Locally s-arc transitive graphs with two different quasiprimitive actions [⋆]

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Abstract

Previous work of the authors has shown that an important class of locally $(G,2)$-arc transitive graphs are those for which G acts faithfully and quasiprimitively on each of its two orbits on vertices. In this paper we give a complete classification in the case where the two quasiprimitive actions of G are of different types. The graphs obtained have amalgams previously unknown to the authors and involve both an almost simple 2-transitive action and an affine 2-transitive action on the neighbourhoods of vertices.

Key words: locally s–arc transitive graphs, quasiprimitive

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1 Introduction

An *s-arc* in a graph Γ is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that each v_i is adjacent to v_{i+1} while $v_i \neq v_{i+2}$. Given $G \leqslant \mathrm{Aut}(\Gamma)$ we say that Γ is *locally (G,s)-arc transitive* if for each vertex v, the stabiliser G_v acts transitively on the set of s-arcs starting at v. Provided that all vertices of Γ

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have valency at least three, a locally (G, s)-arc transitive graph is also locally $(G, s-1)$-arc transitive. Throughout this paper we assume that this is the case for Γ. If G is not vertex transitive, then a locally $(G, 2)$-arc transitive graph is a bipartite graph and the two parts Δ_1, Δ_2 of the bipartition are G-orbits.

The study of locally (G, s)-arc transitive graphs goes back to Tutte [15,16] who showed that if Γ has valency three and G is vertex transitive then $s \leq 5$. This was extended by Weiss [17] who showed that if Γ has valency at least three and G is vertex transitive then $s \leq 7$. Stellmacher [14] has proved that if G is vertex intransitive and all vertices have valency at least three then $s \leq 9$. This inequality is sharp as demonstrated by the incidence graphs of the classical generalised octagons associated with the simple groups ${}^2F_4(q)$.

In [4], the authors initiated a global analysis of locally (G, s)-arc transitive graphs for which G has two orbits on vertices. This extended the work of the third author [11] in the vertex transitive case. It was shown that the important graphs to study are those where G acts faithfully on both G-orbits and quasiprimitively on at least one. (A transitive permutation group G is *quasiprimitive* if every nontrivial normal subgroup is transitive.) In the case where G acts quasiprimitively on both orbits, the possible quasiprimitive types were studied and it was shown that either the two quasiprimitive actions are of the same type, or one is of Simple Diagonal type and the other is of Product Action type [4, Theorem 1.2]. (A description of these two actions will be given in Subsection 2.1.) We say that graphs in the latter case are of $\{\mathrm{SD}, \mathrm{PA}\}$-type. The first example of a locally 2-arc transitive graph of $\{\mathrm{SD}, \mathrm{PA}\}$-type was given in [4, Example 4.1]. Five infinite families were provided in [6, Example 4.2] while an infinite family of locally 5-arc transitive graphs of $\{\mathrm{SD}, \mathrm{PA}\}$-type was given in [5].

In this paper we give a general construction (Construction 3.3) of locally $(G, 2)$-arc transitive graphs of $\{\mathrm{SD}, \mathrm{PA}\}$-type which proves the following theorem.

Theorem 1.1 *For each simple group* $T = \mathrm{PSL}(n, q)$ *(*$n \geq 2$*),* $\mathrm{PSU}(3, q)$*,* $Ree(q)'$ *or* $Sz(q)$*, and positive integer* d *such that, if* $T = \mathrm{PSL}(n, q)$ *with* $n \geq 3$*, then* $d = 1$*, there exists a locally* $(G, 2)$*-arc transitive graph* Γ *with the following properties.*

(1) $\mathrm{soc}(G) = T^{q^d}$
(2) G *induces* $\mathrm{A\Gamma L}(d, q)$ *on the set of* q^d *simple direct factors of* $\mathrm{soc}(G)$*.*
(3) G *has two orbits* Δ_1 *and* Δ_2 *on* $V\Gamma$ *such that* G *acts quasiprimitively of type* SD *on* Δ_1 *and quasiprimitively of type* PA *on* Δ_2*.*
(4) For $v \in \Delta_1$*, we have* $G_v^{\Gamma(v)} = \mathrm{P\Gamma L}(n, q)$*,* $\mathrm{P\Gamma U}(3, q)$*,* $\mathrm{Aut}(Ree(q))$ *or* $\mathrm{Aut}(Sz(q))$*, and* $|\Gamma(v)| = (q^n - 1)/(q - 1)$*,* $q^3 + 1$*,* $q^3 + 1$ *and* $q^2 + 1$ *respectively.*

(5) For $w \in \Delta_2$, we have $G_w^{\Gamma(w)} = \mathrm{A\Gamma L}(d,q)$ and $|\Gamma(w)| = q^d$.

We construct the group G in Section 3.1 while G_v is the group L constructed in Section 3.1 and G_w is the group R constructed in Section 3.2. The amalgams (G_v, G_w, G_{vw}) were previously unknown to the authors.

In Construction 3.10 we find that certain quotients of the graphs yielded by Construction 3.3 are also of $\{\mathrm{SD}, \mathrm{PA}\}$-type. We prove that all locally $(G,2)$-arc transitive graphs of $\{\mathrm{SD}, \mathrm{PA}\}$-type can be constructed in this way.

Theorem 1.2 *Let Γ be a locally $(G,2)$-arc transitive graph of $\{\mathrm{SD}, \mathrm{PA}\}$-type. Then Γ is isomorphic to a graph arising from Construction 3.3 or to a normal quotient of such a graph as in Construction 3.10.*

An important tool in the proof of this theorem is the classification in [3, Theorem 1.1] of all codes C containing the constant code E which have a weight preserving group of automorphisms H which acts transitively on the nontrivial cosets of E in C. One consequence of Theorem 1.2 is that we can bound the degree of local s-arc transitivity for graphs of $\{\mathrm{SD}, \mathrm{PA}\}$-type.

Corollary 1.3 *Let Γ be a locally (G,s)-arc transitive graph of $\{\mathrm{SD}, \mathrm{PA}\}$-type which is not locally $(G,s+1)$-arc transitive. Then either $s = 5$ or $s \leq 3$. Moreover, there exists a locally $(G,5)$-arc transitive graph of $\{\mathrm{SD}, \mathrm{PA}\}$-type.*

This paper is set out as follows. In Section 2 we give the necessary background. We outline the global analysis of locally s-arc transitive graphs in Subsection 2.1 and describe the two quasiprimitive types crucial to this paper. In Subsection 2.2 we give an overview of constructing graphs via cosets and in Subsection 2.3 we collate some required information about 2-transitive groups. Section 3 provides the two general constructions of locally $(G,2)$-arc transitive graphs of $\{\mathrm{SD}, \mathrm{PA}\}$-type and we show that the constructions produce graphs which are at most locally $(G,5)$-arc transitive. Finally, in Section 4 we prove that all locally 2-arc transitive graphs of $\{\mathrm{SD}, \mathrm{PA}\}$-type can be obtained from Construction 3.3 or Construction 3.10.

2 Preliminaries

2.1 Quotient graphs and quasiprimitive types

First we give an outline of the global analysis initiated in [4] to which the reader is referred for the details. Let Γ be a locally $(G,2)$-arc transitive graph such that G has two orbits Δ_1 and Δ_2 on vertices and each vertex has valency at least three. Suppose that G has a nontrivial normal subgroup N which acts intransitively on Δ_1 and on Δ_2, and choose N to be a maximal such subgroup. Let Γ_N be the *quotient graph* of Γ whose vertex set is the set of N-orbits on $V\Gamma$ such that two orbits B_1 and B_2 are adjacent if there exist $v_1 \in B_1$ and $v_2 \in B_2$ such that v_1 and v_2 are adjacent in Γ. Then Γ_N is locally $(G/N, s)$-arc transitive such that G/N has two orbits on $V\Gamma_N$. Moreover, Γ is a *cover* of Γ_N, that is, if B_1 and B_2 are adjacent in Γ_N then for all $v \in B_1$ there exists a unique $u \in B_2$ which is adjacent to v in Γ (see [4, Theorem 1.1]). If G/N is not faithful on one of its orbits then Γ_N is a complete bipartite graph. Also, by the maximality of N, G/N acts quasiprimitively on at least one of its two orbits on the vertex set of Γ_N. Hence the "basic" graphs to study are those for which G acts faithfully on both orbits and quasiprimitively on at least one.

In [11] an O'Nan–Scott-like theorem for the structure of quasiprimitive groups was given by the third author. We follow the subdivision into 8 disjoint types and the notation given in [12]. An investigation of the possible quasiprimitive types for G for a locally $(G, 2)$-arc transitive graph was undertaken in [4] and it was proved that in the case where G is quasiprimitive on both orbits, either G is of the same type on both orbits and only 4 of the 8 types occur, or one action is of Simple Diagonal (SD) type and the other is of Product Action (PA) type. Moreover, it is not possible to have G acting with SD type on both orbits. We now give a description of these two important types.

Let G be a quasiprimitive group acting on a set Ω and suppose that G has a unique minimal normal subgroup $N \cong T^k$ for some finite nonabelian simple group T and positive integer $k \geq 2$. Then G is quasiprimitive of type SD if and only if for all $\alpha \in \Omega$, $N_\alpha \cong T$ and G transitively permutes the k simple direct factors of N. We now give a more explicit description of actions of SD type. Let $N = T^k$ act on the set Ω of right cosets of

$$N_\alpha = \{(t, t, \ldots, t) \mid t \in T\}$$

in N. Then $\{(t_1, t_2, \ldots, t_{k-1}, 1) \mid t_i \in T\}$ is a set of coset representatives for N_α in N and so we can identify Ω with T^{k-1}. Each element $\tau \in \mathrm{Aut}(T)$ acts on Ω via

$$(t_1, t_2, \ldots, t_{k-1}, 1)^\tau = (t_1^\tau, t_2^\tau, \ldots, t_{k-1}^\tau, 1)$$

and note that if $\tau \in \mathrm{Inn}(T)$ and is conjugation by the element t, then the action of τ is induced by $(t, t, \ldots, t) \in N$. Each $\sigma \in S_k$ also acts on Ω via

$$(t_1, t_2, \ldots, t_{k-1}, 1)^\sigma = (t_{k\sigma^{-1}}^{-1} t_{1\sigma^{-1}}, \ldots, t_{k\sigma^{-1}}^{-1} t_{(k-1)\sigma^{-1}}, 1)$$

where $t_k = 1$. Let $W = \langle N, \mathrm{Aut}(T), S_k\rangle$. Then $W \cong T^k.(\mathrm{Out}(T) \times S_k)$, W is the normaliser of N in $\mathrm{Sym}(\Omega)$ and the stabiliser in W of the coset N_α is $\mathrm{Aut}(T) \times S_k$. Each quasiprimitive group of type SD with socle T^k is equivalent to a group G acting on Ω such that $N \lhd G \leqslant W$ and G acts transitively by conjugation on the k simple direct factors of N. Such a group G is primitive of type SD if and only if G acts primitively on the k simple direct factors of N.

A quasiprimitive group G with a unique minimal normal subgroup $N = T^k$ is of type PA if and only if $N_\alpha \neq 1$ and is not isomorphic to T^l for any $l \leq k$. For a quasiprimitive group of type PA on a set Ω, there exists a partition $\mathcal{P}$ of Ω (possibly with blocks of size 1) such that G acts faithfully on $\mathcal{P}$ and preserves a product structure. Thus G is isomorphic to a subgroup of $H \operatorname{wr} S_k$, where H is an almost simple group with socle T. Furthermore, there exists $R < T$ such that N_α is a subdirect subgroup of R^k, that is, N_α projects onto R in each coordinate. Moreover, for the block B of $\mathcal{P}$ containing α, we have $N_B = R^k$.

2.2 Coset graphs

Construction 3.3 defines a graph in terms of the cosets of subgroups of a group G. We collect a few results concerning coset graphs here. See for example [4, Lemma 3.7] for proofs. For a subgroup H of a group G, we denote $[G : H] = \{Hg \mid g \in G\}$ and the coset action of G on $[G : H]$ is right multiplication. We say that H is *core-free* in G if H contains no nontrivial normal subgroups of G. For proper subgroups L, R of a group G, $Cos(G, L, R)$ is the graph with vertex set the disjoint union of $[G : L]$ and $[G : R]$ with Lx, Ry adjacent if and only if $xy^{-1} \in LR$. Note that the condition $xy^{-1} \in LR$ is equivalent to $Lx \cap Ry \neq \varnothing$.

Lemma 2.1 *For a group G with subgroups $L, R < G$ such that $L \cap R$ is core-free in G, the graph $\Gamma = Cos(G, L, R)$ has the following properties:*

(1) Γ is connected if and only if $\langle L, R \rangle = G$;
(2) $G \leqslant \mathrm{Aut}(\Gamma)$, Γ is G-edge transitive and G has two orbits $[G : L]$ and $[G : R]$ on vertices.
(3) G acts faithfully on both $[G : L]$ and $[G : R]$ if and only if both L and R are core-free.
(4) Γ is locally $(G, 2)$-arc transitive if and only if both the L-coset action on $[L : L \cap R]$ and the R-coset action on $[R : L \cap R]$ are 2-transitive.

