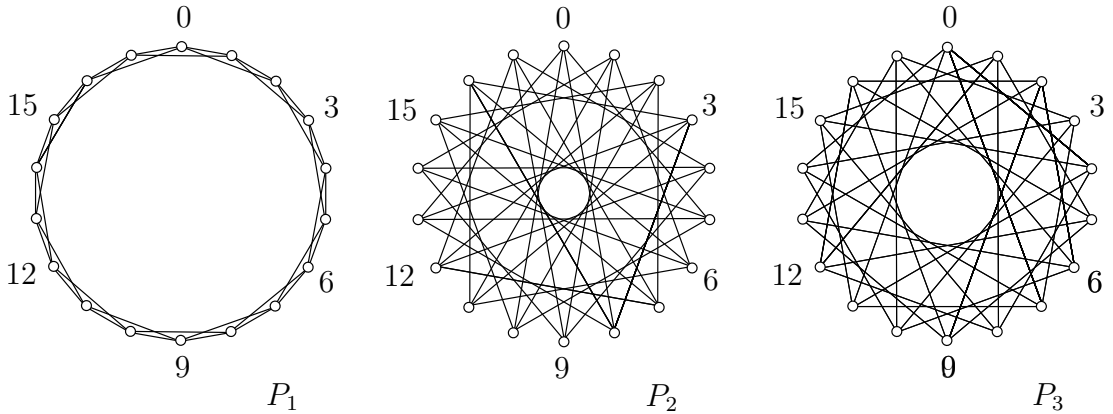


Fig. 2. The homogeneous factorisation resulting from Example 8.6 using  $X = \{1, 2\}$

factorisation are given in Figure 1. Not all edges are included for  $P_2$  and  $P_3$ . In these cases, each edge indicates a copy of  $K_{2,2}$  between the two corresponding sets of size two. From Figure 1, it is evident that the factors are isomorphic to the lexicographic product  $C_9[\overline{K_2}]$ , and therefore admit a common arc-transitive group of automorphisms isomorphic to  $\text{Aut}(K_2) \text{ wr } \text{Aut}(C_9) \cong C_2 \text{ wr } D_{18}$  (see for example [14, Theorem 6.13]). Hence  $(\overline{M}, \overline{M} \rtimes H, \Gamma, \{P_1, P_2, P_3\})$  is also a homogeneous factorisation of index 3.

We note that the choice of  $X$  is crucial to the existence of  $\overline{M}$ . If we had chosen  $X = \{1, 2\}$  instead of  $X = \{1, 10\}$ , then  $P_1$  would be the 9-antiprism (see for example [25, p270]) and the factorisation would be as in Figure 2.

## 8.2 Proof of Theorem 1.5

Before beginning the proof of Theorem 1.5, we prove the following useful proposition. As usual, for a positive integer  $n$  and a prime  $p$ ,  $n_p$  denotes the largest power of  $p$  dividing  $n$ .

**Proposition 8.7** *Let  $(M, G, K_{s \times t}, \mathcal{P})$  be an  $M$ -circulant homogeneous factorisation of index  $k$  and let  $p$  be a prime dividing  $s$ . Further, let  $M_p$  be the unique Sylow  $p$ -subgroup of  $M$ . Then there exists a partition  $\mathcal{B}_p$  of the arc set of  $K_{s_p \times t_p}$  and a group  $G_p$  containing  $M_p$  such that  $(M_p, G_p, K_{s_p \times t_p}, \mathcal{B}_p)$  is an  $M_p$ -circulant homogeneous factorisation of index  $k$  with  $G^{\mathcal{P}}$  permutationally isomorphic to  $(G_p)^{\mathcal{B}_p}$ . Furthermore, if  $(M, G, K_{s \times t}, \mathcal{P})$  is symmetric then so is  $(M_p, G_p, K_{s_p \times t_p}, \mathcal{B}_p)$ .*

PROOF. Let  $\Gamma = K_{s \times t}$ . Now  $M_p \triangleleft G$  and so the set of orbits of  $M_p$  on  $V\Gamma$  forms a system of imprimitivity for  $G$ . Let  $B$  be an  $M_p$ -orbit on  $V\Gamma$ . Now  $B$  consists of  $s_p$  parts of size  $t_p$  and the subgraph  $\Gamma'$  of  $\Gamma$  induced on  $B$  is the complete multipartite graph  $K_{s_p \times t_p}$ .

