

Involution graphs where the product of two adjacent vertices has order three

Alice Devillers and Michael Giudici

Université Libre de Bruxelles
Département de Mathématiques
Géométrie- CP 216
Boulevard du Triomphe
B-1050 Bruxelles Belgique
School of Mathematics and Statistics
The University of Western Australia
35 Stirling Highway
Crawley WA 6009
Australia

Abstract

An S_3 -involution graph for a group G is a graph with vertex set a union of conjugacy classes of involutions of G such that two involutions are adjacent if they generate an S_3 -subgroup in a particular set of conjugacy classes. We investigate such graphs in general and also for the case where $G = \text{PSL}(2, q)$.

1 Introduction

In [11] an interesting tower of graphs was discovered and investigated which is associated with the subgroup chain

$$A_5 < \text{PSL}(2, 11) < M_{11} < M_{12}.$$

The smallest graph in the tower is the line graph of the Petersen graph while the largest graph is the Johnson graph $J(12, 4)$. (The *Johnson graph* $J(n, k)$ is the graph with vertices the k -subsets of an n -set such that two k -subsets are adjacent if their intersection has size $k - 1$.) The graphs associated with $\text{PSL}(2, 11)$ and M_{11} are related to the Witt designs on 11 and 12 points. A uniform description of the graphs in the tower was achieved via involutions and S_3 -subgroups of the groups in the subgroup chain. This leads to the following definition.

Definition 1.1. Let G be a group with a nonempty set X of involutions closed under conjugation and a nonempty set \mathcal{S} of S_3 -subgroups also closed under conjugation. The S_3 -involvement graph $\Gamma(G, X, \mathcal{S})$ of G with respect to X and \mathcal{S} is the graph with vertices the elements of X such that two vertices x, y are adjacent if and only if $\langle x, y \rangle \in \mathcal{S}$. In order to avoid degeneracies, we always require that X is the set of all involutions contained in elements of \mathcal{S} .

The tower of graphs is then given by a series of S_3 -involvement graphs for A_5 , $\text{PSL}(2, 11)$, M_{11} and M_{12} where for each group, both X and \mathcal{S} are single conjugacy classes. The existence of this tower suggests the following natural problem.

Problem 1.2. Investigate S_3 -involvement graphs for other families of groups.

This paper is intended as an initial investigation of S_3 -involvement graphs in general and we also investigate the S_3 -involvement graphs arising from the simple groups $\text{PSL}(2, q)$, for $q \geq 4$.

In Section 2, after describing some examples of S_3 -involvement graphs, we investigate automorphisms, connectivity, and triangles. Given an S_3 -involvement graph $\Gamma(G, X, \mathcal{S})$ the three involutions of each $S \in \mathcal{S}$ give rise to a triangle in the graph. The following gives a sufficient condition for these to be the only triangles. See Subsection 2.3 for a discussion about the converse.

Theorem 1.3. *Let G be a finite group with conjugacy class X of involutions and union of conjugacy classes \mathcal{S} of S_3 -subgroups. If G has no subgroups of the form $C_3^2 \rtimes C_2$ or $C_p^2 \rtimes S_3$ for some prime p then the only triangles of $\Gamma(G, X, \mathcal{S})$ are those given by subgroups in \mathcal{S} .*

In Section 3 we analyse the S_3 -involvement graphs for $\text{PSL}(2, q)$. In particular we determine the full automorphism groups (Theorems 3.9 and 3.11) and show that there is a duality with the graph induced on S_3 -triangles if and only if $q = 11$ and 13 (Theorem 3.8). We also give the following determination of the size of the largest cliques.

Theorem 1.4. *Let $G = \text{PSL}(2, q)$ for $q \geq 4$, let X be the unique conjugacy class of involutions in G and \mathcal{S} be a conjugacy class of S_3 -subgroups. The size of the largest clique is 3^e if $q = 9^e$, 4 if $q = 25^e$ and 3 otherwise.*

The definition of an S_3 -involvement graph is reminiscent of Fischer's *3-transposition groups*, that is, groups generated by a conjugacy class X of involutions such that any pair of noncommuting elements of X generates an S_3 . The elements of X are called *3-transpositions*. Fischer's investigation of such groups [14, 13] led to the discovery of three new sporadic simple groups. If G is a 3-transposition group with class X of 3-transpositions and \mathcal{S} is the set of all S_3 -subgroups generated by a pair of noncommuting 3-transpositions, the S_3 -involvement graph $\Gamma(G, X, \mathcal{S})$ is called the *diagram* of X and was used in [10, 13, 16] in the study of 3-transposition groups. In fact, a 3-transposition group with class X of 3-transpositions is a quotient of the Coxeter group with Coxeter diagram the diagram of X . We are interested in S_3 -involvement graphs for arbitrary groups, indeed the groups A_5 , $\text{PSL}(2, 11)$, M_{11} and M_{12} are not 3-transposition groups.

Given a 3-transposition group G with a class X of 3-transpositions and \mathcal{S} the set of all S_3 -subgroups generated by pairs of elements of X , one can construct a partial linear

space known as a *Fischer space*, whose points are the elements of X and lines are the sets of three involutions contained in an S_3 in \mathcal{S} . Moreover, Fischer spaces are precisely the partial linear spaces such that each plane is either an affine plane over $\text{GF}(3)$ or a dual affine plane over $\text{GF}(2)$ (see [9, 10]). The S_3 -involution graph $\Gamma(G, X, \mathcal{S})$ is the collinearity graph of the Fischer space.