Conversely, if Γ is a G-edge transitive but not G-vertex transitive graph, and v and w are adjacent vertices then $\Gamma \cong Cos(G, G_v, G_w)$.

2.3 2-transitive groups

We will require the following result about 2-transitive permutation groups which follows from the classification of all 2-transitive groups, see for example

Table 1
Possibilities for T and H

T	H	$\|\Omega\|$	$\|M\|$
$\mathrm{PSL}(2,q)$, for $q \geq 4$	$[q] \rtimes C_{(q-1)/(2,q-1)}$	$q+1$	q
$\mathrm{PSL}(n,q)$, for $n \geq 3$	$[q^{n-1}] \rtimes (C_{q-1} \circ \mathrm{SL}(n-1,q)).C_{(q-1,n-1)}$	$\frac{q^n-1}{q-1}$	q^{n-1}
$\mathrm{PSU}(3,q)$, for $q \geq 3$	$[q^3] \rtimes C_{(q^2-1)/(3,q+1)}$	q^3+1	q
$Ree(q)$, $q = 3^{2m+1} \geq 27$	$[q^3] \rtimes C_{q-1}$	q^3+1	q
$\mathrm{PSL}(2,8)$	$[3^2] \rtimes C_2$	28	3
$Sz(q)$, $q = 2^{2m+1} \geq 8$	$[q^2] \rtimes C_{q-1}$	q^2+1	q

[2]. We use $[n]$ to denote a group of order n.

Theorem 2.2 *Let T be the socle of an almost simple group A which acts 2-transitively on a set Ω. Suppose that $H = T_\alpha$ for some $\alpha \in \Omega$ and that H is not almost simple. Then T, H and $|\Omega|$ are given in Table 1. Furthermore, H has a unique minimal normal subgroup M and the order of M is given in Table 1.*

We make several observations.

Remark 2.3
(1) T is 2-transitive except in the case where $T = \mathrm{PSL}(2,8)$ and $|\Omega| = 28$. Here $A = \mathrm{P\Gamma L}(2,8) \cong Ree(3)$ is 2-transitive on 28 points and $T = A'$. From now on we include this case with the Ree groups.

(2) M is isomorphic to the additive group of the field $\mathrm{GF}(q)$, except when $T = \mathrm{PSL}(n,q)$ with $n \geq 3$. (Here we regard $\mathrm{PSL}(2,8)$ in line 5 as $Ree(3)'$.) In this exceptional case, M is isomorphic to the additive group of an $(n-1)$-dimensional vector space over $\mathrm{GF}(q)$.

(3) H induces $\mathrm{GF}(q)$-linear automorphisms of M.

(4) H acts transitively by conjugation on the nontrivial elements of M except in the case where $T = \mathrm{PSL}(2,q)$ for q odd. In this case, if we let $A = \mathrm{PGL}(2,q)$ then A_α acts transitively on $M \backslash \{1\}$.

(5) If $|M| = q^m$ and $B = N_{\mathrm{Sym}(\Omega)}(T)$, then M is the unique minimal normal subgroup of B_α and B_α induces $\Gamma\mathrm{L}(m,q)$ on M.

We also collect the following information in the case where $T \neq \mathrm{PSL}(n,q)$ with $n \geq 3$, that is, for the rank one groups of Lie type.

Remark 2.4 Let $A = \mathrm{PGL}(2,q)$, $Ree(q)$, $Sz(q)$ or $\mathrm{PGU}(3,q)$ and $\alpha \in \Omega$.
(1) Then $A_\alpha = O_p(A_\alpha) \rtimes P$, where $q = p^e$, $O_p(A_\alpha)$ is the largest normal

p-subgroup of A_α and $P = A_{\alpha,\alpha_2}$ for some $\alpha_2 \in \Omega\backslash\{\alpha\}$. Furthermore, the subgroup M of Theorem 2.2 is the centre of $O_p(A_\alpha)$ and $O_p(A_\alpha)$ acts regularly on $\Omega\backslash\{\alpha\}$.

(2) In each case $|M| = q$ and P induces $\mathrm{GF}(q)$-multiplication on M. Moreover, $|P| = q-1$, except for $A = \mathrm{PGU}(3,q)$, in which case $|P| = q^2-1$ and P has a subgroup of order $q+1$ which acts trivially on M. In the $\mathrm{PGU}(3,q)$ case, P is isomorphic to the multiplicative group of $\mathrm{GF}(q^2)$ and the action of P on M is given by $\lambda : x \mapsto \lambda^{q+1}x$, for all $x \in M$. See for example [10, Lemma 1.11(ii)].

(3) Let $A \neq \mathrm{PGU}(3,q)$ and $B = N_{\mathrm{Sym}(\Omega)}(T)$. Let ϕ be the Frobenius automorphism of $\mathrm{GF}(q)$, that is, ϕ raises each field element to its p^{th} power. Then ϕ defines an automorphism of A, and $B = A \rtimes \langle\phi\rangle$. Moreover, since A is 2-transitive on Ω we can choose $\alpha, \alpha_2 \in \Omega$ such that $B_{\alpha,\alpha_2} = P \rtimes \langle\phi\rangle$. The group automorphism ϕ induces the field automorphism ϕ on M.

(4) Let $A = \mathrm{PGU}(3,q)$ and $B = N_{\mathrm{Sym}(\Omega)}(T)$. Let φ be the Frobenius automorphism of $\mathrm{GF}(q^2)$, that is φ raises each field element to its p^{th} power. (Note that $\mathrm{GU}(3,q)$ consists of matrices whose elements lie in $\mathrm{GF}(q^2)$.) Then φ defines an automorphism of A, and $B = A \rtimes \langle\varphi\rangle$. Moreover, since A is 2-transitive on Ω we can choose $\alpha, \alpha_2 \in \Omega$ such that $B_{\alpha,\alpha_2} = P \rtimes \langle\varphi\rangle$. Note that we still have $|M| = \mathrm{GF}(q)$. Looking at the matrix representation for M given in [10] we see that φ^e, where $q = p^e$, induces multiplication by -1 on M.

We also have the following lemma concerning subgroups of $\mathrm{A\Gamma L}(1,q)$.

Lemma 2.5 *Let K be a subgroup of* $\mathrm{A\Gamma L}(1,q)$ *whose order is divisible by $q(q-1)$. Then K acts transitively on* $\mathrm{GF}(q)$*.*

PROOF. Let $q = p^e$ where p is prime and let N be the unique minimal normal subgroup of $\mathrm{A\Gamma L}(1,q)$. Then $N \cong C_p^e$ and note that $|\,\mathrm{A\Gamma L}(1,q)| = q(q-1)e$. If $e = 1$ then $K = \mathrm{AGL}(1,q)$ and so the result holds. Thus we assume that $e \geq 2$.

Suppose that $(p,e) \neq (2,6)$ and that if $e = 2$ then p is not a Mersenne prime. Let r be a primitive prime divisor of $p^e - 1$. Such an r exists by [18]. Then K has an element g of order r, and as r is coprime to e it follows that $g \in \mathrm{AGL}(1,q)$. Then as q is coprime to r (conjugating K by an element of $\mathrm{AGL}(d,q)$ if necessary) we may assume that $g \in \mathrm{GL}(1,q)$. Such a g normalises no proper nontrivial subgroup of N, and so $K \cap N = 1$ or $N \leqslant K$. The first is not possible as $e < q$ and q divides $|K|$ and so $N \leqslant K$. As N acts transitively on $\mathrm{GF}(q)$ it follows that so does K.

Suppose now that $e = 2$ and p is a Mersenne prime. Then q is odd and N is the unique subgroup of $\mathrm{A\Gamma L}(1,q)$ of order q. Hence $N \leqslant K$ and again we have that K is transitive. This leaves us to consider the case where $p = 2$ and $e = 6$. As 2^6 divides $|K|$ it follows that $|K \cap N| \geq 2^5$. However, K contains an element g of order 7 which we may assume belongs to $\mathrm{GL}(1, 2^6)$. Such an element does not normalise an index 2 subgroup of N and so $N \leqslant K$ and we are done.

3 Construction

In this section we give a general method for constructing locally $(G, 2)$-arc transitive graphs of type $\{\mathrm{SD}, \mathrm{PA}\}$. We start with a general construction where $\mathrm{soc}(G) = T^{q^d}$ for $T = \mathrm{PSL}(n, q)$, $\mathrm{PGU}(3, q)$, $Ree(q)'$ or $Sz(q)$, and d a positive integer such that if $T = \mathrm{PGL}(n, q)$ for $n \geq 3$ we have $d = 1$. The group induced by G by conjugation on the set of simple direct factors of $\mathrm{soc}(G)$ is $\mathrm{A\Gamma L}(d, q)$. Moreover, if $g \in G$ induces an element of $\mathrm{A\Gamma L}(d, q)$ on coordinates whose associated field automorphism is nontrivial, then it also induces a corresponding field automorphism of T on each direct factor. We construct two subgroups L and R of G with L corresponding to constant functions $f : \mathrm{GF}(q)^d \to T$ and R related to certain affine functions with domain $\mathrm{GF}(q)^d$. Then for every subgroup $K \leqslant \mathrm{A\Gamma L}(d, q)$ whose projection onto $\Gamma\mathrm{L}(d, q)$ is transitive on the set of 1-spaces of $\mathrm{GF}(q)^d$, we find a subgroup G_K of G which induces K on the simple direct factors of $\mathrm{soc}(G)$. If K is not transitive on $\mathrm{GF}(q)^d$ then G_K has a normal subgroup which is intransitive on each part of the bipartition and the quotient graph with respect to this normal subgroup is also of $\{\mathrm{SD}, \mathrm{PA}\}$ type. We now proceed with the details so that we can define G, L and R precisely.

3.1 The groups G and L

Let A be one of $\mathrm{PGL}(n, q)$, $\mathrm{PGU}(3, q)$, $Ree(q)$ or $Sz(q)$ so that A is an almost simple group with socle T. For each choice of A, there exists a set Ω such that A acts 2-transitively on Ω and for $\alpha \in \Omega$, A_α is not almost simple. Up to permutational isomorphism, the choice of Ω is unique. Note that there are two possible A with the same socle $T = \mathrm{PSL}(2, 8)$, one being $\mathrm{PGL}(2, 8)$ and the other $Ree(3) \cong \mathrm{P\Gamma L}(2, 8)$.

Let d be a positive integer with the added restriction that if $A = \mathrm{PGL}(n, q)$ for $n \geq 3$, then $d = 1$. We define the group

$$F = \{f : \mathrm{GF}(q)^d \to A\} \tag{3.1}$$

with multiplication defined pointwise, that is, $(fg)(\mathbf{a}) = f(\mathbf{a})g(\mathbf{a})$. Then $F \cong A^{q^d}$. We take $\mathrm{GF}(q)^d$ to consist of column vectors and sometimes regard each

$f \in F$ as the q^d-tuple given by the evaluation of f at each element $\mathbf{a} \in \mathrm{GF}(q)^d$. Note that $N := \mathrm{soc}(F) = \{f : \mathrm{GF}(q)^d \to T\}$, where $T = \mathrm{soc}(A)$. For each $h \in A$ we let $f_h \in F$ denote the constant function with value h.

Each $g \in \mathrm{A\Gamma L}(d,q)$ defines an automorphism σ_g of F via the action

$$f^{\sigma_g}(\mathbf{a}) = f(\mathbf{a}^{g^{-1}})$$

for all $\mathbf{a} \in \mathrm{GF}(q)^d$. Let ϕ be the Frobenius automorphism of $\mathrm{GF}(q)$. Then ϕ defines a semilinear map on $\mathrm{GF}(q)^d$ and there is an associated automorphism σ_ϕ of F.