For each  $P_i \in \mathcal{P}$ , let  $B_i = P_i \cap A\Gamma'$ . As  $\Gamma'$  contains edges, at least one  $B_i$  is nonempty. Also  $M$  fixes each  $P_i$  setwise, and so  $M_p$  fixes each  $B_i$  setwise. Let  $G_p = G_B^B$ , the permutation group induced by  $G_B$  on the set  $B$ . Since  $M_p$  acts faithfully on  $B$ , we have  $M_p \triangleleft G_p$  and note that  $M_p$  is cyclic. Since  $G = G_B M$ , it follows that  $G_B^{\mathcal{P}}$  is transitive. For each  $g \in G_B$ ,  $(B_i)^g = (P_i)^g \cap A\Gamma'$  and so  $G_p$  acts transitively on  $\mathcal{B}_p = \{B_1, \dots, B_k\}$  and hence each  $B_i$  is nonempty, that is,  $\mathcal{B}_p$  is a  $G_p$ -invariant partition of  $A\Gamma'$  of size  $k$ . Thus  $(M_p, G_p, K_{s_p \times t_p}, \mathcal{B}_p)$  is an  $M_p$ -circulant homogeneous factorisation of index  $k$ . Furthermore,  $(G_p)^{\mathcal{B}_p}$  is permutationally isomorphic to  $G^{\mathcal{P}}$ . If  $(M, G, K_{s \times t}, \mathcal{P})$  is symmetric, it follows from the definition of the  $B_i$  that  $(M_p, G_p, K_{s_p \times t_p}, \mathcal{B}_p)$  is also symmetric. ■

Note that if  $p$  is a prime dividing  $n$  which does not divide  $s$ , then the subgraph of  $\Gamma$  induced on an orbit of a Sylow  $p$ -subgroup of  $M$  contains no edges.

We can now prove Theorem 1.5.

PROOF. Let  $s, t$  and  $k$  be integers such that  $t \geq 1$  and  $s, k > 1$ , and let  $\Gamma = K_{s \times t}$ . Suppose  $n = st$  has prime power factorisation  $p_1^{d_1} p_2^{d_2} \dots p_l^{d_l}$ , and  $s = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$  for some  $r \leq l$ . When  $s$  is even we let  $p_1 = 2$ . We shall first prove the sufficiency of the conditions in Theorem 1.5 for the existence of an appropriate factorisation.

When  $s$  is a power of 2 and  $k$  divides  $2^{d_1 - e_1}$ , the existence of a homogeneous factorisation follows from Corollary 8.4.

Assume next that we have one of the following:

- (1)  $s$  is odd and  $k$  divides  $(p_i - 1)p^{d_i - e_i}$  for each  $i = 1, 2, \dots, l$ , or
- (2)  $s$  is even but not a power of two, and, either (I)  $k = 2$  and  $t$  is even, or (II)  $k = 2^f \geq 4$ ,  $2k$  divides  $2^{d_1 - e_1}$ , and  $k$  divides  $(p_i - 1)$  for each odd prime  $p_i$  dividing  $s$