Another graph with vertex set a conjugacy class of 3-transpositions was also used in the investigation of 3-transposition groups. Given a group G and conjugacy class X of involutions, the *commuting involution graph* $\mathcal{C}(G, X)$ is the graph with vertices the elements of X such that two vertices are adjacent if they commute. If G is a 3-transposition group such that X is a class of 3-transpositions and \mathcal{S} is the set of all S_3 -subgroups generated by pairs of elements of X then $\mathcal{C}(G, X)$ is the complement of $\Gamma(G, X, \mathcal{S})$. Commuting involution graphs for groups other than 3-transposition groups have recently been studied in [2, 3, 4].

2 General Theory

We begin this section with a few simple examples.

Example 2.1. Let $G = S_n$, X the conjugacy class of transpositions and \mathcal{S} the conjugacy class of S_3 -subgroups generated by two transpositions. Note that X is a class of 3-transpositions. The map from X to the set of 2-subsets of an n -set which maps each transposition x to the set of two points moved by x yields a one-to-one correspondence between X and the vertex set of $J(n, 2)$. Moreover, two transpositions generate an S_3 if and only if their 2-cycles have a unique point in common. Thus $\Gamma(G, X, \mathcal{S}) \cong J(n, 2)$.

Example 2.2. Let V be a $(2n)$ -dimensional vector space of $\text{GF}(2)$ equipped with an alternating form (\cdot, \cdot) . Let $G = \text{Sp}(2n, 2)$ be the group of all linear transformations of V which preserve (\cdot, \cdot) and let X be the set of all transvections contained in G , that is all maps

$$\begin{aligned} t_v : V &\rightarrow V \\ x &\mapsto x + (x, v)v \end{aligned}$$

where v is a nonzero vector of V . Calculations show that if $(v, u) = 0$ then $t_v t_u$ has order two, otherwise $t_u t_v$ has order three and $t_u^{t_v} = t_{u+v}$. Thus X is a class of 3-transpositions for G . Letting \mathcal{S} be the set of all S_3 -subgroups of G generated by pairs of noncommuting elements of X it follows that $\Gamma(G, X, \mathcal{S})$ is isomorphic to the graph with vertices the nonzero vectors of V such that two vertices u, v are adjacent if and only if $(v, u) = 1$, that is if and only if they lie in a hyperbolic line; moreover the third vector of that line corresponds to the third involution of $\langle t_u, t_v \rangle$. Hence the Fischer space of G is the partial linear space whose points are the nonzero vectors of V and whose lines are the hyperbolic lines of V . This space is called a symplectic copolar space over $\text{GF}(2)$ and is denoted by $\overline{W(2n-1, 2)}$ or $\overline{\text{Sp}(2n, 2)}$.

Similar graphs can be constructed from the other classical groups which are also 3-transposition groups.

Example 2.3. Let $G = M_{11}$. Then G has a unique conjugacy class X of involutions and two classes of S_3 -subgroups. Now $|X| = 165$ and there exists a bijection from X

to the set of 3-subsets of an 11-set where each involution is mapped to its set of fixed points. Also G has two conjugacy classes $\mathcal{S}_1, \mathcal{S}_2$ of S_3 -subgroups with $|\mathcal{S}_1| = 220$ and $|\mathcal{S}_2| = 660$.

The graph $\Gamma(G, X, \mathcal{S}_1)$ is the graph associated with M_{11} in the tower investigated in [11]. By [11, Theorem 1.1] it has valency 8 and M_{11} as its full automorphism group. Moreover, $\Gamma(G, X, \mathcal{S}_1)$ is isomorphic to the graph with vertex set the set of 3-subsets of an 11-set such that two vertices are adjacent if they are disjoint and the complement of their union is a pentad in the Witt design on 11 points associated with M_{11} .

Each S_3 -subgroup in the class \mathcal{S}_2 fixes 2 points of an 11-set. Thus if two involutions are adjacent in $\Gamma(G, X, \mathcal{S}_2)$ then they have two fixed points in common, that is, their sets of fixed points are adjacent in $J(11, 3)$. Each involution of G is contained in twelve S_3 -subgroups of \mathcal{S}_2 and so is adjacent to 24 involutions in $\Gamma(G, X, \mathcal{S}_2)$. This is the valency of $J(11, 3)$ and so $\Gamma(G, X, \mathcal{S}_2) \cong J(11, 3)$.

Example 2.4. Let $G = \text{AGL}(1, 3^n)$ for some positive integer n . Then G is the group of all maps

$$\begin{aligned} t_{a,b} : \text{GF}(3^n) &\rightarrow \text{GF}(3^n) \\ x &\mapsto ax + b \end{aligned}$$

for any $a, b \in \text{GF}(3^n)$ with $a \neq 0$. Let $X = \{t_{-1,b} \mid b \in \text{GF}(3^n)\}$, the unique conjugacy class of involutions of G . Let x be the involution $t_{-1,0}$. Note that any involution $t_{-1,b} = t_{1,-b}t_{-1,0}$ is of the form hx where h is an element of order 3. Now G contains $(3^n - 1)/(3 - 1)$ subgroups of order three and each element of order 3 is inverted by any involution in X . Thus G has $3^n(3^n - 1)/6$ subgroups isomorphic to S_3 . Moreover, G acts transitively by conjugation on the set of subgroups of order three while given $h, h_1, h_2 \in G$ of order 3, the S_3 -subgroups $\langle h, h_1x \rangle$ and $\langle h, h_2x \rangle$ are conjugate under the element $h_1^{-1}h_2$. Thus G has a unique conjugacy class \mathcal{S} of S_3 -subgroups. Moreover, given two distinct involutions $x_1, x_2 \in X$ we have that x_1x_2 has order three. Hence $\Gamma(G, X, \mathcal{S})$ is the complete graph K_{3^n} on 3^n vertices.

