Semiregular automorphisms of vertex-transitive graphs

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Semiregular automorphisms

A semiregular automorphism of a graph is a nontrivial automorphism, all of whose cycles have the same length.

Existence is equivalent to a fixed point free element of prime order.

Circulants and metacirculants

A Cayley graph for a cyclic group is called a circulant.

A generator of the cyclic group is semiregular with one cycle.

A metacirculant is a graph with a vertex-transitive group $\langle \rho, \sigma \rangle$ of automorphisms where

- ρ is semiregular with m cycles of length n,
- σ normalises ρ and cyclically permutes the orbits of ρ such that σ^m has at least one fixed vertex.

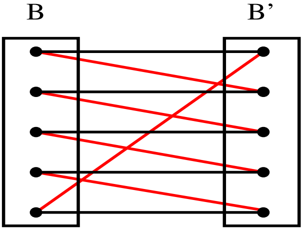
Quotienting

Let Γ be a graph with semiregular automorphism g.

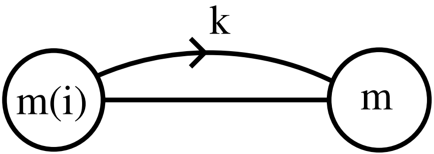
Γ/g is the quotient graph of Γ with respect to the orbits of $\langle g \rangle$.

The subgraph induced on each $\langle g \rangle$-orbit is a circulant.

If B, B' are adjacent orbits in Γ/g then any edge between a vertex of B and B' can be spun under g to give a complete matching.



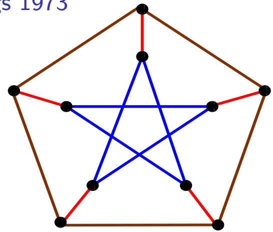
Frucht's Notation

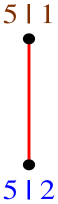


- Each circle denotes m vertices: $v_0, \ldots, v_{m-1}$ and $u_0, \ldots, u_{m-1}$.
- $m(i)$ denotes $v_j \sim v_{j+i}$ with addition modulo m.
- m denotes $\overline{K_m}$.
- $m(i, l, \ldots)$
- The undirected edge denotes $v_j \sim u_j$ for each j.
- The directed edge labelled k denotes $v_j \sim u_{j+k}$.

A slight variation

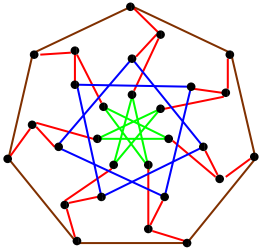
Biggs 1973

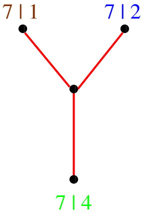




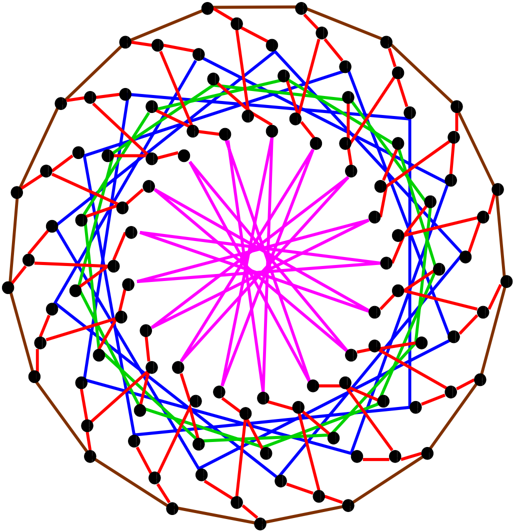
Petersen graph

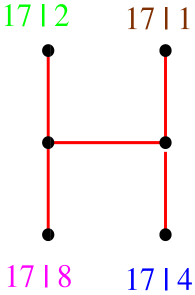
Coxeter graph





Nice representations II





Biggs-Smith graph

Uses of semiregular elements

Lift paths and cycles in quotient to Hamiltonian paths and cycles in original graph in certain cases (eg Marušič and Parsons, Alspach, Kutnar and Marušič).

Used in the enumeration of all vertex-transitive graphs of small degrees (eg McKay and Royle).

A question of Marušič

Marušič (1981): Are there any vertex-transitive digraphs with no semiregular automorphisms?

Independently posed by Jordan in 1988.

A digraph is a Cayley digraph if and only if its automorphism group contains a regular subgroup.

Does group theory help?