When $A = \mathrm{PGL}(d,q)$, $Ree(q)$ or $Sz(q)$, the Frobenius automorphism ϕ defines an automorphism of A, which we will also denote by ϕ. Note that our notation does not distinguish between the field automorphism ϕ, the related automorphism of A and the semilinear map induced on $\mathrm{GF}(q)^d$. The meaning should be clear from the context. For each integer i, we define the constant map

$$\begin{aligned} f_{\phi^i} : \mathrm{GF}(q)^d &\to \mathrm{Aut}(A) \\ \mathbf{a} &\mapsto \phi^i. \end{aligned}$$

Then $(f_\phi)^{-1} = f_{\phi^{-1}}$ and f_ϕ defines an automorphism of F by

$$f^{f_\phi}(\mathbf{a}) = (f(\mathbf{a}))^{\phi}.$$

Let $\rho = f_\phi \sigma_\phi$, where multiplication is composition in $\mathrm{Aut}(F)$. Then for all $f \in F$, $g \in \mathrm{AGL}(d,q)$, and $\mathbf{a} \in \mathrm{GF}(q)^d$,

$$\begin{aligned} f^{\rho^{-1}\sigma_g\rho}(\mathbf{a}) &= (f^{\rho^{-1}\sigma_g}(\mathbf{a}^{\phi^{-1}}))^{\phi} \\ &= (f^{\rho^{-1}}((\mathbf{a}^{\phi^{-1}})^{g^{-1}}))^{\phi} \\ &= f(\mathbf{a}^{\phi^{-1}g^{-1}\phi}) \\ &= f^{\sigma_{\phi^{-1}g\phi}}(\mathbf{a}) \end{aligned}$$

and so $\rho^{-1}\sigma_g\rho = \sigma_{\phi^{-1}g\phi}$. Let

$$K_0 = \langle \rho, \sigma_g \mid g \in \mathrm{AGL}(d,q) \rangle \leqslant \mathrm{Aut}(F). \tag{3.2}$$

Then

$$\begin{aligned} \Phi : K_0 &\to \mathrm{A\Gamma L}(d,q) \\ \sigma_g &\mapsto g \\ \rho &\mapsto \phi. \end{aligned} \tag{3.3}$$

is an isomorphism from K_0 to $\mathrm{A\Gamma L}(d,q)$. Note that K_0 induces the group $\mathrm{A\Gamma L}(d,q)$ on the q^d simple direct factors of N.

Field automorphisms of $\mathrm{PGU}(3,q)$ arise from field automorphisms of $\mathrm{GF}(q^2)$ and so we need to treat this case slightly differently. Let φ be the Frobenius automorphism of $\mathrm{GF}(q^2)$. Then φ defines a field automorphism of $A = \mathrm{PGU}(3,q)$ which we also denote by φ. For each integer i, let f_{φ^i} be the constant function

$$\begin{aligned} f_{\varphi^i} : \mathrm{GF}(q)^d &\to \mathrm{Aut}(A) \\ \mathbf{a} \quad &\mapsto \quad \varphi^i. \end{aligned}$$

Then $(f_\varphi)^{-1} = f_{\varphi^{-1}}$ and f_φ defines an automorphism of F by

$$f^{f_\varphi}(\mathbf{a}) = (f(\mathbf{a}))^\varphi.$$

Furthermore, φ fixes setwise the subfield $\mathrm{GF}(q)$ of $\mathrm{GF}(q^2)$ and induces on $\mathrm{GF}(q)$ the field automorphism ϕ. Let $\rho_u = f_\varphi \sigma_\phi$, where multiplication is composition in $\mathrm{Aut}(F)$. Then for all $f \in F$, $g \in \mathrm{AGL}(d,q)$, and $\mathbf{a} \in \mathrm{GF}(q)^d$,

$$\begin{aligned} f^{\rho_u^{-1}\sigma_g\rho_u}(\mathbf{a}) &= (f^{\rho_u^{-1}\sigma_g}(\mathbf{a}^{\phi^{-1}}))^\varphi \\ &= (f^{\rho_u^{-1}}((\mathbf{a}^{\phi^{-1}})^{g^{-1}}))^\varphi \\ &= f(\mathbf{a}^{\phi^{-1}g^{-1}\phi}) \\ &= f^{\sigma_{\phi^{-1}g\phi}}(\mathbf{a}) \end{aligned}$$

and so $\rho_u^{-1}\sigma\rho_u = \sigma_{\phi^{-1}g\phi}$. Let

$$K_u = \langle \rho_u, \sigma_g \mid g \in \mathrm{AGL}(d,q)\rangle \leqslant \mathrm{Aut}(F) \tag{3.4}$$

Then

$$\begin{aligned} \Phi_u : K_u &\to \mathrm{A\Gamma L}(d,q) \\ \sigma_g &\mapsto g \\ \rho_u &\mapsto \phi \end{aligned} \tag{3.5}$$

is a homomorphism from K_u onto $\mathrm{A\Gamma L}(d,q)$ with kernel $\langle \rho_u^e \rangle$, where $q = p^e$. Note that $\rho_u^e = f_{\varphi^e}$ and so $K_u \cong 2.\,\mathrm{A\Gamma L}(d,q)$ where the extension is split if and only if e is odd. Moreover, K_u induces $\mathrm{A\Gamma L}(d,q)$ on the q^d simple direct factors of N.

We are now in a position to define G and L. Note that when $A \neq \mathrm{PGU}(3,q)$, the group K_0 of automorphism of F normalises the subgroups N and $\{f_h \mid h \in A\}$, while when $A = \mathrm{PGU}(3,q)$, the subgroups N and $\{f_h \mid h \in A\}$ are normalised by K_u. Thus when $A \neq \mathrm{PGU}(3,q)$ we let

$$G = \langle N, f_h \mid h \in A\rangle \rtimes K_0 \tag{3.6}$$

and

$$L = \{f_h \mid h \in A\} \rtimes K_0. \tag{3.7}$$

Note that $G \cong (T^{q^d}.(A/T)) \rtimes \mathrm{A\Gamma L}(d,q)$ while $L \cong A \rtimes \mathrm{A\Gamma L}(d,q)$. When $A = \mathrm{PGU}(3,q)$ we let

$$G = \langle N, f_h \mid h \in A\rangle \rtimes K_u \tag{3.8}$$

and

$$L = \{f_h \mid h \in A\} \rtimes K_u. \tag{3.9}$$

Then $G \cong (T^{q^d}.(A/T)) \rtimes (2.\,\mathrm{A\Gamma L}(d,q))$ and $L \cong A \rtimes (2.\,\mathrm{A\Gamma L}(d,q))$.

3.2 The group R

When $A = \mathrm{PGL}(d,q), Sz(q)$ or $Ree(q)$, there is a point $\alpha \in \Omega$ such that the group automorphism ϕ normalises A_α. Hence $\langle A, \phi\rangle$ acts 2-transitively on Ω with point stabiliser $\langle A_\alpha, \phi\rangle$. Similarly, when $A = \mathrm{PGU}(3,q)$ there exists $\alpha \in \Omega$ such that $\langle A, \varphi\rangle$ acts 2-transitively with point stabiliser $\langle A_\alpha, \varphi\rangle$. Now for all A, the point stabiliser A_α has a unique minimal normal subgroup M given in Table 1. In each case M is isomorphic to the additive group of an m-dimensional vector space over $\mathrm{GF}(q)$ and the group of automorphisms induced by conjugation by A_α on M is $\mathrm{GL}(m,q)$. See Remark 2.3. Note that $m = 1$ except when $A = \mathrm{PGL}(n,q)$, in which case $m = n-1$. Each $\lambda \in \mathrm{GF}(q)^*$ induces an automorphism of M, this being scalar multiplication, and we denote the image of each $l \in M$ under this automorphism by l^λ. We also define $l^0 = 1_M$ for all $l \in M$. Then for $\lambda, \mu \in \mathrm{GF}(q)$, the distributivity of scalar multiplication implies that $l^{\lambda+\mu} = l^\lambda l^\mu$. Each $h \in A_\alpha$ also induces an automorphism of M. Note that A_α acts transitively on the nontrivial elements of M and induces $\mathrm{GF}(q)$-linear automorphisms, that is, for all $h \in A_\alpha$, $\lambda \in \mathrm{GF}(q)^*$ and $l \in M$,

$$(l^\lambda)^h = (l^h)^\lambda. \tag{3.10}$$

Furthermore when $A \neq \mathrm{PGU}(3,q)$,

$$(l^\lambda)^\phi = (l^\phi)^{\lambda^\phi} \tag{3.11}$$

while when $A = \mathrm{PGU}(3,q)$

$$(l^\lambda)^\varphi = (l^\varphi)^{\lambda^\phi}. \tag{3.12}$$

Recall from Remark 2.4(4) that if $q = p^e$ then φ^e induces multiplication by -1 on M.

As M is a $\mathrm{GF}(q)$-vector space we can study linear functions $f : \mathrm{GF}(q)^d \to M$. Since composition in M is written multiplicatively, the linearity conditions become: for all $\mathbf{a}, \mathbf{b} \in \mathrm{GF}(q)^d$ and $\lambda \in \mathrm{GF}(q)$, we have $f(\mathbf{a}+\mathbf{b}) = f(\mathbf{a})f(\mathbf{b})$ and $f(\lambda\mathbf{a}) = f(\mathbf{a})^\lambda$. We have the following lemma.

Lemma 3.1 *Let*

$$Y = \langle f : \mathrm{GF}(q)^d \to M \mid f \ \textit{constant} \rangle \tag{3.13}$$

and

$$X = \langle f : \mathrm{GF}(q)^d \to M \mid f \ \textit{constant or linear} \rangle. \tag{3.14}$$

Then $Y \cong M$ and $X/Y \cong M^d$. Moreover, when $A \neq \mathrm{PGU}(3,q)$ both X and Y are normalised by K_0, while when $A = \mathrm{PGU}(3,q)$, both X and Y are normalised by K_u (as defined in (3.2) and (3.4)).

PROOF. The first assertion is trivial while the second assertion follows from the fact that the set of all linear functions $f : \mathrm{GF}(q)^d \to M$ is a set of coset representatives for Y in X. When $A \neq \mathrm{PGU}(3,q)$ the group automorphism ϕ normalises M and so K_0 is normalised by Y. Furthermore, it follows from [3, Lemma 2.3] that X is normalised by K_0 (since ρ acts on M^k in the same way that $\tau_\phi \sigma_\phi$ does there). Similar calculations show that when $A = \mathrm{PGU}(3,q)$ the group K_u normalises X and Y.

We have the following lemma.

Lemma 3.2 *Let F be as in (3.1). Then F has a subgroup*

$$F_L = \langle X, f_h \mid h \in A_\alpha \rangle \cong M^{d+1}.(A_\alpha / M).$$

If $A \neq \mathrm{PGU}(3,q)$ then F_L is normalised by K_0 while if $A = \mathrm{PGU}(3,q)$ then F_L is normalised by K_u.

PROOF. Let $h \in A_\alpha$. If $f \in X$ is a constant function then as h normalises M, f^{f_h} is also a constant function in X. If $f \in X$ is linear then for all $\mathbf{a}, \mathbf{b} \in \mathrm{GF}(q)^d$,

$$\begin{aligned} f_h^{-1} f f_h(\mathbf{a} + \mathbf{b}) &= h^{-1} f(\mathbf{a} + \mathbf{b}) h \\ &= h^{-1} f(\mathbf{a}) f(\mathbf{b}) h \\ &= f(\mathbf{a})^h f(\mathbf{b})^h \\ &= f_h^{-1} f f_h(\mathbf{a}) f_h^{-1} f f_h(\mathbf{b}) \end{aligned}$$

and for all $\lambda \in \mathrm{GF}(q)$,

$$\begin{aligned} f_h^{-1} f f_h(\lambda \mathbf{a}) &= h^{-1} f(\lambda \mathbf{a}) h \\ &= h^{-1} f(\mathbf{a})^\lambda h \\ &= (f(\mathbf{a})^\lambda)^h \\ &= (f_h^{-1} f f_h(\mathbf{a}))^\lambda. \end{aligned}$$

Hence $f_h^{-1} f f_h$ is a linear function and so f_h normalises X. For $h \in M$, $f_h \in X$ and so $\langle X, f_h \mid h \in A_\alpha \rangle \cong M^{d+1}.(A_\alpha/M)$. If $A \neq \mathrm{PGU}(3,q)$ then Lemma 3.1 implies that, K_0 normalises X and since K_0 normalises the subgroup $\{f_h \mid h \in A_\alpha\}$ it follows that K_0 normalises F_L. Similarly, when $A = \mathrm{PGU}(3,q)$, K_u normalises X, $\{f_h \mid h \in A_\alpha\}$ and F_L.

We now define the subgroup R. When $A = \mathrm{PGL}(d,q), Ree(q)$ or $Sz(q)$ let

$$R = \langle X, f_h, \mid h \in A_\alpha \rangle \rtimes K_0 \tag{3.15}$$

Note that $R \cong (M^{d+1}.(A_\alpha/M)) \rtimes \mathrm{A\Gamma L}(d,q)$. When $A = \mathrm{PGU}(3,q)$ then let

$$R = \langle X, f_h, \mid h \in A_\alpha \rangle \rtimes K_u \tag{3.16}$$

In this case we have that $R \cong (M^{d+1}.(A_\alpha/M)) \rtimes (2.\,\mathrm{A\Gamma L}(d,q))$.

3.3 The main construction

We can now give our general construction.

Construction 3.3 *We begin with the following:*

- *a 2-transitive almost simple group A on a set Ω, such that $A = \mathrm{PGL}(n,q)$, $\mathrm{PGU}(3,q)$, $Ree(q)$ or $Sz(q)$, and for $\alpha \in \Omega$, A_α has a unique minimal normal subgroup M which is elementary abelian,*
- *a positive integer d, such that if $A = \mathrm{PGL}(n,q)$ with $n \geq 3$, then $d = 1$,*

and we construct a bipartite graph $\Gamma(A,d)$.