In fact the following theorem shows that the only complete graphs which occur as S_3 -involution graphs are those on 3^n vertices.

Theorem 2.5. *Let G be a finite group with X a conjugacy class of involutions and \mathcal{S} a union of conjugacy classes of S_3 -subgroups. If $\Gamma(G, X, \mathcal{S})$ is the complete graph on X then $|X| = 3^n$ for some positive integer n . Moreover, for each positive integer n there exists a group G with an S_3 -involution graph isomorphic to K_{3^n} .*

Proof. Suppose $\Gamma(G, X, \mathcal{S})$ is a complete graph. Then for all $x, y \in X$, xy has order three. Thus $\langle X \rangle$ is a 3-transposition group and by [1, (8.6)], $\langle X \rangle = N \rtimes \langle x \rangle$ for $x \in X$ and N a 3-subgroup. Thus $|x^{\langle X \rangle}|$ is a power of three. Moreover, by Sylow's Theorem, $X = x^{\langle X \rangle}$ and so the first part follows. Example 2.4 provides the required examples for the second part. \square

Our next two examples arise from subgroups of $\text{AGL}(1, 3^n)$.

Example 2.6. Let n be an even positive integer and G be the index two subgroup of $\text{AGL}(1, 3^n)$ given by all maps

$$\begin{aligned} t_{a,b} : \text{GF}(3^n) &\rightarrow \text{GF}(3^n) \\ x &\mapsto ax + b \end{aligned}$$

for any $a, b \in \text{GF}(3^n)$ with a a nonzero square. This group is isomorphic to $C_3^n \rtimes C_{(3^n-1)/2}$ and is a point stabiliser in $\text{PSL}(2, 3^n)$. Then G still has a unique conjugacy class $X = \{t_{-1,b} \mid b \in \text{GF}(3^n)\}$ of involutions. However, G has two conjugacy classes of S_3 -subgroups: those for which the elements of order three are of the form $t_{1,a}$ with a a nonzero square and those for which the elements of order three are of the form $t_{1,a}$ with a a nonsquare. Let \mathcal{S}_1 be the first class. Then $\langle t_{-1,b_1}, t_{-1,b_2} \rangle \in \mathcal{S}_1$ if and only if $b_2 - b_1$ is a nonzero square. Since X is in one-to-one correspondence with the elements of $\text{GF}(3^n)$ it follows that $\Gamma(G, X, \mathcal{S}_1)$ is isomorphic to the graph with vertices the elements of $\text{GF}(3^n)$ such that two elements are adjacent if and only if their difference is a square. (Note that $3^n \equiv 1 \pmod{4}$ and so this relation is symmetric.) Thus $\Gamma(G, X, \mathcal{S}_1)$ is the Paley graph of $\text{GF}(3^n)$.

Example 2.7. Let $G = C_3^2 \rtimes C_2$ where elements of order two in G invert each element of order three. Then G has a unique conjugacy class X of involutions and four conjugacy classes \mathcal{S}_i ($i = 1, 2, 3, 4$) of S_3 -subgroups where each elements from the same conjugacy class share a common C_3 -subgroup. For each i , $\Gamma(G, X, \mathcal{S}_i) \cong 3K_3$ while if $\mathcal{S} = \cup_i \mathcal{S}_i$ then $\Gamma(G, X, \mathcal{S}) \cong K_9$.

2.1 Automorphisms

Given an S_3 -involution graph $\Gamma(G, X, \mathcal{S})$, the group G acts on the set of vertices by conjugation and preserves adjacency. The following lemma collects some information about the action of G .

Lemma 2.8. *Let G be a group with a set X of involutions closed under conjugation and a set \mathcal{S} of S_3 -subgroups closed under conjugation. Let $\Gamma = \Gamma(G, X, \mathcal{S})$.*

- (1). *The orbits of G on the set of vertices are the conjugacy classes in X .*
- (2). *The kernel of the action of G on the set of vertices is $C_G(\langle X \rangle)$.*
- (3). *The orbits of G on the set of arcs are $\{(x, y) \mid \langle x, y \rangle \in \mathcal{S}_i\}$ for each conjugacy class $\mathcal{S}_i \subseteq \mathcal{S}$.*

Proof. Part (1) is trivial. Part (2) follows as an element lies in the the kernel if and only if it centralises each element of X and hence of $\langle X \rangle$. Part (3) follows from the fact that two arcs lie in the same G -orbit if and only if the S_3 -subgroups generated by their vertices are conjugate. \square

The next lemma gives a natural way to find extra automorphisms of S_3 -involution graphs.

Lemma 2.9. *Let G be a group with conjugacy classes X_1, X_2 of involutions and $\mathcal{S}_1, \mathcal{S}_2$ of S_3 -subgroups. If there exists $g \in \text{Aut}(G)$ such that $X_1^g = X_2$ and $\mathcal{S}_1^g = \mathcal{S}_2$ then $\Gamma(G, X_1, \mathcal{S}_1) \cong \Gamma(G, X_2, \mathcal{S}_2)$.*

Proof. The element g provides the isomorphism. \square

Corollary 2.10. *If $g \in \text{Aut}(G)$ fixes X_1 and \mathcal{S}_1 setwise then g induces an automorphism of $\Gamma(G, X_1, \mathcal{S}_1)$.*

By [8, Theorem 9.1.2], if $n \geq 5$ then $\text{Aut}(J(n, 2)) = S_n$ and so the graphs in Example 2.1 give examples where the full automorphism group of $\Gamma(G, X, \mathcal{S})$ is G . The line graph of the Petersen graph is the S_3 -involution graph of A_5 and has full automorphism group S_5 . This provides examples of automorphism provided by Corollary 2.10. We saw in Example 2.3 that M_{11} has an S_3 -involution graph isomorphic to $J(11, 3)$. Thus its full automorphism group is S_{11} which is much bigger than G . Similarly, M_{12} has an S_3 -involution graph isomorphic to $J(12, 4)$ whose full automorphism group is S_{12} .