For a finite group G acting on a set Ω,

$$\# \text{ of orbits} = \frac{1}{|G|} \sum_{g \in G} |\operatorname{fix}(g)|$$

If G is transitive then the average number of fixed points of elements of G is 1.

1_G has $|\Omega|$ fixed points.

So if $|\Omega| > 1$, G must contain a fixed point free element.

What are they?

Fein-Kantor-Schacher (1981): G has a fixed point free element of prime power order.

We cannot replace 'prime power' with 'prime'.

For example, M_{11} acting on 12 points has no fixed point free elements of prime order.

Equivalently, has no semiregular elements.

Elusive groups

We say that a transitive permutation group is elusive if it contains no semiregular elements.

G is elusive on Ω if and only if every conjugacy class of elements of prime order meets G_ω nontrivially.

Fein-Kantor-Schacher examples

$\mathrm{AGL}(1, p^2)$, for p a Mersenne prime, acting on the set of $p(p+1)$ lines of the affine plane $\mathrm{AG}(2, p)$.

All elements of order 2 and p fix a line so action is elusive.

$\mathrm{A\Gamma L}(1, p^2)$ is also elusive in this action.

More constructions

Cameron-MG-Jones-Kantor-Klin-Marušič-Nowitz (2002)

Suppose G_1, G_2 are elusive groups on the sets Ω_1, Ω_2 respectively.

Then

- $G_1 \times G_2$ is elusive on $\Omega_1 \times \Omega_2$.
- $G_1 \operatorname{wr} G_2$ is elusive on $\Omega_1 \times \Omega_2$.
- $G_1 \operatorname{wr} S_n$ is elusive on Ω^n.

General affine construction

Cameron-MG-Jones-Kantor-Klin-Marušič-Nowitz (2002)

- V a vector space over a field of characteristic p,
- $G_1 \leqslant \mathrm{GL}(V)$ with order prime to p,
- W a subspace of V,
- $H_1 < G_1$ fixes W setwise.

The action of $V \rtimes G_1$ on the set of right cosets of $W \rtimes H_1$ is elusive if and only if

1. the images of W under G_1 cover V, and
2. every conjugacy class of elements of prime order in G_1 meets H_1.

FKS examples again

$\mathrm{AGL}(1, p^2) \cong C_p^2 \rtimes C_{p^2-1}$

Stabiliser of a line is isomorphic to $C_p \rtimes C_{p-1}$

$G_1 = C_{p^2-1} \leqslant \mathrm{GL}(2, p)$, W a 1-dimensional subspace, $H_1 = C_{p-1}$

1. G_1 acts transitively on nontrivial elements of V.
2. Only primes dividing $p^2 - 1$ are those dividing $p - 1$.

Affine construction can be used to build many soluble examples.

2-closures

The 2-closure $G^{(2)}$ of G is the group of all permutations of Ω which fix setwise each orbit of G on $\Omega \times \Omega$.

If G is 2-transitive on Ω then $G^{(2)} = \mathrm{Sym}(\Omega)$.

$G^{(2)}$ preserves all systems of imprimitivity for G.

We say that G is 2-closed if $G = G^{(2)}$.

The full automorphism group of a digraph is 2-closed.

Polycirculant conjecture

Klin (1997) extended the question of Marušič to 2-closed groups.

Polycirculant Conjecture

Every finite transitive 2-closed permutation group has a semiregular element.

In action on 12 points, $(M_{11})^{(2)} = S_{12}$.

2-closure of each known elusive group contains a semiregular element.

Early results

Marušič (1981): All transitive permutation groups of degree p^k or mp, for some prime p and $m < p$, have a semiregular element of order p.

Marušič and Scapellato (1993):

- All cubic vertex-transitive graphs have a semiregular element.
- All vertex-transitive digraphs of order $2p^2$ have a semiregular element of order p.

Primitive groups and generalisations

Let $G \leqslant \mathrm{Sym}(\Omega)$ transitive

- G is primitive if there are no nontrivial partitions of Ω preserved by G.
- G is quasiprimitive if all nontrivial normal subgroup of G are transitive.
- G is biquasiprimitive if G is not quasiprimitive and all nontrivial normal subgroups have at most two orbits.

Classifications

Theorem (MG 2003)

The only almost simple elusive groups are M_{11} *and* $M_{10} = A_6 {}^{\cdot} 2$ *acting on 12 points.*

Theorem (MG 2003)

Let G *be an elusive permutation group with a transitive minimal normal subgroup. Then* $G = M_{11} \,\mathrm{wr}\, K$ *acting on* 12^k *points with* K *a transitive subgroup of* S_k.