Recall the definition of F from (3.1), and let $N = \mathrm{soc}(F) \cong T^{q^d}$, where $T = \mathrm{soc}(A)$. Let X be the group generated by the set of constant or linear functions $f : \mathrm{GF}(q)^d \to M$. Let $f_h \in F$ be the constant function with value h. If $A \neq \mathrm{PGU}(3,q)$ recall K_0 from (3.2) and define

$$\begin{aligned} G &= \langle N, f_h \mid h \in A \rangle \rtimes K_0, \\ L &= \langle f_h \mid h \in A \rangle \rtimes K_0, \ \text{ and} \\ R &= \langle X, f_h \mid h \in A_\alpha \rangle \rtimes K_0, \end{aligned}$$

while when $A = \mathrm{PSU}(3,q)$, recall K_u from (3.4) and define

$$\begin{aligned} G &= \langle N, f_h \mid h \in A\rangle \rtimes K_u, \\ L &= \langle f_h \mid h \in A\rangle \rtimes K_u, \text{ and} \\ R &= \langle X, f_h \mid h \in A_\alpha\rangle \rtimes K_u. \end{aligned}$$

We can then construct the bipartite graph

$$\Gamma(A,d) := Cos(G,L,R)$$

as defined in Subsection 2.2.

If $d = 1$ and $A = \mathrm{PGL}(2,q)$ with $q = p^e$, then the graph $\Gamma(A,1)$ is the graph $\mathcal{G}(p,e)$ constructed and studied in [5].

Let $\Gamma = \Gamma(A,d)$ as yielded by Construction 3.3. Then G has two orbits on $V\Gamma$, these being $\Delta_1 = [G : L]$ and $\Delta_2 = [G : R]$. Now

$$|\Delta_1| = |G : L| = T^{q^d - 1}$$

and G acts quasiprimitively of type SD on Δ_1. Also,

$$|\Delta_2| = |G : R| = \frac{|\Omega||T|^{q^d-1}}{|M|^d}.$$

Furthermore, G acts quasiprimitively of type PA on Δ_2.

Let v be the vertex of Γ given by the coset L. Then

$$|\Gamma(v)| = |L : L \cap R| = |A : A_\alpha| = |\Omega|,$$

and $G_v^{\Gamma(v)} = \mathrm{P\Gamma L}(n,q)$, $\mathrm{P\Gamma U}(3,q)$, $\mathrm{Aut}(Ree(q))$ or $\mathrm{Aut}(Sz(q))$. Let w be the vertex of Γ given by the coset R. Then $w \in \Gamma(v)$ and

$$|\Gamma(w)| = |R : L \cap R| = |M|^d.$$

Moreover, $G_w^{\Gamma(w)} = \mathrm{A\Gamma L}(d,q)$. Thus Γ has valency $\{|\Omega|, |M|^d\}$. Also $G_v = L$, $G_w = R$ and if $A \neq \mathrm{PGU}(3,q)$ then

$$G_{vw} = L \cap R = \{f_h \mid h \in A_\alpha\} \rtimes K_0,$$

while if $A = \mathrm{PGU}(3,q)$ we have

$$G_{vw} = L \cap R = \{f_h \mid h \in A_\alpha\} \rtimes K_u.$$

Moreover, $G = NG_{vw}$ as $A = TA_\alpha$. Thus N acts transitively on the set of edges of Γ and so by Lemma 2.1, $\Gamma \cong Cos(N, N_v, N_w)$.

Collecting this information together we have the following lemma.

Lemma 3.4 *Each graph* $\Gamma(A,d)$ *yielded by Construction 3.3 is bipartite of valency* $\{|\Omega|, |M|^d\}$ *and* G *acts quasiprimitively of type SD on* Δ_1 *and quasiprimitively of type PA on* Δ_2*. Moreover,* $G = N(L\cap R)$*, that is,* N *is transitive on edges.*

Our next lemma shows that Γ is connected. A subgroup D of $N = T^k$ is *subdirect* if D projects onto T in each of the k simple direct factors of N, and is a *full diagonal subgroup* if it is subdirect and isomorphic to T.

Lemma 3.5 *Each graph* $\Gamma(A,d)$ *yielded by Construction 3.3 is connected.*

PROOF. Let $\Gamma = \Gamma(A,d)$ and L, R, N and G be as obtained from Construction 3.3. Let $D = \langle L, R\rangle \cap N$. By Lemma 3.4, it follows that $L \cap N$ is a full diagonal subgroup of $N = T^{q^d}$ and hence D is a subdirect subgroup of N. Thus, it follows from a well known lemma, (see for example [13, p 328]), that there is a partition $\mathcal{I}$ of $\mathrm{GF}(q)^d$ with $D = \prod_{I\in\mathcal{I}} T_I$, where each T_I is a group of functions $f : \mathrm{GF}(q)^d \to T$ for which $f(\mathbf{a}) = 1$ for all $\mathbf{a} \notin I$, the image of each $\mathbf{a} \in I$ under T_I is equal to T and $T_I \cong T$. Choose $I \in \mathcal{I}$ and let $I = \{\mathbf{b}_1, \ldots, \mathbf{b}_s\}$. Now for each $i \le s$ there exists $\tau_i \in \mathrm{Aut}(T)$, with $\tau_1 = 1$, such that for each $t \in T$ there exists a unique $f \in T_I$ with $f(\mathbf{b}_i) = t^{\tau_i}$ for all $\mathbf{b}_i \in I$.

Suppose that $s \ge 2$ and let $f_h \in L$ be a constant function whose value is h, for some $h \in T$. Then for all $i = 1, \ldots, s$, we have $f^{f_h}(\mathbf{b}_i) = (t^{\tau_i})^h$. Since L normalises D, we have $f^{f_h} \in T_I$ and so it follows that $(t^{\tau_i})^h = f^{f_h}(\mathbf{b}_i) = (f^{f_h}(\mathbf{b}_1))^{\tau_i} = (t^h)^{\tau_i}$. Hence for all $h, t \in T$ we have $t^{\tau_i h} = t^{h\tau_i}$. Thus each τ_i centralises $\mathrm{Inn}(T)$ and so each $\tau_i = 1$, that is, all functions in T_I are constant on I and trivial elsewhere. Furthermore, each function in D is constant on I. However, there exists a linear function $f \in X \leqslant D$ which is not constant on I, a contradiction. Thus $s = 1$ and $N = D \leqslant \langle L, R\rangle$. As $G = NL$ it follows that $\langle L, R\rangle = G$ and so by Lemma 2.1, Γ is connected.

3.4 Local s-arc transitivity

Recall the homomorphisms Φ and Φ_u from K_0 and K_u respectively, onto $\mathrm{A\Gamma L}(d,q)$. For any $K \leqslant \mathrm{A\Gamma L}(d,q)$ we have $\Phi^{-1}(K) \leqslant K_0$ and $\Phi_u^{-1}(K) \leqslant K_u$. Thus when $A \neq \mathrm{PGU}(3,q)$, G has a subgroup

$$G_K = \langle N, f_h \mid h \in A_\alpha\rangle \rtimes \Phi^{-1}(K), \tag{3.17}$$

while when $A = \mathrm{PGU}(3,q)$, G has a subgroup

$$G_K = \langle N, f_h \mid h \in A_\alpha \rangle \rtimes \Phi_u^{-1}(K). \tag{3.18}$$

Note that in both cases, $G = G_{\mathrm{A\Gamma L}(d,q)}$. We wish to determine the largest s such that $\Gamma(A,d)$ is locally (G_K, s)-arc transitive.

For each $g \in \mathrm{A\Gamma L}(d,q)$ we can uniquely write $g = g_1 g_2$ for some translation g_1 and some $g_2 \in \Gamma\mathrm{L}(d,q)$. Since the subgroup of all translations is normal in $\mathrm{A\Gamma L}(d,q)$ we can define the projection map $\pi : \mathrm{A\Gamma L}(d,q) \to \Gamma\mathrm{L}(d,q)$ which takes $g = g_1 g_2$ to g_2. Moreover, there exists a field automorphism ϕ^i such that for all $\mathbf{a} \in \mathrm{GF}(q)^d$ and $\lambda \in \mathrm{GF}(q)$, we have $(\lambda \mathbf{a})^{g_2} = \lambda^{\phi^i} \mathbf{a}^{g_2}$. We call ϕ^i the *field automorphism associated with* g_2 and g.

We will require the following lemma.

Lemma 3.6 *Let* $A = \mathrm{PGL}(2,q)$, $\mathrm{PGU}(3,q)$, *Ree*(q) *or* $Sz(q)$, *act* 2*-transitively on a set* Ω *of size* $q+1$, q^3+1, q^3+1 *or* q^2+1 *respectively. Let* $P = A_{\alpha,\alpha_2}$ *for* $\alpha, \alpha_2 \in \Omega$ *with* $\alpha \neq \alpha_2$, *and let* M *be the centre of* $O_p(A_\alpha)$. *Let* $K \leqslant \mathrm{A\Gamma L}(d,q)$ *for* $d \geq 2$ *such that* $\pi(K)$ *acts transitively on the set of* 1*-spaces of* $\mathrm{GF}(q)^d$. *If* $A \neq \mathrm{PGU}(3,q)$ *then* $\{f_h \mid h \in P\} \rtimes \Phi^{-1}(K)$ *acts transitively on the set of nontrivial elements of* X/Y. *If* $A = \mathrm{PGU}(3,q)$ *then* $\{f_h \mid h \in P\} \rtimes \Phi_u^{-1}(K)$ *acts transitively on the set of nontrivial elements of* X/Y.

PROOF. Let $f_1, f_2 : \mathrm{GF}(q)^d \to M$ be linear functions, and recall that M is a 1-dimensional vector space over $\mathrm{GF}(q)$. Then $W_1 = \ker(f_1)$ and $W_2 = \ker(f_2)$ both have dimension $d-1$. Since $\pi(K)$ acts transitively on the set of 1-spaces, by Block's Lemma [1], it also acts transitively on the set of hyperplanes of $\mathrm{GF}(q)^d$. Hence there exists $g = g_1 g_2 \in K$ such that g_1 is a translation and $g_2 \in \Gamma\mathrm{L}(d,q)$, and $(W_1)^{g_2^{-1}} = W_2$.

Suppose first that $A \neq \mathrm{PGU}(3,q)$. Hence $(Yf_1)^{\Phi^{-1}(g)} = Y(\lambda f_2)$ for some $\lambda \in \mathrm{GF}(q)$. Now there exists $h \in P$ such that h induces the automorphism of M corresponding to scalar multiplication by λ^{-1} (Remark 2.4) and hence $(Yf_1)^{\Phi^{-1}(g) f_h} = Yf_2$. Thus $\{f_h \mid h \in P\} \rtimes \Phi^{-1}(K)$ acts transitively by conjugation on the set of nontrivial elements of X/Y.

Suppose now that $A = \mathrm{PGU}(3,q)$. Then Φ_u is not an isomorphism so $\Phi_u^{-1}(g)$ is not a unique element. Let ϕ^i be the field automorphism of $\mathrm{GF}(q)$ associated with $g_2 \in \Gamma\mathrm{L}(d,q)$. Then $\Phi_u(f_{\varphi^i}\sigma_g) = g$ and $(Yf_1)^{f_{\varphi^i}\sigma_g} = Y(\lambda f_2)$ for some $\lambda \in \mathrm{GF}(q)$. Then taking the image under a suitable element f_h, $h \in P$, it follows that $\{f_h \mid h \in P\} \rtimes \Phi_u^{-1}(K)$ acts transitively by conjugation on the set of nontrivial elements of X/Y.

We can now prove the following lemma which completes the proof of Theorem 1.1.

Lemma 3.7 *Each graph* $\Gamma(A,d)$ *yielded by Construction 3.3 is locally* $(G,2)$*-arc transitive. Furthermore, let* $K \leqslant \mathrm{A\Gamma L}(d,q)$ *such that* $\pi(K)$ *is transitive on the set of* 1*-spaces of* $\mathrm{GF}(q)^d$*. Then* $\Gamma(A,d)$ *is locally* $(G_K,2)$*-arc transitive.*

PROOF. Let $\Gamma = \Gamma(A,d)$, and let v be the vertex corresponding to the coset L and w be the vertex corresponding to the coset R. When $A \neq \mathrm{PGU}(3,q)$, the action of $(G_K)_v$ on $\Gamma(v)$ is equivalent to the action of $\langle A,\phi\rangle$ on Ω and so in this case $(G_K)_v$ acts 2-transitively on $\Gamma(v)$. When $A=\mathrm{PGU}(3,q)$, the action of $(G_K)_v$ on $\Gamma(v)$ is equivalent to the action of $\langle A,\varphi\rangle = \mathrm{P\Gamma U}(3,q)$ on Ω and so in all cases $(G_K)_v$ acts 2-transitively on $\Gamma(v)$.