2.2 Connectivity

We begin with the following lemma.

Lemma 2.11. *Let $\Gamma = (G, X, \mathcal{S})$ be an S_3 -involution graph. If $X = \dot{\cup} X_i$ with each X_i a G -conjugacy class then Γ is the vertex disjoint union of the graphs $\Gamma(G, X_i, \mathcal{S})$.*

Proof. Suppose that $x, y \in X$ lie in the same connected component of Γ . Then there exists a path $x = x_0, x_1, \dots, x_d = y$ in Γ . For each $i = 0, \dots, d-1$, we have $\langle x_i, x_{i+1} \rangle \cong S_3$ and so x_i is conjugate to x_{i+1} . Hence x is conjugate to y and the result follows. \square

Note that if X is a single conjugacy class then $\Gamma(G, X, \mathcal{S})$ is not necessarily connected. For example, let $G = S_3 \text{ wr } S_2$ and $X = \{(x, 1), (1, y) \mid x, y \in S_3, o(x) = o(y) = 2\}$, a conjugacy class of involutions. If $\mathcal{S} = \{S_3 \times 1, 1 \times S_3\}$ then $\Gamma(G, X, \mathcal{S})$ consists of two disjoint triangles. (Note that in this example $\langle X \rangle \neq G$ and X is not a $\langle X \rangle$ -conjugacy class).

We have the following lemma.

Lemma 2.12. *Let $N \triangleleft G$ and let $X \subset N$. Then $\Gamma(G, X, \mathcal{S}) = \Gamma(N, X, \mathcal{S})$. In particular, $\Gamma(G, X, \mathcal{S}) = \Gamma(\langle X \rangle, X, \mathcal{S})$.*

Proof. Since the involutions of elements of \mathcal{S} lie in X and hence N , it follows that for $S \in \mathcal{S}$ we have $S \leq N$. \square

Lemma 2.12 leads to the following necessary condition for connectedness.

Lemma 2.13. *If $\Gamma(G, X, \mathcal{S})$ is connected, then X is both a G -conjugacy class and an $\langle X \rangle$ -conjugacy class.*

Proof. By Lemma 2.12, $\Gamma(G, X, \mathcal{S}) = \Gamma(\langle X \rangle, X, \mathcal{S})$. By Lemma 2.11, if $\Gamma(G, X, \mathcal{S})$ is connected then X is a G -conjugacy class, while if $\Gamma(\langle X \rangle, X, \mathcal{S})$ is connected then X is an $\langle X \rangle$ -conjugacy class. \square

Example 2.7 is an example where X is both a G -conjugacy class and an $\langle X \rangle$ -conjugacy class but $\Gamma(G, X, \mathcal{S}_i)$ is still disconnected.

To obtain a necessary and sufficient condition for an S_3 -involution graph $\Gamma(G, X, \mathcal{S})$ to be connected with X a single conjugacy class we need to introduce a general method for constructing vertex-transitive graphs, see for example [20].

Let G be a group with subgroup H and D a union of double cosets of H in G which is closed under inverses. We can define a graph $\mathcal{G}(G, H, D)$ with vertices the right cosets of H in G such that Hg_1 is adjacent to Hg_2 if and only if $g_1g_2^{-1} \in D$. Then G acts

vertex-transitively on $\mathcal{G}(G, H, D)$ by right multiplication preserving adjacency. Moreover, $\mathcal{G}(G, H, D)$ is connected if and only if D generates G [20, Theorem 7]. Conversely, given a G -vertex-transitive graph Γ with arbitrary vertex v , $\Gamma \cong \mathcal{G}(G, H, D)$ where $H = G_v$ and D is the set of all elements of G which map v to a vertex adjacent to v [20, Theorem 1].

Lemma 2.14. *Let X be a conjugacy class of involutions in G and let $\mathcal{S} = \mathcal{S}_1 \cup \dots \cup \mathcal{S}_t$ be a union of t conjugacy classes of S_3 -subgroups of G such the involutions in elements of \mathcal{S} are contained in X . Let $x \in X$ and for each i , let $y_i \in X$ such that $\langle x, y_i \rangle \in \mathcal{S}_i$. Then $\Gamma(G, X, \mathcal{S})$ is connected if and only if $\langle C_G(x), y_1, \dots, y_t \rangle = G$.*

Proof. Let $\Gamma = \Gamma(G, X, \mathcal{S})$ and let $H = C_G(x)$. For each i , $\{x, y_i, y_i x y_i\}$ forms a triangle in Γ and y_i maps x to the adjacent vertex $y_i x y_i$. Moreover, the double coset $H y_i H$ is the set of all elements of G mapping x to a neighbour u of x such that $\langle x, u \rangle \in \mathcal{S}_i$. Thus letting $D = H y_1 H \cup \dots \cup H y_t H$, [20, Theorem 1] implies that $\Gamma \cong \mathcal{G}(G, H, D)$. Hence by [20, Theorem 7], Γ is connected if and only if D generates G . Since $\langle D \rangle = \langle C_G(x), y_1, \dots, y_t \rangle$ the result follows. \square

2.3 Triangles

Let $S \in \mathcal{S}$ and $T(S) = \{x, y, z\}$ be the set of three involutions in S . Then $T(S)$ is a triangle in $\Gamma(G, X, \mathcal{S})$. In particular, note that all S_3 -involution graphs have girth three.