Let  $s_1 = p_1^{e_1}$  and  $t_1 = p_1^{d_1 - e_1} p_{r+1}^{d_{r+1}} \dots p_l^{d_l}$ . Also for each  $i = 2, 3, \dots, r$ , let  $s_i = p_i^{e_i}$  and  $t_i = p_i^{d_i - e_i}$ . Then letting  $\Gamma_i = K_{s_i \times t_i}$ ,  $n_i = s_i t_i$ ,  $M_i = \mathbb{Z}_{n_i}$ ,  $G_i = M_i \rtimes \text{Aut}(M_i)$ , we have from Corollary 8.4 that for each  $i$ , there exists a homogenous factorisation  $(M_i, G_i, \Gamma_i, \mathcal{P}_i)$  of index  $k$  such that  $G_i^{\mathcal{P}_i} \cong C_k$ . Then using Construction 7.1, we obtain a homogeneous factorisation  $(M, G, \Gamma, \mathcal{P})$  of index  $k$  where  $M = M_1 \times \dots \times M_l$ . Since each  $M_i$  is cyclic and  $(|M_i|, |M_j|) = 1$  for  $i \neq j$ , it follows that  $M$  is cyclic of order  $st$  and so we have found an appropriate factorisation.

Assume now that the condition in part (ii) of Theorem 1.5 holds, that is,  $2k$  divides  $(p_i - 1)p_i^{d_i - e_i}$  for each  $i = 1, \dots, l$ . Let  $s_1 = p_1^{e_1}$  and  $t_1 = p_1^{d_1 - e_1} p_{r+1}^{d_{r+1}} \dots p_l^{d_l}$ . Also for  $i = 2, 3, \dots, r$ , let  $s_i = p_i^{e_i}$  and  $t_i = p_i^{d_i - e_i}$ . Then letting  $\Gamma_i = K_{s_i \times t_i}$ ,  $n_i = s_i t_i$ ,  $M_i = \mathbb{Z}_{n_i}$  and  $G_i = M_i \rtimes \text{Aut}(M_i)$ , we have from Corollary 8.4 that for each  $i$ , there exists a symmetric homogeneous factorisation  $(M_i, G_i, \Gamma_i, \mathcal{P}_i)$  of index  $k$  such that  $G_i^{\mathcal{P}_i} \cong C_k$ . Then using Construction 7.1, we obtain a symmetric homogeneous factorisation  $(M, G, \Gamma, \mathcal{P})$  of index  $k$  where  $M = M_1 \times \dots \times M_l$  is a cyclic group of order  $n$ .

It remains to show the necessity of the conditions in Theorem 1.5 for the existence of an  $M$ -circulant homogeneous factorisation. Let  $(M, G, \Gamma, \mathcal{P})$  be an  $M$ -circulant homogeneous factorisation of index  $k \geq 2$ . First we prove Theorem 1.5 for the case where  $s = p^e > 1$  and  $t = p^f \geq 1$  for some prime  $p$ . Let  $n = st = p^{e+f}$ . We may assume without loss of generality that  $M$  is the additive group of the ring  $\mathbb{Z}_n$ , and that  $\Gamma = \text{Cay}(M, S)$ , where  $S$  is the set-theoretical complement of the ideal  $\langle s \rangle \triangleleft \mathbb{Z}_n$ . Further, we may assume that  $G = M \rtimes H$  for some subgroup  $H \leq \mathbb{Z}_n^*$ , where  $M$  acts on itself by addition and  $H$  acts on  $M$  by multiplication. Let  $x = p^{e-1} \in M$ . Observe that the (additive) order of  $x$  is  $p^{f+1}$  and  $x \in S$ . By Lemma 8.2, the index of  $H_x$  in  $H$  divides  $(p-1)p^f$ . On the other hand, since  $x \in S$ , by Theorem 1.2 and Lemma 8.1, there exists a subgroup  $R$  of index  $k$  in  $H$ , which contains  $H_x$ . Thus,  $k$  divides the index of  $H_x$  in  $H$ . This shows that  $k$  divides  $(p-1)p^f$ . In particular, if  $p = 2$  then Theorem 1.5(i)(a) holds and if  $p$  is odd then Theorem 1.5(i)(b) holds. Furthermore, by Lemma 8.2, if  $p = 2$ , then  $G^{\mathcal{P}}$  is a section of  $C_2 \times C_{2^{f-1}}$  of order  $k$ , while if  $p$  is odd then  $G^{\mathcal{P}}$  is cyclic of order  $k$ .