Now $(G_K)_w = X(G_K)_{vw}$ and so X acts transitively on $\Gamma(w)$. Furthermore, X is abelian and $X_v = Y$. Hence as $X \lhd (G_K)_w$ we can identify $\Gamma(w)$ with X/Y such that X acts by right multiplication and $(G_K)_{vw}$ acts by conjugation. We may take the set of linear functions as a set of coset representatives for Y in X. When $A \neq \mathrm{PGU}(3,q)$, we have $(G_K)_{vw} = \{f_h \mid h \in A_\alpha\} \rtimes \Phi^{-1}(K)$ while when $A=\mathrm{PGU}(3,q)$ we have $(G_K)_{vw} = \{f_h \mid h\in A_\alpha\} \rtimes \Phi_u^{-1}(K)$.

Suppose first that $d=1$ and let $\mathbf{a}\in \mathrm{GF}(q)\backslash\{\mathbf{0}\}$. Then each linear function $f:\mathrm{GF}(q)\to M$ is determined by $f(\mathbf{a})$. Furthermore, for each $h\in A_\alpha$, let f_h be the constant function with value h. Then $f^{f_h}(\mathbf{a}) = f(\mathbf{a})^h$ and as A_α acts transitively by conjugation on the nontrivial elements of M it follows that $(G_K)_{vw}$ acts transitively on the set of nontrivial elements of X/Y and hence acts transitively on $\Gamma(w)\backslash\{v\}$.

Suppose now that $d\geq 2$. Note that $A\neq \mathrm{PGL}(n,q)$ with $n\geq 3$ in this case. Then by Lemma 3.6, when $A\neq\mathrm{PGU}(3,q)$ the subgroup $\{f_h \mid h\in P\}\rtimes\Phi^{-1}(K)$ of $(G_K)_{vw}$ acts transitively on the set of nontrivial elements of X/Y while when $A=\mathrm{PGU}(3,q)$ the subgroup $\{f_h\mid h\in P\}\rtimes\Phi_u^{-1}(K)$ acts transitively on the set of nontrivial cosets of Y in X. Hence in all cases $(G_K)_{vw}$ acts transitively on $\Gamma(w)\backslash\{v\}$ and so for all values of d, $(G_K)_w$ acts 2-transitively on $\Gamma(w)$. Thus Γ is locally $(G_K,2)$-arc transitive. Moreover, since $G = G_{\mathrm{A\Gamma L}(d,q)}$ and $\mathrm{\Gamma L}(d,q)$ acts transitively on the set of 1-spaces of $\mathrm{GF}(q)^d$, it follows that $\Gamma(A,d)$ is locally $(G,2)$-arc transitive.

We split the rest of our analysis into case where A is a rank one Lie group and the case where the rank of A is greater than one.

Lemma 3.8 *Let* $A=\mathrm{PGL}(n,q)$ *with* $n\geq 3$*. Then each graph* $\Gamma(A,1)$ *yielded by Construction 3.3 is not locally* $(G,3)$*-arc transitive.*

PROOF. Let $\Gamma = \Gamma(A,1)$, let v be the vertex corresponding to the coset L and w be the vertex corresponding to the coset R. Then there exists $u \in \Gamma(v)\backslash\{w\}$ and $\alpha_2 \in \Omega\backslash\{\alpha\}$ such that

$$G_{uvw} = \{f_h \mid h \in A_{\alpha,\alpha_2}\} \rtimes \Phi^{-1}(K).$$

Now A_{α,α_2} normalises a unique index q subgroup of $M = O_p(A_\alpha)$, where $q = p^e$, and so $\{f_h \mid h \in A_{\alpha,\alpha_2}\} \rtimes \Phi^{-1}(K)$ does not act transitively by conjugation on the set of nontrivial elements of X/Y. Hence G_{uvw} does not act transitively on $\Gamma(w)\backslash\{v\}$ and so Γ is not locally $(G,3)$-arc transitive.

We have the following proposition.

Proposition 3.9 *(1) Each graph* $\Gamma(A,d)$*, where* $A = \mathrm{PGL}(2,q)$, $\mathrm{PGU}(3,q)$, $Sz(q)$ *or Ree*(q)*, is locally* $(G_K,3)$*-arc transitive.*

(2) The graph $\Gamma(A,d)$ *is locally* $(G,4)$*-arc transitive if and only if* $A = \mathrm{PGL}(2,q)$, $d=1$ *and* q *is even. Furthermore, when* q *is even, the graph* $\Gamma(\mathrm{PGL}(2,q),1)$ *is locally* 5*-arc transitive but not locally* 6*-arc transitive.*

(3) If $\Gamma(\mathrm{PGL}(2,2^e),1)$ *is locally* $(G_K,4)$*-arc transitive then* K *acts transitively on* $\mathrm{GF}(q)$*.*

PROOF. Let $\Gamma = \Gamma(A,d)$. Choose $u \in \Gamma(v)\backslash\{w\}$ such that when $A \neq \mathrm{PGU}(3,q)$,

$$(G_K)_{uvw} = \{f_h \mid h \in P\} \rtimes \Phi^{-1}(K)$$

and when $A = \mathrm{PGU}(3,q)$,

$$(G_K)_{uvw} = \{f_h \mid h \in P\} \rtimes \Phi_u^{-1}(K).$$

Note that u corresponds to the point $\alpha_2 \in \Omega\backslash\{\alpha\}$ such that $A_{\alpha,\alpha_2} = P$. If $d=1$, then $\{f_h \mid h \in P\}$ acts transitively by conjugation on the set of nontrivial elements of X/Y and so $(G_K)_{uvw}$ acts transitively on $\Gamma(w)\backslash\{v\}$. When $d>1$, we have from Lemma 3.6 that in all cases, the group $(G_K)_{uvw}$ acts transitively on the set of nontrivial elements of X/Y. Hence for all possibilities for A, $(G_K)_{uvw}$ acts transitively on $\Gamma(w)\backslash\{v\}$. Thus $(G_K)_u$ acts transitively on the set of 3-arcs starting at u.

Let $\{\mathbf{e}_1,\ldots,\mathbf{e}_d\}$ be a basis for $\mathrm{GF}(q)^d$ such that the semilinear map ϕ fixes each $\mathbf{e}_i$. Fix $l_1 \in M\backslash\{1\}$ such that l_1 is centralised by the group automorphism ϕ of A (φ when $A = \mathrm{PGU}(3,q)$) and let $f_1 \in X$ be the linear function which maps $\mathbf{e}_1$ to l_1 and sends each $\mathbf{e}_i, i \geq 2$, to 1_M. Let $x = Lf_1 \in \Gamma(w)$. Since $O_p(A_\alpha)$ centralises M (Remark 2.4(1)), we have

$$\{f_h \mid h \in O_p(A_\alpha)\} \leqslant (G_K)_{xwv}.$$

Then as $O_p(A_\alpha)$ acts transitively on $\Omega\backslash\{\alpha\}$ (Remark 2.4), it follows that $(G_K)_{xwv}$ acts transitively on $\Gamma(v)\backslash\{w\}$. Hence $(G_K)_x$ acts transitively on the set of 3-arcs starting at x and so Γ is locally $(G_K,3)$-arc transitive. This completes the proof of part 1.

Next we investigate 4-arcs. Now $G_{uvwx}\leqslant G_{uvw}$. Let $f_h\sigma\in G_{uvwx}$ where $\sigma\in K_0=\Phi^{-1}(\mathrm{A\Gamma L}(d,q))$ if $A\neq \mathrm{PGU}(3,q)$ while $\sigma\in K_u=\Phi_u^{-1}(\mathrm{A\Gamma L}(d,q))$ if $A=\mathrm{PGU}(3,q)$. Then

$$\begin{aligned} Lf_1 &= Lf_1f_h\sigma \\ &= L(\lambda f_1)\sigma \text{ for some } \lambda\in \mathrm{GF}(q). \end{aligned}$$

Then as K_0 and K_u induce semilinear maps on X it follows that σ fixes setwise

$$\{L(\mu f_1)\mid \mu\in\mathrm{GF}(q)\}.$$

Since A acts 2-transitively on Ω there exists $t\in A$ such that $\alpha^t=\alpha_2$ and $\alpha_2^t=\alpha$. Furthermore, we can choose t such that t is centralised by the group automorphism ϕ (alternatively by φ when $A=\mathrm{PGU}(3,q)$). This can be done by choosing $t\in\mathrm{PGL}(2,p),\mathrm{PGU}(3,p),Ree(p)$, or $Sz(p)$ appropriately. Let $f_t:\mathrm{GF}(q)^d\to A$ be the constant function with value t. Then $f_t\in G_v$ and interchanges u and w. Thus f_t maps $\Gamma(w)$ to $\Gamma(u)$. Then

$$\Gamma(u)=\Gamma(w)^{f_t}=\{Lff_t\mid f \text{ linear}\}.$$

Also note that t normalises $P=A_{\alpha,\alpha_2}$. Let $f_h\sigma\in G_{uvwx}$, where $\sigma\in K_0$ if $A\neq\mathrm{PGU}(3,q)$ while $\sigma\in K_u$ if $A=\mathrm{PGU}(3,q)$. Then $Lf_1f_t\in\Gamma(u)$ and

$$\begin{aligned} Lf_1f_tf_h\sigma &= Lf_1f_tf_h\sigma f_t^{-1}f_t \\ &= Lf_1f_{h^t}\sigma f_t \text{ as } \sigma \text{ centralises } f_t \\ &= L(\xi f_1)\sigma f_t \text{ for some } \xi\in\mathrm{GF}(q), \text{ since } h^t\in P. \end{aligned}$$

Now as $f_h\sigma\in G_{uvwx}$, we have seen that σ fixes the set

$$\{L(\mu f_1)\mid \mu\in\mathrm{GF}(q)\}$$

setwise and so each $f_h\sigma\in G_{uvwx}$ fixes

$$\{L(\mu f_1)f_t\mid \mu\in\mathrm{GF}(q)\}\subset\Gamma(u)$$

setwise. Hence if $d>1$, then G_{uvwx} is not transitive on $\Gamma(u)\backslash\{v\}$. Thus $\Gamma(A,d)$ is not locally $(G,4)$-arc transitive for $d\geq 2$.

Suppose now that $d=1$. Then G_{uvw} acts transitively on the set $\Gamma(w)\backslash\{v\}$ of size $q-1$. If $A\neq\mathrm{PGU}(3.q)$ it follows that $|G_{uvwx}|=|P|q(q-1)e/(q-1)=qe|P|$. On the other hand if $A=\mathrm{PGU}(3,q)$ then $|G_{uvwx}|=|P|2q(q-1)e/(q-1)|=2qe|P|$.

If $A = Ree(q)$ or $Sz(q)$, then $|P| = q-1$ and $|\Omega| = q^3+1$ or q^2+1 respectively. Thus in all cases $|\Gamma(x)\backslash\{w\}| = |\Omega| - 1$ does not divide $|G_{uvwx}|$, so G_{uvwx} is not transitive on $\Gamma(x)\backslash\{w\}$. Hence $\Gamma(A, 1)$ is not locally $(G, 4)$-arc transitive for either of these choices for A.

If $A = \mathrm{PGU}(3, q)$ then $|P| = q^2 - 1$ and $|\Omega| = q^3 + 1$. Thus $|\Gamma(x)\backslash\{w\}| = |\Omega| - 1 = q^3$ does not divide $|G_{uvwx}|$, so G_{uvwx} is not transitive on $\Gamma(x)\backslash\{w\}$. Hence $\Gamma(\mathrm{PSU}(3, q), 1)$ is not locally $(G, 4)$-arc transitive.

This leaves us to investigate the case where $A = \mathrm{PGL}(2, q)$. Then $|(G_K)_{uvw}| = |P||K|$ and since $(G_K)_{uvw}$ acts transitively on the set $\Gamma(w)\backslash\{v\}$ of size $q-1$ it follows that $|(G_K)_{uvwx}| = |P||K|/(q-1)$. Here $|P| = q-1$ and $|\Omega| = q+1$. Thus $|(G_K)_{uvwx}| = |K|$ divides $q(q-1)e$, and $|\Gamma(x)\backslash\{w\}| = q$. Also $|\Gamma(u)\backslash\{v\}| = q-1$ and so for $(G_K)_{uvwx}$ to be transitive on both $\Gamma(x)\backslash\{w\}$ and $\Gamma(u)\backslash\{v\}$ we require that $q(q-1)$ divides $|(G_K)_{uvwx}|$. Hence K is a subgroup of $\mathrm{A\Gamma L}(1, q)$ whose order is divisible by $q(q-1)$. By Lemma 2.5, K acts transitively on $\mathrm{GF}(q)$. Hence if $\Gamma(\mathrm{PGL}(2, q), 1)$ is locally $(G_K, 4)$-arc transitive then K is transitive on $\mathrm{GF}(q)$. This completes the proof of part (3). Furthermore, the graph $\Gamma(\mathrm{PGL}(2, q), 1)$ where $q = p^e$, is the graph $\mathcal{G}(p, e)$ constructed in [5]. By [5, Theorem 1.1] this is locally $(G, 4)$-arc transitive if and only if $p = 2$. Furthermore, for q even, the graph $\Gamma(\mathrm{PGL}(2, q), 1, \mathrm{GF}(q))$ is locally 5-arc transitive but not locally 6-arc transitive. This completes the proof of part (2).