Given a graph Γ and partition \mathcal{P} of the edge-set of Γ , we say that (Γ, \mathcal{P}) is a G -arc-symmetrical decomposition if G preserves \mathcal{P} , $G^{\mathcal{P}}$ is transitive, G acts transitively on the set of arcs of Γ and for $P \in \mathcal{P}$, G_P is transitive on the set of arcs of P . Arc-symmetrical decompositions were introduced in [15].

Lemma 2.15. *Let G be a group with conjugacy class X of involutions and conjugacy class \mathcal{S} of S_3 -subgroups containing elements of X . Let $\Gamma = \Gamma(G, X, \mathcal{S})$ and $\mathcal{P} = \{T(S) \mid S \in \mathcal{S}\}$. Then (Γ, \mathcal{P}) is a G -arc-symmetrical decomposition.*

Proof. Each edge of Γ lies in a unique triangle $T(S)$, $S \in \mathcal{S}$ and so \mathcal{P} is a partition of $E\Gamma$ preserved by G . Since \mathcal{S} is a G -conjugacy class, $G^{\mathcal{P}}$ is transitive. Moreover, given $P \in \mathcal{P}$, $G_P^P \cong S_3$ which is transitive on the six arcs of P . Hence (Γ, \mathcal{P}) is a G -arc-symmetrical decomposition. \square

An S_3 -involution graph may or may not contain triangles other than those of the form $T(S)$ for $S \in \mathcal{S}$. The following theorem determines what subgroups arise in G if $\Gamma(G, X, \mathcal{S})$ has extra triangles.

Theorem 2.16. *Let G be a finite group with conjugacy class X of involutions and union of conjugacy classes \mathcal{S} of S_3 -subgroups. If $\{x, y, z\}$ is a triangle in $\Gamma(G, X, \mathcal{S})$ then $\langle x, y, z \rangle \cong C_n^2 \rtimes S_3$ or $(C_{3n} \times C_n) \rtimes S_3$ for some $n \geq 1$.*

Proof. Let $x, y \in S$ be involutions and hence $\{x, y\}$ is an edge of $\Gamma = \Gamma(G, X, \mathcal{S})$. Suppose that $z \in X$ such that $\{x, y, z\}$ is a triangle of Γ not obtained by a subgroup in \mathcal{S} and let $R = \langle x, y, z \rangle$. Note that $\langle x, y \rangle < R$ and R satisfies the relations $x^2 = y^2 = z^2 = (xy)^3 = (xz)^3 = (yz)^3 = 1$, and hence is a finite quotient of the affine Coxeter

group \tilde{A}_2 , which is isomorphic to $\mathbb{Z}^2 \rtimes S_3$. We model this Coxeter group by the group $H = L \rtimes W$ where $L = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1 + x_2 + x_3 = 0\}$ and $W \cong S_3$ acts on L by naturally permuting the coordinates. Note that L is generated by $(1, -1, 0)$ and $(0, -1, 1)$. The normal subgroups of H are determined in [21, Theorem 0.2] and are either subgroups of L or are kernels of homomorphisms from H to a Coxeter group induced by a homomorphism from the Coxeter graph of H . As the Coxeter graph for H is a triangle, the only possible images of such homomorphisms have Coxeter graph a single edge or a single vertex and so the quotients obtained are C_2 or S_3 . Since R contains the proper subgroup $\langle x, y \rangle \cong S_3$ it follows that R is the quotient of H by a proper subgroup of L . By [21, Proposition 7.2], the normal subgroups contained in L are integral multiples of one of the lattices $\Lambda_1 = L + \mathbb{Z}\omega$ and $\Lambda_3 = 3L + \mathbb{Z}\omega$ where $\omega = (1, 1, -2) \in L$. Since $\omega \in L$ we have $\Lambda_1 = L$ and so nontrivial integral multiples of Λ_1 give the quotients $C_n^2 \rtimes S_3$ for some integer $n \geq 2$. Now $L/\Lambda_3 = \langle (1, -1, 0) + \Lambda_3 \rangle \cong C_3$. Let $W = \langle \sigma, \tau \rangle$ with $\sigma^3 = \tau^2 = 1$ such that $(1, -1, 0)^\sigma = (0, 1, -1)$ and $(1, -1, 0)^\tau = (-1, 1, 0)$. Then $((1, -1, 0) + \Lambda_3)^\sigma = (1, -1, 0) + \Lambda_3$ and $((1, -1, 0) + \Lambda_3)^\tau = (2, -2, 0) + \Lambda_3$. Hence $H/\Lambda_3 \cong C_3^2 \rtimes C_2$. For $n \geq 2$, we have $L/(n\Lambda_3) = \langle (1, -1, 0) + n\Lambda_3, (1, 1, -2) + n\Lambda_3 \rangle \cong C_{3n} \times C_n$ and hence $H/(n\Lambda_3) \cong (C_{3n} \times C_n) \rtimes S_3$. \square

We can now prove Theorem 1.3.

