Transitive decompositions of graphs

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A *decomposition* of a graph is a partition of the edge set. One can also look at partitions of the arc set but in this talk we restrict our attention to edges. If each part of the decomposition is a spanning subgraph then we call the decomposition a *factorisation* and the parts are called *factors*. Decompositions are especially interesting when the subgraphs induced by each part are pairwise isomorphic. Such decompositions are known as *isomorphic decompositions*. Decompositions of graphs have been widely studied and much attention has been paid to determining when a given graph can be decomposed into copies of a certain subgraph, for example, cycles or 1-factors. See for example [2, 8, 9, 10, 16].

A special class of decompositions is transitive decompositions. A *G-transitive decomposition* of a graph Γ is a decomposition which is invariant under some group G of automorphisms of Γ such that G acts transitively on the set of parts of the decomposition. This class of decompositions has been widely studied in many different guises. A *partial linear space* is a set of points and a set of subsets of the point set called *lines*, such that each pair of points is contained in at most one line. Given a decomposition of a graph into complete subgraphs, we can form a partial linear space with point set the set of vertices and line set the set of parts of the decomposition. Since each edge lies in only one part, every pair of points lies in at most one line, and so we indeed have a partial linear space. If the decomposition is G-transitive, then the partial linear space is G-line-transitive, and if G is also transitive on the set of arcs of the graph then the partial line space is G-flag-transitive. Conversely, given a G-line-transitive partial linear space we can construct a G-transitive decomposition of the collinearity graph of the partial linear space, that is, of the graph with vertices the set of points such that two points are adjacent if they lie on the same line. In the special case where the original graph is a complete graph then we have a linear space, that is, every two points lie on a unique line. Flag-transitive linear spaces were classified in [3].

If G is an arc-transitive group of automorphisms of the complete graph K_n then G acts 2-transitively on a set of size n. Cameron and Korchmáros [4] have determined all factorisations of complete graphs into 1-factors with a 2-transitive automorphism group, while Sibley [15] has extended this to a classification of all G-transitive decompositions of complete graphs such that G is 2-transitive. Sibley calls such decompositions *2-transitive edge-coloured graphs*. Recently all G-transitive decompositions of graphs with G rank 3 of product action [1] have been characterised.

A special class of transitive decompositions are *homogeneous factorisations*. These are G-transitive decompositions such that the kernel of the action of G on the partition is vertex-transitive. This forces all parts of the decomposition to be spanning subgraphs and so we do indeed have a factorisation. Homogeneous factorisations of complete graphs were first introduced in [13] as a generalisation of vertex-transitive self-complementary graphs. The concept was extended to arbitrary graphs in [11].

The **Johnson graph** $J(n,k)$ is the graph with vertex set the set of k-subsets of an n-set such that two vertices are adjacent if they intersect in a $(k-1)$-set. Since $J(n,k) \cong J(n,n-k)$ we always assume that $2k \leq n$. If $2k < n$ then $\mathrm{Aut}(J(n,k)) = S_n$ while when $n = 2k$ we have $\mathrm{Aut}(J(n,k)) = S_n \times S_2$.

Cuaresma studied homogeneous factorisations of Johnson graphs in her PhD thesis [5]. In particular, she proved that there are no homogeneous factorisations of $J(n,k)$ for $k \geq 4$ and showed that homogeneous factorisations of $J(n,3)$ exist if and only if $n = 8$ or $n = 2^l + 1$ for l a power of an odd prime. Moreover, all such homogeneous factorisations were determined. The homogeneous factorisations of $J(n,2)$ were also completely determined, except for a couple of unresolved cases where G is an affine group. When G is not affine, the only homogeneous factorisations of $J(n,2)$ have index two and occur for $n - 1 = q \equiv 1 \pmod 4$ and G a 3-transitive subgroup of $\mathrm{P\Gamma L}(2,q)$.

Despite the scarcity of homogeneous factorisations of Johnson graphs there are three infinite families of transitive decompositions of $J(n,k)$.

Construction 1. *Let X be an n-set. For each $(k-1)$-subset Y of X, let P_Y be the complete subgraph of $J(n,k)$ of size $n-k+1$ induced on the set of k-subsets containing Y. Since the symmetric group S_n acts transitively on the set of all $(k-1)$-subsets of X, it follows that*

$$\mathcal{P}_\cap = \{P_Y \mid Y \text{ a } (k-1)\text{-subset of } X\}$$

is an S_n-transitive decomposition of $J(n,k)$.

Construction 2. *Let X be an n-set. For each $(k+1)$-subset W of X, let Q_W be the complete subgraph of $J(n,k)$ of size $k+1$ induced on the set of k-subsets contained in W. Then*

$$\mathcal{P}_\cup = \{Q_W \mid W \text{ a } (k+1)\text{-subset of } X\}$$

is an S_n-transitive decomposition of $J(n,k)$.

Construction 3. *Let X be an n-set and $\{a,b\} \subseteq X$. Then*

$$M_{\{a,b\}} = \{\{\{a\} \cup Y, \{b\} \cup Y\}\} \mid Y \text{ a } (k-1)\text{-subset of } X \backslash \{a,b\}\}$$

is a matching with $\binom{n-2}{k-1}$ edges. Since S_n acts 2-transitively on X it follows that

$$\mathcal{P}_\ominus = \{M_{\{a,b\}} \mid \{a,b\} \subseteq X\}$$

is an S_n-transitive decomposition.

Constructions 1 and 2 were first pointed out to us by Michael Orrison and Construction 1 is used in [14] for the analysis of unranked data. After my talk at the conference, Misha Klin brought to my attention that Constructions 1 and 2 were used in [12] to aid in determining maximal subgroups of the symmetric groups.