3.5 Quotients

Given $K \leqslant \mathrm{A\Gamma L}(d, q)$ recall G_K defined in (3.17) and (3.18), and that the group induced by G_K on the q^d simple direct factors of $N = \mathrm{soc}(G) \leqslant G_K$ is K. Hence if K is not transitive on the set of q^d vectors of $\mathrm{GF}(q)^d$, our group G_K is not quasiprimitive on Δ_1 or on Δ_2. However, by Lemma 3.7 if $\pi(K)$, the projection of K onto $\Gamma\mathrm{L}(d, q)$, is transitive on the set of 1-spaces of $\mathrm{GF}(q)^d$, then $\Gamma(A, d)$ is locally $(G_K, 2)$-arc transitive. This allows us to give the following construction of more locally 2-arc transitive graphs of $\{\mathrm{SD}, \mathrm{PA}\}$ type.

Construction 3.10 *We begin with*

- *a 2-transitive almost simple group A on a set Ω, such that $A = \mathrm{PGL}(n, q)$, $\mathrm{PGU}(3, q)$, $Ree(q)$ or $Sz(q)$,*
- *a positive integer d, such that if $A = \mathrm{PGL}(n, q)$ with $n \geq 3$, then $d = 1$,*
- *the locally $(G, 2)$-arc transitive graph $\Gamma(A, d)$ obtained from Construction 3.3,*
- *$K \leqslant \mathrm{A\Gamma L}(d, q)$ such that $\pi(K)$ acts transitively on the set of 1-spaces of $\mathrm{GF}(q)^d$ but K is intransitive on $\mathrm{GF}(q)^d$.*

- *an orbit S of K on* $\mathrm{GF}(q)^d$ *of length* $k > 1$.

Now $N = \mathrm{soc}(G) \cong T^{q^d}$ *contains a subgroup*

$$N_S = \{f : \mathrm{GF}(q)^d \to T \mid f(\mathbf{a}) = 1 \textit{ for all } \mathbf{a} \in S\}.$$

Then $N_S \cong T^{q^d-k}$ *and* $N_S \lhd G_K$. *Moreover,* N_S *acts intransitively on both* Δ_1 *and* Δ_2. *We construct the graph* $\Gamma(A, d, S)$ *to be the quotient graph of* $\Gamma(A, d)$ *with respect to the orbits of* N_S.

We collect the following remarks about the choice of K.

Remark 3.11
(1) If S is an orbit of two groups $K_1, K_2 \leqslant \mathrm{A\Gamma L}(d, q)$ on $\mathrm{GF}(q)^d$ then Construction 3.10 yields the same graph using either K_1 or K_2.

(2) If $d = 1$ then we can take K to be any intransitive subgroup of $\mathrm{A\Gamma L}(1, q)$ and S any nontrivial orbit.

(3) Suppose now that $d \geq 2$. The group of all translations in $\mathrm{A\Gamma L}(d, q)$ is normalised by K. Then since $\pi(K)$ acts transitively on the set of 1-spaces of $\mathrm{GF}(q)^d$ but K acts intransitively on $\mathrm{GF}(q)^d$, it follows that K does not contain any nontrivial translations. Hence $K \cong \pi(K) \leqslant \mathrm{\Gamma L}(d, q)$. One possibility for K is $\mathrm{\Gamma L}(d, q)$ and $S = \mathrm{GF}(q)^d \backslash \{0\}$. If $\pi(K) < \mathrm{\Gamma L}(d, q)$ and Z is the group of scalars in $\mathrm{\Gamma L}(d, q)$ then $\pi(K)Z$ acts transitively on $\mathrm{GF}(q)^d \backslash \{0\}$ and so is the stabiliser of a point in an affine 2-transitive group. All such groups are known, see [7–9].

For each $f \in F$ let $f_{|S}$ denote the restriction of f to S. Then when $A \neq \mathrm{PGU}(3, q)$

$$G_{K,S} := G_K/N_S \cong \langle f_{|S}, (f_h)_{|S} \mid f \in N, h \in A\rangle \rtimes \Phi^{-1}(K)$$

while when $A = \mathrm{PGU}(3, q)$

$$G_{K,S} := G_K/N_S \cong \langle f_{|S}, (f_h)_{|S} \mid f \in N, h \in A\rangle \rtimes \Phi_u^{-1}(K).$$

Note that $G_{K,S}$ is isomorphic to either $T^k.(A/T) \rtimes K$ or $T^k.(A/T) \rtimes (2.K)$ depending on A.

Since K fixes S setwise, $\Phi^{-1}(K)$ and $\Phi_u^{-1}(K)$ induce automorphisms of

$$F_S = \langle f_{|S}, (f_h)_{|S} \mid f \in N, h \in A\rangle.$$

As $\pi(K)$ acts transitively on the set of 1-spaces of $\mathrm{GF}(q)^d$, it follows that S spans $\mathrm{GF}(q)^d$. Hence if $g \in K$ acts trivially on S then $g \notin \mathrm{AGL}(d, q)$ and so has a nontrivial associated field automorphism ϕ^i for some $i = 1, \ldots, e-1$, where $q = p^e$. If $A \neq \mathrm{PGU}(3, q)$ then $\Phi^{-1}(g) = f_{\phi^i}\sigma_g$. Hence $\Phi^{-1}(g)$ induces

$(f_{\phi^i})_{|S}$ on F_S and so $\Phi^{-1}(K)$ acts faithfully on F_S. Similarly, if $A = \mathrm{PGU}(3,q)$ then $\Phi_u^{-1}(K)$ acts faithfully on F_S.

Proposition 3.12 *Suppose that* $\Gamma(A,d)$ *is locally* (G_K, s)*-arc transitive for some* $s \geq 2$*. Then the graph* $\Gamma(A,d,S)$ *obtained from Construction 3.10 is locally* $(G_{K,S}, s)$*-arc transitive. Moreover,* $\Gamma(A,d,S)$ *is connected,* $\Gamma(A,d)$ *is a cover of* $\Gamma(A,d,S)$*, and* $G_{K,S}$ *is quasiprimitive of type* $\{\mathrm{SD},\mathrm{PA}\}$*.*

PROOF. By Lemma 3.5, $\Gamma(A,d)$ is connected and so $\Gamma(A,d,S)$ is connected. By [4, Lemma 5.1], $\Gamma(A,d,S)$ is locally $(G_K/N_S, s)$-arc transitive, $\Gamma(A,d)$ is a cover of $\Gamma(A,d,S)$ and $G_{K,S} = G_K/N_S \leqslant \mathrm{Aut}(\Gamma(A,d,S))$. Moreover, $\Gamma(A,d,S)$ is a bipartite graph with bipartition $\{\Delta_1', \Delta_2'\}$, where Δ_1' is the set of N_S-orbits on Δ_1 and Δ_2' is the set of N_S-orbits on Δ_2. Let $W = \mathrm{soc}(G_{K,S}) \cong T^k$. Then as K acts transitively on S it follows that $G_{K,S}$ acts transitively on the k simple direct factors of W. Given an N_S-orbit $B \in \Delta_1'$ with $v \in B$ we have that $W_B = N_vN_S/N_S \cong N_v/(N_v \cap N_S) \cong N_v \cong T$. Hence $G_{K,S}$ acts quasiprimitively of type SD on Δ_1'. Since S spans $\mathrm{GF}(q)^d$ we have $N_w \cap N_S = 1$ and so, given an N_S-orbit $C \in \Delta_2'$ with $w \in C$ it follows that $W_C = N_wN_S/N_S \cong N_w/(N_w \cap N_S) \cong N_w$. Since G acts quasiprimitively of type PA on Δ_1 it follows that $N_w \neq 1$ and is not isomorphic to T^l for any $l \leq k$. Hence $G_{K,S}$ acts quasiprimitively of type PA on Δ_2' and so $\Gamma(A,d,S)$ is of $\{\mathrm{SD},\mathrm{PA}\}$-type.

The examples in [6, Example 4.2] can be obtained from Construction 3.10 by taking K to be a subgroup of $\mathrm{A\Gamma L}(1,q)$ of order 2 and S a nontrivial orbit.

We also need the following lemma which combined with Propositions 3.12 and 3.9, and Lemma 3.8 determines the largest s for which $\Gamma(A,d,S)$ is locally $(G_{K,S}, s)$-arc transitive.

Lemma 3.13 *Let* Γ *be a graph,* $G \leqslant \mathrm{Aut}(\Gamma)$*,* $N \lhd G$ *intransitive on* $V\Gamma$ *and suppose that* Γ *is a cover of* Γ_N*. If* Γ_N *is locally* $(G/N, s)$*-arc transitive then* Γ *is locally* (G, s)*-arc transitive.*

PROOF. Let $(v_0, v_1, \ldots, v_s)$ and $(v_0, w_1, \ldots, w_s)$ be s-arcs in Γ starting at v_0. Let B_i be the N-orbit containing v_i and C_i be the N-orbit containing w_i. Then $(B_0, B_1, \ldots, B_s)$ and $(B_0, C_1, \ldots, C_s)$ are s-arcs in Γ_N starting at B_0. Thus there exists $g \in G$ such that $(B_0, B_1, \ldots, B_s)^{gN} = (B_0, C_1, \ldots, C_s)$. Then $v_0^g \in B_0$ and since B_0 is an N-orbit, there exists $n \in N$ such that $v_0^{gn} = v_0$. Moreover, $v_1^{gn} \in \Gamma(v_0) \cap C_1 = \{w_1\}$ as Γ is a cover of Γ_N. Hence $v_1^{gn} = w_1$. Similarly, we see that $v_i^{gn} = w_i$ for all $i = 2, \ldots, s$ and so $(v_0, v_1, \ldots, v_s)^{gn} = (v_0, w_1, \ldots, w_s)$. Thus Γ is locally (G, s)-arc transitive.

Corollary 3.14 *(1) Each graph* $\Gamma(\mathrm{PGL}(n,q),1,S)$*, for* $n \geq 3$*, is locally* $(G_{K,S},2)$*-arc transitive but not locally* $(G_{K,S},3)$*-arc transitive.*

(2) If A *is one of* $\mathrm{PGU}(3,q)$*, Ree(q) or Sz(q), then each graph* $\Gamma(A,d,S)$ *is locally* $(G_{K,S},3)$*-arc transitive but not locally* $(G_{K,S},4)$*-arc transitive.*

(3) Each graph $\Gamma(\mathrm{PGL}(2,q),d,S)$ *is locally* $(G_{K,S},3)$*-arc transitive. Moreover, if either* $d \geq 2$*,* q *is odd, or* $S \neq \mathrm{GF}(q)$ *then* $\Gamma(\mathrm{PGL}(2,q),d,S)$ *is not locally* $(G_{K,S},4)$*-arc transitive.*

4 Proof of Theorem 1.2

We make the following general hypothesis.

(SDPA) Γ *is a locally* $(G,2)$*-arc transitive connected graph such that* G *has two orbits* Δ_1 *and* Δ_2 *on vertices,* G *acts faithfully on both orbits, quasiprimitively on* Δ_1 *with type SD and quasiprimitively on* Δ_2 *with type PA. Furthermore,* $N = \mathrm{soc}(G) \cong T^k$ *for some finite nonabelian simple group* T *and positive integer* $k \geq 2$*. Each vertex of* Γ *has valency at least 3 and* v, w *are a pair of adjacent vertices with* $v \in \Delta_1$ *and* $w \in \Delta_2$*.*

We denote by π_i the projection homomorphism of N onto its i^{th} coordinate. We also let T_i be the normal subgroup of N for which $\pi_i(T_i) = T$ and $\pi_j(T_i) = 1$ for all $j \neq i$. Then for any subset I of $\{1,\ldots,k\}$ we let $T_I = \prod_{i\in I} T_i$.

First we note [4, Lemma 6.2].

Lemma 4.1 *Let* Γ *be a locally* (G,s)*-arc transitive graph with* $s \geq 2$ *such that* G *acts quasiprimitively and faithfully on both orbits on vertices. Let* $N \lhd G$*. If* N *is not regular on* Δ_1 *then* $N_v^{\Gamma(v)}$ *is transitive for all* $v \in V\Gamma$*.*

Corollary 4.2 *Under the hypothesis of Lemma 4.1,* $\Gamma \cong Cos(N, N_v, N_w)$*.*

PROOF. Since N_v acts transitively on $\Gamma(v)$ and N acts transitively on Δ_1, it follows that N is edge transitive. Thus the result follows from Lemma 2.1.