Proof of Theorem 1.3. By Theorem 2.16, if G does not contain any subgroups of the form $C_n^2 \rtimes S_3$ for $n \geq 2$, or $(C_{3n} \times C_n) \rtimes S_3$ for some $n \geq 1$ then the only triangles in $\Gamma(G, X, \mathcal{S})$ are those arising from subgroups in \mathcal{S} . Note that if p is a prime dividing n then $C_n^2 \rtimes S_3$ contains $C_p^2 \rtimes S_3$. Also $R = (C_{3n} \times C_n) \rtimes S_3$ contains a subgroup $C_3^2 \rtimes C_2$ seen as follows. Using the notation from the proof of Theorem 1.3, if $R = H/(n\Lambda_3)$ then R contains the subgroup $\langle (n, -n, 0) + n\Lambda_3 \rangle \cong C_3$ which is normalised by W and centralised by σ . Hence $H/(n\Lambda_3)$ contains a subgroup isomorphic to $C_3^2 \rtimes C_2$. \square

The converse of Theorem 1.3 is not true. For example, we saw in Example 2.7 that if $G = C_3^2 \rtimes C_2$ then G has four conjugacy classes of S_3 -subgroups and for each of the classes \mathcal{S}_i , $\Gamma(G, X, \mathcal{S}_i) \cong 3K_3$ with the three triangles arising from the three subgroups in \mathcal{S}_i . The problem is that if in the proof of Theorem 1.3, $R = C_3^2 \rtimes C_2$ then not all S_3 -subgroups of R are R -conjugate and so are not necessarily contained in \mathcal{S} . Similarly, if $R = C_n^2 \rtimes S_3$ with 3 dividing n . On the other hand, if $(3, n) = 1$ then all S_3 -subgroups of $R = C_n^2 \rtimes S_3$ are conjugate in R and hence contained in \mathcal{S} , so in this case we definitely obtain extra triangles.

3 PSL(2, q) graphs

In this section we investigate the S_3 -involution graphs for PSL(2, q). Note that PSL(2, 2) $\cong S_3$ while PSL(2, 3) $\cong A_4$, which does not contain any S_3 -subgroups.

Table 1 collates information about the involutions and S_3 -subgroups of PSL(2, q). This mostly follows from a theorem of Dickson [12, p285–286] (see [17, Theorem 2.1]). When there are two conjugacy classes of S_3 -subgroups the two classes are fused in PGL(2, q). Lemma 3.1 determines the conjugacy classes of S_3 -subgroups when q is a power of three.

Table 1: $\text{PSL}(2, q)$ information

$q \pmod{12}$	$ X $	$C_G(x), x \in X$	# of classes of S_3 's	$ \mathcal{S} $
4, 8	$q^2 - 1$	C_2^r	1	$ G /6$
1	$q(q+1)/2$	D_{q-1}	2	$ G /12$
3	$q(q-1)/2$	D_{q+1}	0	
5	$q(q+1)/2$	D_{q-1}	1	$ G /6$
7	$q(q-1)/2$	D_{q+1}	1	$ G /6$
11	$q(q-1)/2$	D_{q+1}	2	$ G /12$
9	$q(q+1)/2$	D_{q-1}	2	$ G /6$

Lemma 3.1. *Let $G = \text{PSL}(2, 3^r)$. Then the number and sizes of conjugacy classes of S_3 -subgroups is given by the last two columns of Table 1. In particular, G contains S_3 -subgroups if and only if r is even.*

Proof. Let $q = 3^r$ and consider an S_3 -subgroup S , which contains a C_3 -subgroup C . Since the projective line contains $q+1$ points and a C_3 cannot fix 4 points, it follows that C must be contained in a unique point stabiliser $P \cong C_3^r \rtimes C_{(q-1)/2}$ where P is the index two subgroup of $\text{AGL}(1, 3^r)$ given in Example 2.6. Since S normalises C , it fixes the unique fixed point of C and hence $S \leq P$. Thus $(q-1)/2$ is even and so r is even. We saw in Example 2.6 that P has two conjugacy classes of S_3 -subgroups and hence so does G . \square

First we show that the S_3 -involution graphs for $\text{PSL}(2, q)$ are always connected.

Lemma 3.2. *Let $G = \text{PSL}(2, q)$, X the set of involutions in G and \mathcal{S} a conjugacy class of S_3 -subgroups. Then $\Gamma(G, X, \mathcal{S})$ is connected.*

Proof. By Lemma 2.14, we need to prove that in each case $\langle C_G(x), y \rangle = G$ for some $x, y \in X$ such that $\langle x, y \rangle \in \mathcal{S}$.

Suppose first that $q \equiv 1 \pmod{4}$. Then $C_G(x) \cong D_{q-1}$, which is maximal in G for $q \geq 13$. Since $y \notin C_G(x)$, we are finished if $q \geq 13$. If $G = \text{PSL}(2, 5)$, then $C_G(x) \cong C_2^2$ is contained only in maximal subgroups isomorphic to A_4 . However, C_2^2 contains all involutions of the A_4 -subgroup, and so $y \notin A_4$. Hence $\langle C_G(x), y \rangle = G$. If $G = \text{PSL}(2, 9)$, then $C_G(x) \cong D_8$ is contained only in maximal subgroups isomorphic to S_4 . Looking at the permutation representation on 4 points, we see that a central involution in a D_8 cannot be in an S_3 -subgroup of S_4 . Therefore $y \notin S_4$ and so $\langle C_G(x), y \rangle = G$.

Next suppose $q \equiv 3 \pmod{4}$. Then $C_G(x) \cong D_{q+1}$, which is maximal in G for $q \geq 11$, and so $y \notin C_G(x)$ and we are finished. If $G = \text{PSL}(2, 7)$, then $C_G(x) \cong D_8$ is contained only in maximal subgroups isomorphic to S_4 and we can conclude as above.