Recently Devillers, Praeger, Li and the speaker have been involved in a project to determine all G-transitive decompositions of the Johnson graphs $J(n,k)$ with G arc-transitive [6]. We will now outline some of the results in the case where $G \leqslant S_n$. The case where $n=2k$ and $G \not\leqslant S_n$ is dealt with in [6]. We first need the possibilities for G. Given a subset A of a set X, we use $\overline{A}$ to denote the complement of A in X.

Proposition 4. *[5, Proposition 3.2] Let $\Gamma = J(n,k)$ and $G \leqslant S_n$. The graph Γ is G-arc transitive if and only if G is k-homogeneous and, for A any k-subset, G_A is transitive on $A \times \overline{A}$.*

Corollary 5. *If $G \leqslant S_n$ is $(k+1)$-transitive, then Γ is G-arc transitive. If Γ is G-arc transitive, then $G \leqslant S_n$ is $k-$ and $(k+1)$-homogeneous.*

Suppose that $\mathcal{P}$ is a G-transitive decomposition of Γ such that $G^{\mathcal{P}}$ is imprimitive. Then there is a partition $\mathcal{P}'$ of the edge-set of Γ refined by $\mathcal{P}$ and preserved by G such that $G^{\mathcal{P}'}$ is primitive. Thus we may restrict our attention to studying only those G-transitive decompositions for which G acts primitively on the edge-partition. We call such decompositions *G-primitive decompositions.*

The first group to look at is $G = S_n$. We have the following classification.

Theorem 6. *[6] The S_n-primitive decompositions of $J(n,k)$ are:*

- $\mathcal{P}_{\cap}$,
- $\mathcal{P}_{\cup}$ *for* $n \neq 2k+2$,
- $\mathcal{P}_{\ominus}$ *for* $(n,k) \neq (4,2)$,
- *a partition with parts isomorphic to* $2K_{k+1}$ *for* $n = 2k+2$,
- *a partition with parts isomorphic to* $6K_2$ *for* $(n,k) = (6,3)$, *and*
- *a partition with parts isomorphic to* C_4 *for* $(n,k) = (4,2)$.

We note that the partition with parts isomorphic to $2K_{k+1}$ arises since for $n = 2k+2$, the stabiliser of a $(k+1)$-subset is not maximal in S_n and so $\mathcal{P}_{\cup}$ is not an S_n-primitive decomposition in this case. Similarly, the partition with parts isomorphic to C_4 arises as the stabiliser of a 2-set is not maximal in S_4.

If $G = A_n$ then we find two further primitive decompositions.

Theorem 7. *[6] The A_n-primitive decomposition of $J(n,k)$ which are not preserved by S_n are:*

- *a partition with parts isomorphic to* C_5 *for* $(n,k) = (5,2)$, *and*
- *a partition with Petersen graphs as parts for* $(n,k) = (6,3)$.

By Corollary 5, the only subgroups of S_n which are arc-transitive on $J(n,k)$ for $k \geq 5$ are A_n and S_n. For $k=4$ there are two additional arc-transitive groups: M_{24} when $n=24$ and M_{12} when $n=12$.

Theorem 8. *[6] The M_{24}-primitive decompositions of $J(24,4)$ are:*

- $\mathcal{P}_{\cap}$, *with parts* K_{21},

- $\mathcal{P}_{\ominus}$, *with parts* $\binom{22}{3} K_2$,
- *a partition with parts* $21K_5$,
- *a partition with parts* $J(8,4)$.

The third partition occurs as the stabiliser of a 5-subset is not maximal in M_{24} and so $\mathcal{P}_{\cup}$ is not an M_{24}-primitive decomposition. Moreover, in the case of the fourth partition, we get one part for each octad of the Witt design $S(5,8,24)$.

Theorem 9. *[6] The* M_{12}*-primitive decompositions of* $J(12,4)$ *are:*

- $\mathcal{P}_{\cap}$, *with parts* K_9,
- $\mathcal{P}_{\ominus}$, *with parts* $\binom{10}{3} K_2$,
- *a partition with parts* $66K_5$,
- *a partition with parts* $16K_2$,
- *a partition with parts* $36K_2$,
- *two partitions with parts* $12K_3$,
- *a partition with parts* $2J(6,4)$,
- *a partition with parts isomorphic to a graph on 165 vertices with valency 8 and automorphism group* M_{11}.

The third partition occurs as the stabiliser of a 5-subset is not maximal in M_{12} and so $\mathcal{P}_{\cup}$ is not an M_{12}-primitive decomposition. In the seventh case, we get one part for each pair of complementary hexads in the Witt design $S(5,6,12)$. The graph in the last decomposition is very intriguing and is studied further in [7].

The G-primitive decompositions of $J(n,2)$ and $J(n,3)$ with G arc-transitive are determined in [6]. Some of the interesting ones include:

- $(n,k) = (23,3)$, $G = M_{23}$ and parts $J(7,3)$.
- $(n,k) = (11,3)$, $G = M_{11}$ and parts $J(5,2)$.
- $(n,k) = (11,3)$, $G = M_{11}$ and parts a graph on 55 vertices, valency 6 and automorphism group $\mathrm{PSL}(2,11)$.
- $(n,k) = (22,2)$, $G = M_{22}$ and parts $J(6,2)$.
- $(n,k) = (2^d,2)$, $G = \mathrm{AGL}(d,2)$ and parts $2^{d-2}K_{2,2,2}$.
- $(n,k) = (q+1,2)$, $G = \mathrm{PGL}(2,q)$ and parts $J(q_0,2)$ where $q = q_0^r$.

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