Lemma 4.3 *Let* Γ *be as in (SDPA). Then replacing* G*, if necessary, by a conjugate in* $\mathrm{Sym}(\Delta_1 \cup \Delta_2)$*, we may assume that the following all hold.*

(1) $N_v = \{(t,\ldots,t) \mid t \in T\}$*.*

(2) $N_{vw} = \{(t,\ldots,t) \mid t \in H\}$ *for some maximal subgroup* H *of* T *and the action of* T *on the set of right cosets of* H *is the action of the socle of a 2-transitive almost simple group.*

(3) N_w is a subdirect subgroup of H^k, acts transitively on $\Gamma(w)$ and $N_w \neq N_{vw}$.

PROOF. 1). As the action of G on Δ_1 is quasiprimitive of type SD, replacing G if necessary by a conjugate in $\mathrm{Sym}(\Delta_1 \cup \Delta_2)$, we may choose v such that $N_v = \{(t, \ldots, t) \mid t \in T\}$.

2). By Lemma 4.1,

$$1 \neq N_v^{\Gamma(v)} \lhd G_v^{\Gamma(v)}$$

and $G_v^{\Gamma(v)}$ is a 2-transitive group. As N_v is simple it follows that $N_v^{\Gamma(v)} \cong T$. Then a theorem of Burnside (see [2, Theorem 4.3]) implies that $N_v^{\Gamma(v)}$ is primitive and $G_v^{\Gamma(v)}$ is an almost simple 2-transitive group with socle T. Thus (2) holds.

3). Now $N_{vw} \leqslant N_w$ and so $H = \pi_i(N_{vw}) \leqslant \pi_i(N_w)$ for each $i = 1, \ldots, k$. As G is quasiprimitive of type PA on Δ_2, $\pi_i(N_w) \neq T$ for all i and so the maximality of H in T yields that $\pi_i(N_w) = H$ for all i. Hence N_w is a subdirect subgroup of H^k. By Lemma 4.1, $N_w^{\Gamma(w)}$ is transitive on $\Gamma(w)$. Since $|\Gamma(w)| \geq 3$ it follows that $N_w \neq N_{vw}$.

Lemma 4.4 *Let Γ be as in (SDPA). Then G acts by conjugation on the set $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$, and the permutation groups $G^{\mathcal{T}}$, $G_v^{\mathcal{T}}$, $G_w^{\mathcal{T}}$ and $G_{vw}^{\mathcal{T}}$ are all transitive and pairwise permutationally isomorphic.*

PROOF. Since N is a minimal normal subgroup of G, the group $G^{\mathcal{T}}$ is transitive. Furthermore, N acts transitively on Δ_1, and so $G = NG_v$. Then as N acts trivially on $\mathcal{T}$ it follows that $G^{\mathcal{T}}$ and $G_v^{\mathcal{T}}$ are permutationally isomorphic. Similarly, $G^{\mathcal{T}}$ and $G_w^{\mathcal{T}}$ are permutationally isomorphic. Furthermore, by Lemma 4.3, N_w acts transitively on $\Gamma(w)$ and so $G_w = N_w G_{vw}$. Hence $G_w^{\mathcal{T}}$ and $G_{vw}^{\mathcal{T}}$ are permutationally isomorphic.

We now have the following theorem which determines the possibilities for T and H.

Theorem 4.5 *Let Γ be as in (SDPA), N be as in Lemma 4.3 and H be the subgroup of T isomorphic to N_{vw}. Then T and H are as in one of the rows of Table 1.*

PROOF. Suppose that H is an almost simple group with socle M and let S be the socle of N_w. Then $S \lhd G_w$, as it is a characteristic subgroup of N_w. By Lemma 4.3, N_w is a subdirect subgroup of H^k and so $\pi_i(S) \lhd \pi_i(H)$ for each $i = 1, \ldots, k$. By Lemma 4.4, G_w acts transitively by conjugation on the k simple direct factors of N, so the $\pi_i(S)$, for $i = 1, 2, \ldots, k$, are pairwise isomorphic. Then as M is the unique minimal normal subgroup of H, it follows that $M \leqslant \pi_i(S)$ for all i. Furthermore, if $1 \neq R \lhd N_w$ then $R \cap M^k \lhd N_w$, and also (since S is the socle of N_w,) $R \cap S \neq 1$ from which it follows that $R \cap M^k \neq 1$. Thus every minimal normal subgroup of N_w is contained in M^k and so S is also contained in M^k. Hence S is a subdirect subgroup of M^k. Moreover, a well known lemma, (see for example [13, Page 328]) together with the facts that G_w normalises S and is transitive on the simple direct factors of N, imply that there exists a divisor r of k such that $S = D_1 \times \cdots \times D_r$, where $D_1 = \{(l, l^{\sigma_2}, \ldots, l^{\sigma_{k/r}}) \mid l \in M\} \leqslant M^{k/r}$, for some automorphisms σ_i of M, and the D_j are permuted transitively by G_w. Hence $S \cong M^r$.

Let K be the kernel of the action of S on $\Gamma(w)$. As K is the intersection of S and the kernel of the action of G_w on $\Gamma(w)$, both of which are normal in G_w, we have $K \lhd G_w$. Now G_w acts transitively on the k simple direct factors of N and as $S = D_1 \times \cdots \times D_r$, where each D_i is a nonabelian simple group isomorphic to M, either $K = 1$ or $K = S$. If $K = 1$ then $S^{\Gamma(w)} \cong S \cong M^r$. Now $S^{\Gamma(w)} \lhd G_w^{\Gamma(w)}$, a 2-transitive group, and so by a theorem of Burnside, (see [2, Theorem 4.3], $S^{\Gamma(w)} \cong M^r$ is a nonabelian simple group. Hence in the case where $K = 1$ we have $r = 1$. If on the other hand $K = S$, then $K \leqslant N_{vw} \cong H$. As M is the unique minimal normal subgroup of H, and M is simple, it follows that $r = 1$ in this case also. Thus in either case $M \cong S \lhd N_w$. As $S = \mathrm{soc}(N_w)$ and M is nonabelian, it follows that $C_{N_w}(S) = 1$ and so $N_w \lesssim \mathrm{Aut}(M)$. Hence $N_w \cong H$ and $N_w = N_{vw}$. This contradicts Lemma 4.3 and so H is not almost simple. Then by Theorem 2.2, T and H are as in one of the lines of Table 1.

By Theorem 2.2, each possibility for H given in Theorem 4.5 has a unique minimal normal subgroup M and M is the centre of $O_p(H)$, where p is the characteristic of the field over which T is defined. Recall the Remarks 2.3 and 2.4, especially that we count $\mathrm{PSL}(2, 8)$ twice, once as $\mathrm{PSL}(2, 8)$ of degree 9 and once as $Ree(3)'$ of degree 28. Let $A = \mathrm{PGL}(n, q)$, $\mathrm{PGU}(3, q)$, $Sz(q)$ or $Ree(q)$ such that $\mathrm{soc}(A) = T$. If we let $\Omega = [T : H]$ and $B = N_{\mathrm{Sym}(\Omega)}(T)$, then B acts 2-transitively on Ω with socle T. We can choose $\alpha \in \Omega$ such that $T_\alpha = H$ and B_α induces $\Gamma\mathrm{L}(m, q)$ on M, where M has order q^m. It transpires that the subgroup $X = N_w \cap M^k$ is crucial to our analysis.

Proposition 4.6 *Let Γ be as in (SDPA), N be as in Lemma 4.3, B be as above and let $X = N_w \cap M^k$ and $Y = N_{vw} \cap M^k$. Furthermore, suppose that G is the largest subgroup of* $\mathrm{Aut}(\Gamma)$ *of type SD with socle N. Then, using the notation introduced above, the following all hold.*

(1) X is normal in G_w and acts transitively on $\Gamma(w)$, and $Y = \{(l, l, \ldots, l) \mid l \in M\}$.

(2) $G_{vw} = N_F(X)$, where $F = \{(h, \ldots, h) \mid h \in B_\alpha\} \times S_k$.

(3) The action of G_{vw} induced on X/Y by conjugation is equivalent to the action of G_{vw} on $\Gamma(w)$.

(4) There exists a positive integer d such that X is a $(d+1)$-dimensional vector space over a field of size $|M|$ and $|\Gamma(w)| = |M|^d$.

(5) There does not exist $I \subseteq \{1, 2, \ldots, k\}$ with $|I| \geq 2$ such that every element of X is constant on I.

PROOF. 1). As M is the unique minimal normal subgroup of H, it is characteristic in H and so $X = N_w \cap M^k$ is characteristic in N_w. Thus $X \lhd G_w$. Now G_w acts 2-transitively on $\Gamma(w)$ and so X acts either transitively or trivially on $\Gamma(w)$. By Lemma 4.3, $N_{vw} = \{(h, \ldots, h) \mid h \in H\}$ and so $Y = N_{vw} \cap M^k = \{(l, \ldots, l) \mid l \in M\}$.

Suppose X acts trivially on $\Gamma(w)$. Then $X \leqslant N_{vw}$ and so $X = Y = \{(l, \ldots, l) \mid l \in M\}$. By Lemma 4.3(3), $N_{vw} \neq N_w$. Thus there exists a proper subset I of $\{1, \ldots, k\}$ such that $N_w \cap T_I \neq 1$ and we may choose I to be a minimal such set. Without loss of generality we may assume that $I = \{1, \ldots, t\}$ for some $t < k$. As I is minimal there exists a nontrivial subgroup K of H such that

$$N_w \cap T_I = \{(l, l^{\sigma_2}, \ldots, l^{\sigma_t}, 1 \ldots, 1) \mid l \in K\}$$

for some $\sigma_i \in \mathrm{Aut}(K)$. Now $N_w \cap T_I \lhd N_w$ so for each $i \in I$,

$$1 \neq K \cong \pi_i(N_w \cap T_I) \lhd \pi_i(N_w) = H.$$

Then as M is the unique minimal normal subgroup of H we have $M \leqslant K$ and

$$\{(l, l^{\sigma_2}, \ldots, l^{\sigma_t}, 1 \ldots, 1) \mid l \in M\} \leqslant N_w \cap T_I \leqslant N_w \cap M^k = X.$$

This contradicts the fact that $X = \{(l, \ldots, l) \mid l \in M\}$ and so X acts transitively on $\Gamma(w)$.

2). Since G acts transitively on Δ_1 with quasiprimitive type SD and $N_v = \{(t, \ldots, t) \mid t \in T\}$ we have

$$G_v \leqslant N_{\mathrm{Sym}(\Delta_1)}(N_v) = \{(t, \ldots, t) \mid t \in \mathrm{Aut}(T)\} \times S_k.$$

Furthermore, G_v acts on $\Gamma(v)$ and $G_v^{\Gamma(v)}$ is a 2-transitive group with socle T. Thus

$$G_v \leqslant \{(t, \ldots, t) \mid t \in B\} \times S_k,$$

where $B = N_{\mathrm{Sym}(\Omega)}(T)$, and $G_{vw} \leqslant F = \{(h, \ldots, h) \mid h \in B_\alpha\} \times S_k$. By Part 1, $X \lhd G_w$ and so is normalised by G_{vw}. Hence $G_{vw} \leqslant N_F(X)$ and it remains to prove equality. By Lemma 2.1, $\Gamma \cong Cos(N, N_v, N_w)$. Let $g \in N_F(X)$. Then as $g \in \mathrm{Aut}(T)^k \rtimes S_k$ we have that g normalises N. Furthermore, as B_α normalises T it follows that g normalises N_v. Finally, as $g \in N_F(X)$ and B_α normalises $H = T \cap B_\alpha$ we have that g normalises $N_w = XN_{vw}$. Thus g acts on $\Delta_1 = [N : N_v]$ via the action $g : N_v x \mapsto N_v x^g$ and on $\Delta_2 = [N : N_w]$ via $g : N_w y \mapsto N_w y^g$ for all $x, y \in N$. Now $(x^g)(y^g)^{-1} = (xy^{-1})^g$ which in turn belongs to $N_v N_w$ if and only if $xy^{-1} \in N_v N_w$. Hence g preserves adjacency in Γ. Then as G is the largest subgroup of $\mathrm{Aut}(\Gamma)$ of type SD with socle N we have $g \in G_{vw}$. Thus $G_{vw} = N_F(X)$.

3). Since X is abelian and acts transitively on $\Gamma(w)$ with point stabiliser Y we can identify the elements of $\Gamma(w)$ with the cosets of Y in X such that v corresponds to the coset Y. Furthermore, $G_w = XG_{vw}$ with X acting on $\Gamma(w)$ by right multiplication and G_{vw} acting by conjugation.