Finally, suppose $q = 2^r$. Then $C_G(x) \cong C_2^r$, which lies in a unique maximal subgroup $H \cong C_2^r \rtimes C_{q-1}$. If $y \in H$, then y lies in the unique Sylow 2-subgroup of H , and so commutes with x . Therefore $y \notin H$ and hence $\langle C_G(x), y \rangle = G$. \square

Theorem 3.3. *Let $G = \text{PSL}(2, q)$, X the set of involutions in G and \mathcal{S} a conjugacy class of S_3 -subgroups. Then the valency of $\Gamma(G, X, \mathcal{S})$ is given by Table 2 according to the value of $q \pmod{12}$. Moreover if there are two conjugacy classes of S_3 -subgroups in G , the corresponding graphs are isomorphic.*

Table 2: Valencies of $\Gamma(\text{PSL}(2, q), X, \mathcal{S})$

$q \pmod{12}$	valency
4, 8	q
1	$(q - 1)/2$
5	$q - 1$
7	$q + 1$
11	$(q + 1)/2$
9	$q - 1$

Proof. Knowing the number $|X|$ of involutions, and hence of vertices, and the number of S_3 -subgroups in a conjugacy class \mathcal{S} , that is, a third of the number of edges, it is immediate to deduce the valency of $\Gamma(G, X, \mathcal{S})$. If there are two conjugacy classes of S_3 -subgroups, the two classes are fused in $\text{PGL}(2, q)$ and hence by Lemma 2.9 the corresponding graphs are isomorphic. \square

3.1 Cliques

First we analyse triangles.

Lemma 3.4. *Let $G = \text{PSL}(2, q)$ with q even or $q \equiv \pm 3 \pmod{8}$. Then the only triangles in $\Gamma(G, X, \mathcal{S})$ are those arising from the elements of \mathcal{S} .*

Proof. If q is a power of 3 and $q \equiv \pm 3 \pmod{8}$ then q is an odd power of 3 and so G contains no S_3 -subgroups. For the values of q given in the statement, we can see from [12, p285–286] that G does not contain any subgroups of the form $C_3^2 \rtimes C_2$ or $C_p^2 \rtimes S_3$. Hence by Theorem 1.3, the only triangles in $\Gamma(G, X, \mathcal{S})$ are those arising from the elements of \mathcal{S} . \square

Corollary 3.5. *Let $G = \text{PSL}(2, q)$ with q even or $q \equiv \pm 3 \pmod{8}$. Then each edge of $\Gamma(G, X, \mathcal{S})$ lies in a unique triangle and the size of the largest clique is 3.*

For the values of q where Γ contains triangles other than the natural ones it is obvious to ask what is the size of the largest clique.

Theorem 3.6. *Let $G = \text{PSL}(2, 9^e)$, X the unique conjugacy class of involutions in G and \mathcal{S} be a conjugacy class of S_3 -subgroups of G . Then the size of the largest clique in $\Gamma(G, X, \mathcal{S})$ is 3^e .*

Proof. Let $\{x, y\}$ be an edge of $\Gamma = \Gamma(G, X, \mathcal{S})$. Let $\{x, y, z\}$ be a triangle such that $\langle x, y, z \rangle \cong S_3$ and suppose that $\{x, y, u\}$ is another triangle. By Theorem 2.16 and the subgroup structure of G , $\langle x, y, u \rangle \cong C_2^2 \rtimes S_3 \cong S_4$ or $\langle x, y, u \rangle \cong C_3^2 \rtimes C_2$. Indeed, $\langle x, y, u \rangle$ is either isomorphic to $C_n^2 \rtimes S_3$, and the only such possibility in G is for $n = 2$, or it contains an abelian subgroup $C_{3n} \times C_n$, and the only such possibility in G is for $n = 1$.

Since G has two conjugacy classes of S_4 -subgroups ([17]) and two conjugacy classes of S_3 -subgroups it follows that all S_4 -subgroups of G containing $\langle x, y \rangle$ are conjugate. There are $|G|/6$ conjugates of $\langle x, y \rangle$ and $|G|/24$ conjugates in each class of S_4 -subgroups. Thus

$\langle x, y \rangle$ lies in a unique S_4 -subgroup. The edge $\{x, y\}$ lies in the two triangles $\{x, y, z\}$ and $\{x, y, u_1\}$ where $\langle x, y, u_1 \rangle \cong S_4$. Under this isomorphism we can make the identifications $x = (1, 2)$, $y = (1, 3)$, $z = (2, 3)$ and $u_1 = (1, 4)$. Hence z is not adjacent to u_1 and so we do not obtain a clique of size 4 in this way.

Now $\langle x, y \rangle$ is contained in a unique parabolic subgroup $P \cong C_3^{2e} \rtimes C_{(9^e-1)/2}$ and if $\langle x, y, u \rangle \cong C_3^2 \rtimes C_2$ then $\langle x, y, u \rangle \leq P$. Now P is isomorphic to the index two subgroup of $\text{AGL}(1, 3^{2e})$ as in Example 2.6. Moreover, the two conjugacy classes of S_3 -subgroups of P remain separate conjugacy classes in G . Thus the restriction of Γ to the involutions of P is the Paley graph of $\text{GF}(9^e)$. By [7] the largest clique in the Paley graph of $\text{GF}(9^e)$ has size 3^e . Moreover, [5] proved that such cliques are affine images of subfields $\text{GF}(3^e)$. Thus a clique in P of size 3^e containing $x = t_{-1,a}$ and $y = t_{-1,b}$ corresponds to $(b-a)\text{GF}(3^e) + a$ containing a, b and $-b-a$ (with notation for elements as in Example 2.6). Therefore $z = x^y = t_{-1,-a-b}$ also lies in this clique. Since z is not adjacent to u_1 we cannot make the clique larger by adding u_1 . Thus the largest clique size of Γ is 3^e . \square

Theorem 3.7. *Let $G = \text{PSL}(2, q)$ with $q \equiv \pm 1 \pmod{8}$ not a power of 3. Then the size of the largest clique in $\Gamma(G, X, \mathcal{S})$ is 4 if $q = 25^e$ and 3 otherwise.*