Classifications II

Theorem (MG-Jing Xu 2007)

Let G be a biquasiprimitive elusive permutation group on Ω. Then one of the following holds:

1. $G = M_{10}$ *and* $|\Omega| = 12$;
2. $G = M_{11} \operatorname{wr} K$ *and* $|\Omega| = 2(12^k)$, *where* $K \leqslant S_k$ *is transitive with an index two subgroup;*
3. $G = M_{11} \operatorname{wr} K$ *and* $|\Omega| = 2(12)^{k/2}$, *where* $K \leqslant S_k$ *is transitive with an index two intransitive subgroup.*

Some consequences

None of the exceptions in the previous two theorems are 2-closed.

Hence:

- All vertex-primitive graphs have a semiregular automorphism.
- All vertex-quasiprimitive graphs have a semiregular automorphism.
- All vertex-transitive bipartite graphs where only system of imprimitivity is the bipartition, have a semiregular automorphism.
- All minimal normal subgroups of a counterexample to the polycirculant conjecture must have at least three orbits.

Edge-primitive graphs

Γ a graph, G primitive on edges.

If $N \lhd G$ has at least three orbits on vertices then N would not be transitive on edges.

Thus edge-primitive graphs are vertex-quasiprimitive or vertex-biquasiprimitive.

Corollary

Every vertex-transitive, edge-primitive graph has a semiregular automorphism.

Locally quasiprimitive graphs

Γ a graph, $G \leqslant \mathrm{Aut}(\Gamma)$.

Γ is *G-locally quasiprimitive* if for all vertices v, G_v is quasiprimitive on $\Gamma(v)$.

A *2-arc* in a graph is a triple (v_0, v_1, v_2) such that $v_0 \sim v_1 \sim v_2$ and $v_0 \neq v_2$.

Γ is *2-arc transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on the set of 2-arcs in Γ.

A vertex-transitive graph is 2-arc-transitive if and only if G_v is 2-transitive on $\Gamma(v)$ for all v.

Locally quasiprimitive graphs II

Theorem (Praeger 1985)

Let Γ be a finite connected G-vertex-transitive, G-locally-quasiprimitive graph and let $1 \neq N \lhd G$. If N has at least three vertex-orbits then it is semiregular.

Corollary

Either G contains a semiregular subgroup, or G is quasiprimitive or biquasiprimitive on vertices.

Locally quasiprimitive graphs III

Theorem (MG-Jing Xu 2007)

Every vertex-transitive, locally quasiprimitive graph has a semiregular automorphism.

Corollary

Every 2-arc-transitive graph has a semiregular automorphism.

Corollary

Every arc-transitive graph of prime valency has a semiregular automorphism.

Some more recent results

Dobson-Malnič-Marušič-Nowitz (2007):

- All quartic vertex-transitive graphs have a semiregular automorphism.
- All vertex-transitive graphs of valency $p+1$ admitting a transitive $\{2, p\}$-group for p odd have a semiregular automorphism.
- There are no elusive 2-closed groups of square-free degree.

Even more results

Jing Xu (2008): All arc-transitive graphs with valency pq, p, q primes, such that $\mathrm{Aut}(\Gamma)$ has a nonabelian minimal normal subgroup N with at least 3 vertex orbits, have a semiregular automorphism.

Kutnar-Marušič (2008): If G is a transitive permutation group such that every Sylow p-subgroup is cyclic then G contains a semiregular element.

What order?

Cameron, Sheehan, Spiga (2006): All cubic vertex-transitive graphs have a semiregular automorphism of order greater than 2.

Li (2008): There is a function f satisfying $f(n) \to \infty$ as $n \to \infty$ such that a vertex-transitive automorphism group of a connected cubic graph on n vertices has a semiregular subgroup of order at least $f(n)$.

Conjectures existence of functions f_k for each valency k with same property.

True in quasiprimitive and biquasiprimitive case.

Open problems and questions

- Prove the polycirculant conjecture.
- Prove the polycirculant conjecture for arc-transitive graphs.
- Distance transitive graphs?
- Find new constructions of elusive groups.
- For which degrees do elusive groups exist? (smallest degree for which existence is unknown is 40.)
- Does the set of all such degrees have density 0?