Suppose now that in addition  $(M, G, K_{s \times t}, \mathcal{P})$  is symmetric. Then by Theorem 1.2 and Lemma 8.1, for every  $x \in S$  satisfying  $x^{-1} \in x^H$  then also  $x^{-1} \in x^R$ . Hence either  $|H|$  is odd or  $-1 \in R$ . When  $|H|$  is odd, then  $k$  is also odd and we have seen that  $k$  divides  $(p-1)p^f$ . Thus  $p \neq 2$  and so  $p-1$  is even. This implies that  $2k$  divides  $(p-1)p^f$ . On the other hand note that if  $-1 \in H_x$  then  $2p^e = 0$ . Thus  $p = 2$  and  $t = 1$ . As  $k$  divides  $(p-1)p^f$ , this contradicts  $k > 1$ . Hence  $H_x$  does not contain  $-1$  and so  $R$  is an index  $k$  subgroup of  $H$  containing  $\langle R, -1 \rangle$ . Thus  $k$  divides  $\frac{1}{2}(p-1)p^f$ . This implies that Theorem 1.5(ii) holds when  $p$  is a prime power. Furthermore, since  $R$  contains  $-1$  it follows that  $G^{\mathcal{P}} \cong H/R$  is a cyclic group of order  $k$ , even when  $p = 2$ .

It remains to prove Theorem 1.5 when  $s$  is not a prime power. For each prime  $p$  dividing  $s$ , let  $s_p$  denote the largest power of  $p$  dividing  $s$  and  $t_p$  denote the largest power of  $p$  dividing  $t$ . Then by Proposition 8.7, there exists an  $M_p$ -circulant homogeneous factorisation  $(M_p, G_p, K_{s_p \times t_p}, \mathcal{B}_p)$  of index  $k$  such that  $(G_p)^{\mathcal{B}_p}$  is permutationally isomorphic to  $G^{\mathcal{P}}$ . Each  $s_p$  and  $t_p$  are powers of  $p$  and  $M_p$  is cyclic. Thus we can now apply the necessary conditions of Theorem 1.5 to each of these homogeneous factorisations. Hence if  $s$  is odd then for every prime  $p$  dividing  $s$ , it follows that  $k$  divides  $(p-1)p^f$ , that is Theorem 1.5(i)(b) holds. If  $s$  is even and not a power of 2, then since we require that each  $(G_p)^{\mathcal{B}_p}$  is permutationally isomorphic to  $G^{\mathcal{P}}$ , it follows that each  $G^{\mathcal{B}_p}$  is cyclic of order  $k$ . Hence  $k$  divides  $(p-1)t_p$  for each odd prime  $p$  and  $C_2 \times C_{t_2/2}$  has a cyclic section of order  $k$ . Thus  $2k$  divides  $t_2$  if  $t_2 \geq 4$  and otherwise  $k = 2$  and  $t_2 = 2$ . Hence, for each odd prime  $p$  dividing  $s$ , since  $k$  is a power of 2 we have  $\gcd(k, t_p) = 1$  and so  $k$  divides  $(p-1)$ . Thus Theorem 1.5(i)(c) holds. Furthermore, if  $(M, G, K_{s \times t}, \mathcal{P})$  is symmetric then each  $(M_p, G_p, K_{s_p \times t_p}, \mathcal{B}_p)$  is a symmetric homogeneous factorisation such that  $(G_p)^{\mathcal{B}_p}$  is cyclic of order  $k$ . Hence it follows that  $2k$  divides  $(p-1)p^f$  for each prime  $p$  dividing  $s$ . Thus Theorem 1.5(ii) holds. ■

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