4). Regarding M as the additive group of a field F_M, we note that if $T \neq \mathrm{PSL}(2, q)$, q odd, then H contains a subgroup H_λ such that the group of automorphisms of M induced by H_λ is the multiplicative group of F_M. Thus in these cases N_{vw} contains the subgroup $\{(h, \dots, h) \mid h \in H_\lambda\}$, which induces the multiplicative group of F_M on M^k and leaves X invariant. (Recall $X = N_{vw} \cap M^k$ and so X is invariant under N_{vw}.) Hence X is a vector space over F_M. If $T = \mathrm{PSL}(2, q)$, q odd, then the group of automorphisms of M induced by H is an index two subgroup of the multiplicative group of F_M, that is, is the multiplicative group of all squares in F_M. Since the additive subgroup of F_M generated by the $(q-1)/2$ squares must have order dividing q, it follows that every element of F_M is a sum of squares. Then as X is closed under multiplication by the squares of F_M and is an additive subgroup of M^k, it follows that X is closed under the multiplicative group of F_M. (We are grateful to Tim Penttila for this argument.) Thus for all T, X is a vector space over F_M. Now consider $Y = N_{vw} \cap M^k$. By Part (1), $Y = \{(l, \dots, l) \mid l \in M\} < X$ and $Y \cong M$. Thus there exists a positive integer d such that $|X| = |M|^{d+1}$. Since X acts transitively on $\Gamma(w)$ with point stabiliser Y it follows that $|\Gamma(w)| = |M|^d$.

5). Suppose that there does exist $I \subseteq \{1, \dots, k\}$ such that $|I| \geq 2$ and every element of X is constant on I. We may assume that I is a maximal such set. By part (1), $N_w = XN_{vw}$ and as every element of N_{vw} is constant on I it follows that each element of N_w is constant on I. Then as $N_w \lhd G_w$ and G_w permutes $\mathcal{T} = \{T_1, \dots, T_k\}$ transitively, it follows that I forms a block of imprimitivity for G_w in its action on $\mathcal{T}$. Let $\mathcal{P}$ be the associated system of imprimitivity. By Lemma 4.4, $G_{vw}^{\mathcal{T}} = G_w^{\mathcal{T}}$. Hence for all blocks $I' \in \mathcal{P}$, there exists $g \in G_{vw} \leqslant \{(h, \dots, h) \mid h \in B_\alpha\} \times S_k$ such that $I' = I^g$. Now $X^g = X$ and it follows that each element of X is constant on I'. Arguing as before we see that each element of N_w is constant on each $I' \in \mathcal{P}$. Furthermore, by Lemma 4.4, $\mathcal{P}$ is also a system of imprimitivity for $G_v^{\mathcal{T}}$. Since $G_v \leqslant \{(t, \dots, t) \mid t \in B\} \times S_k$ it follows that for all $g \in G_v$, each element of N_w^g is constant on each $I' \in \mathcal{P}$. As $N_w \lhd G_w$, it follows that for all $g \in \langle G_v, G_w \rangle$, each element of N_w^g is constant on I. Since Γ is connected, by Lemma 2.1, $\langle G_v, G_w \rangle = G$. Thus each element of $N_0 = \langle N_w^g \mid g \in G \rangle$ is constant on I. Now $N_0 \lhd G$ and, as N is a minimal normal subgroup of G, we deduce that $N_0 = N$. This contradicts the fact that

$|I| \geq 2$. Thus $|I| = 1$.

We can now prove the following theorem which is the crucial part of the proof of Theorem 1.2. Recall that $X = N_w \cap M^k$ and $|X| = |M|^{d+1}$. Recall that $\pi : \mathrm{A\Gamma L}(d,q) \to \mathrm{\Gamma L}(d,q)$ is the projection homomorphism, Φ is the isomorphism defined in (3.3), Φ_u is the homomorphism defined in (3.5), and that $\mathcal{T}$ is the set of simple direct factors of N.

Theorem 4.7 *Let Γ be as in (SDPA), N as in Lemma 4.3 and suppose that $|X| = |M|^{d+1}$, as in Proposition 4.6. Suppose also that G is the largest subgroup of $\mathrm{Aut}(\Gamma)$ of type SD with socle $N = T^k$ and let $A = \mathrm{PGL}(n,q)$, $\mathrm{PGU}(3,q)$, Ree(q) or Sz(q) such that $\mathrm{soc}(A) = T$. Then the following all hold.*

(1) There exists $S \subseteq \mathrm{GF}(q)^d$ of size k which spans $\mathrm{GF}(q)^d$ such that Y is the group consisting of the restrictions to S of all constant functions from $\mathrm{GF}(q)^d$ to M, and X is generated by Y together with the set of restrictions to S of all linear functions from $\mathrm{GF}(q)^d$ to M. Moreover, if $T = \mathrm{PSL}(n,q)$ with $n \geq 3$ then $d = 1$.

(2) If $A \neq \mathrm{PGU}(3,q)$ then there exists $K \leqslant \mathrm{A\Gamma L}(d,q)$ such that $G^{\mathcal{T}} = K^S$, and $\pi(K)$ is transitive on the set of 1-spaces of $\mathrm{GF}(q)^d$. Moreover, if $A \neq \mathrm{PGU}(3,q)$ then

$$G = \langle N, (t,\ldots,t) \mid t \in A\rangle \rtimes \Phi^{-1}(K),$$

while if $A = \mathrm{PGU}(3,q)$ then

$$G = \langle N, (t,\ldots,t) \mid t \in A\rangle \rtimes \Phi_u^{-1}(K).$$

PROOF. Note that $Y \leqslant X \leqslant M^k$ and by Proposition 4.6(3), G_{vw} acts transitively by conjugation on the set of nontrivial cosets of Y in X. Moreover, by Proposition 4.6(2), $G_{vw} = N_F(X)$ where $F = \{(h,\ldots,h) \mid h \in B_\alpha\} \times S_k$. By Remark 2.3(5), B_α induces $\mathrm{\Gamma L}(m,q)$ on M and so there exists a homomorphism $\theta : F \to \mathrm{Aut}(M^k)$ such that $\theta(F) \leqslant \mathrm{\Gamma L}(m,q) \times S_k$ and $\theta(G_{vw})$ fixes X and Y setwise.

Suppose first that $T \neq \mathrm{PSL}(n,q)$ with $n \geq 3$. Then M is the additive group of the field $\mathrm{GF}(q)$. By Proposition 4.6(4), X is a vector space over M and for each $h \in A_\alpha$, the element $(h,h,\ldots,h)$ induces $\mathrm{GF}(q)$-scalar multiplication on X. Thus $Z := \{(h,h,\ldots,h) \mid h \in A_\alpha\} \leqslant G_{vw}$. If, on the other hand $T = \mathrm{PSL}(n,q)$ with $n \geq 3$, we have that M is the additive group of the vector space $\mathrm{GF}(q)^{n-1}$, A_α induces $\mathrm{GL}(n-1,q)$ on M. Moreover, $H = A_\alpha \cap T$ also induces $\mathrm{GL}(n-1,q)$ on M. Recall that $N_{vw} = \{(t,\ldots,t) \mid t \in H\}$ by Lemma 4.3(2). Then for all $h \in A_\alpha$, there exists $h' \in H$ such that h and h' induce the same element of $\mathrm{GL}(n-1,q)$ on M, and so the elements $(h',\ldots,h') \in N_{vw}$ and $(h,\ldots,h) \in F$ act in the same way on M^k. Since N_{vw} fixes $X = N_w \cap M^k$ setwise, it follows that $G_{vw} = N_F(X)$ contains $Z := \{(h,h,\ldots,h) \mid h \in A_\alpha\}$ in this case also.

Now $\theta(Z) = \mathrm{GL}(m,q)$ (with θ as defined in the first paragraph of the proof) and so $\mathrm{GL}(m,q) \leqslant \theta(G_{vw}) \leqslant \Gamma\mathrm{L}(m,q) \times S_k$. By Proposition 4.6 and the fact that G_{vw} acts transitively on $\Gamma(w)\backslash\{v\}$ we have that $\theta(G_{vw})$ acts transitively on the set of nontrivial cosets of Y in X. Hence X and $\theta(G_{vw})$ are determined by [3, Theorem 1.1(1)]. Thus X is as stated in part (1). Moreover, if $T = \mathrm{PSL}(n,q)$ with $n \geq 3$ then M has dimension $m = n-1 \geq 2$ over $\mathrm{GF}(q)$ and so by [3, Theorem 1.1], $d = 1$. This completes the proof of part (1).

To complete the proof of part (2) we need to determine G_{vw} and G from $\theta(G_{vw})$. As we have said, $\theta(G_{vw})$ is determined by [3, Theorem 1.1(1)]. To put the results there in our context we need to establish some notation.

Let $C \cong M^{q^d}$ be the group all functions from $\mathrm{GF}(q)^d$ to M. Then M^k is the restriction to S of all elements of C, Y is the restriction to S of all constant functions in C and X is the restriction to S of all affine functions in C. Also C has a group of automorphisms $\langle \sigma_g \mid g \in \mathrm{A}\Gamma\mathrm{L}(d,q)\rangle$ where $f^{\sigma_g}(\mathbf{a}) = f(\mathbf{a}^{g^{-1}})$ for all $f \in C$ and $\mathbf{a} \in \mathrm{GF}(q)^d$. Moreover, the Frobenius automorphism ϕ of $\mathrm{GF}(q)$ defines a semilinear automorphism τ_ϕ of C such that $f^{\tau_\phi}(\mathbf{a}) = (f(\mathbf{a}))^\phi$. Then there exists an embedding Ψ of $\mathrm{A}\Gamma\mathrm{L}(d,q)$ such that

$$\begin{aligned}\Psi : \mathrm{A}\Gamma\mathrm{L}(d,q) &\to \langle \tau_\phi\rangle \times \langle \sigma_g \mid g \in \mathrm{A}\Gamma\mathrm{L}(d,q)\rangle \\ g \quad &\mapsto \sigma_g \text{ if } g \in \mathrm{AGL}(d,q) \\ \phi \quad &\mapsto \tau_\phi\sigma_\phi.\end{aligned} \tag{4.1}$$

Furthermore, if $K \leqslant \mathrm{A}\Gamma\mathrm{L}(d,q)$ fixes S setwise then $\Psi(K)$ acts on M^k inducing a subgroup of $\Gamma\mathrm{L}(m,q) \times S_k$. Note that if $A \neq \mathrm{PGU}(3,q)$ then $\theta(\Phi^{-1}(K))$ and $\Psi(K)$ induce the same group of automorphisms of M^k. When $A = \mathrm{PGU}(3,q)$ then $\theta(\Phi_u^{-1}(K))$ induces a group O of automorphisms of M^k containing multiplication by -1. (Recall from Remark 2.4(4) that φ^e induces multiplication by -1 on M). Then $O/\langle -1\rangle$ is the the group of automorphism of M^k induced by $\Psi(K)$.

Since $\theta(G_{vw})$ acts transitively on the nontrivial cosets of Y in X, [3, Theorem 1.1(1)], implies that there exists $K \leqslant \mathrm{A}\Gamma\mathrm{L}(d,q)$ with orbit S such that $\pi(K)$ acts transitively on the set of 1-spaces of $\mathrm{GF}(q)^d$ and $\theta(G_{vw}) = \mathrm{GL}(m,q) \rtimes \Psi(K)$. Since $G_{vw} = N_F(X)$ it then follows that $G_{vw} = \theta^{-1}(\mathrm{GL}(m,q) \rtimes \Psi(K))$. Hence if $A \neq \mathrm{PGU}(3,q)$, we have

$$G_{vw} = \{(h,\ldots,h) \mid h \in A_\alpha\} \rtimes \Phi^{-1}(K)$$

while if $A = \mathrm{PGU}(3,q)$ then

$$G_{vw} = \{(h,\ldots,h) \mid h \in A_\alpha\} \rtimes \Phi_u^{-1}(K).$$

Since N acts transitively on Δ_1 we have that $G = NG_v$. Furthermore, as N_v acts transitively on $\Gamma(v)$ we have $G = N_vG_{vw}$ and so $G = NG_{vw}$. Hence $G^{\mathcal{T}} = K$ and as $A = TA_\alpha$, G is as stated in the part (2).

Corollary 4.8 *Let* Γ *be as in (SDPA). Then either* $\Gamma \cong Cos(A,d)$ *as obtained from Construction 3.3 or* $\Gamma \cong Cos(A,d,S)$ *as yielded by Construction 3.10.*

PROOF. By Theorem 4.7 there exists a group $K \leqslant \mathrm{A\Gamma L}(d,q)$ with an orbit S of size k such that S spans $\mathrm{GF}(q)^d$ and $\pi(K)$ acts transitively on the set of 1-spaces of $\mathrm{GF}(q)^d$. Moreover, Y is the set of restrictions to S of all constant functions $f : \mathrm{GF}(q)^d \to M$, and X is generated by Y and all restrictions to S of linear functions $f : \mathrm{GF}(q)^d \to M$. If $S = \mathrm{GF}(q)^d$ then N, N_v and N_w are as in $\Gamma(A,d)$ and so by Corollary 4.2, $\Gamma \cong \Gamma(A,d)$ as obtained from Construction 3.3. If $S \subset \mathrm{GF}(q)^d$ (a proper subset) then N, N_v and N_w are as in $\Gamma(A,d,S)$ and so by Corollary 4.2, $\Gamma \cong \Gamma(A,d,S)$ as obtained from Construction 3.10.

Hence we have completed the proof of Theorem 1.2. Corollary 1.3 then follows from Proposition 3.9 and Corollary 3.14.

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