Proof. Assume $\Gamma(G, X, \mathcal{S})$ contains a clique $\{x, y, z, t\}$ of size 4. Let $S = \langle x, y \rangle \cong S_3$. Then S contains only one more involution, so we may assume without loss of generality that $\{x, y, t\}$ is a triangle not generating an S_3 . By Theorem 2.16 and the subgroup structure of G it follows that $\langle x, y, t \rangle \cong S_4$. Note that G has two classes of S_4 -subgroups each of length $|G|/24$ [12] and has $|G|/6$ S_3 -subgroups (in 1 or 2 conjugacy classes). An easy counting argument shows that S is contained in two S_4 -subgroups B_1 and B_2 . Obviously, all S_3 -subgroups in a given S_4 are conjugate. Without loss of generality, we may assume $\langle x, y, t \rangle = B_1$ and that the only other involution of B_1 adjacent to x and y is x^y . Moreover, t is not adjacent to x^y and so we must have that $\langle x, y, z \rangle = B_2$. Since z, t and x^y are the only elements of X adjacent to both x and y , we conclude that $\{x, y, z, t\}$ is a maximal clique. By symmetry, any three involutions in this clique generate an S_4 -subgroup.

Let us look at the relations in $\langle x, y, z, t \rangle$. Because of the clique structure, we have $1 = x^2 = y^2 = z^2 = t^2 = (xy)^3 = (xz)^3 = (yz)^3 = (xt)^3 = (yt)^3 = (zt)^3$. Since x, y, z are three involutions generating an S_4 with pairs generating S_3 's, we also have $1 = (yxyz)^2$ and the relations obtained from this one by permuting the 3 letters. Of course we have similar relations for any 3-subset of $\{x, y, z, t\}$. Now let $x' = x, y' = x^y, z' = y^z$ and $t' = z^t$. It is easily seen that $\langle x, y, z, t \rangle = \langle x', y', z', t' \rangle$. It is also easily proved from the relations described above that $1 = x'^2 = y'^2 = z'^2 = t'^2 = (x'y')^3 = (y'z')^3 = (z't')^3 = (z'x')^2 = (t'x')^2 = (y't')^2$. These relations yield a Coxeter group of type A_4 , and so we have $\langle x, y, z, t \rangle = \langle x', y', z', t' \rangle \cong S_5$. Now $\text{PSL}(2, q)$ contains a subgroup isomorphic to S_5 if and only if $q = 25^e$ (note that $\text{PGL}(2, 5) \cong S_5$) and if G contains an S_5 then it does indeed have a clique of size 4. Thus the size of the largest clique in $\Gamma(G, X, \mathcal{S})$ is 4 if $q = 25^e$ and 3 otherwise. \square

Theorem 1.4 follows from Corollary 3.5 and Theorems 3.6 and 3.7

3.2 Duality

The *dual graph* of an S_3 -involution graph $\Gamma(G, X, \mathcal{S})$ is the graph whose vertices are the S_3 -triangles of $\Gamma(G, X, \mathcal{S})$ (i.e. which correspond to elements of \mathcal{S}) with two triangles being adjacent if they share a vertex. It was seen in [11] that the graph $\Gamma(\mathrm{PSL}(2, 11), X, \mathcal{S})$, with X a conjugacy class of involutions and \mathcal{S} a conjugacy class of S_3 -subgroups, is isomorphic to its dual graph, with the duality between X and \mathcal{S} induced by elements of $\mathrm{PGL}(2, 11) \setminus \mathrm{PSL}(2, 11)$. We now show that the only other value of q for which this happens is $q = 13$. Note that by Corollary 3.5 in both cases the only triangles in $\Gamma(G, X, \mathcal{S})$ are S_3 -triangles.

Theorem 3.8. *Let $G = \mathrm{PSL}(2, q)$, X the unique conjugacy class of involutions and \mathcal{S} a conjugacy class of S_3 -subgroups. Then $\Gamma(G, X, \mathcal{S})$ is isomorphic to its dual graph if and only if $q = 11$ or 13 .*

Proof. For $\Gamma(G, X, \mathcal{S})$ to have a duality between X and \mathcal{S} , the number of vertices must equal the number of S_3 -subgroups in \mathcal{S} . It follows from Table 1 that $q = 11$ or 13 . It remains to prove that in these two cases we do indeed have a duality.

When $q = 11$ or 13 , the group G has two conjugacy classes \mathcal{S} and \mathcal{S}' of S_3 -subgroups and $\Gamma(G, X, \mathcal{S}) \cong \Gamma(G, X, \mathcal{S}')$. By Theorem 3.3, $\Gamma(G, X, \mathcal{S})$ has valency 6, which means that each involution is contained in three subgroups of \mathcal{S} and in three subgroups of \mathcal{S}' . The centraliser in G of an involution x is isomorphic to D_{12} and each D_{12} -subgroup contains a unique S_3 -subgroup in each conjugacy class. We will denote the unique subgroup of \mathcal{S} (resp. \mathcal{S}') in $C_G(x)$ by $s(x)$ (resp. $s'(x)$). Moreover, each subgroup $S \in \mathcal{S} \cup \mathcal{S}'$ is contained in a unique subgroup isomorphic to D_{12} , whose central involution will be denoted by $i(S)$. Notice that s and i are inverse bijections between involutions and elements of \mathcal{S} and s' and i are inverse bijections between involutions and elements of \mathcal{S}' .

We claim that for $x \in X$, $T \in \mathcal{S}$, we have $x \in T$ if and only if x and $i(T)$ commute and $T \cap s'(x) = 1$.

First suppose that $x \in T$. Then $i(T)$ commutes with all elements in the D_{12} containing T and hence with x . We have $C_T(x) = \langle x \rangle$ and $x \notin s'(x) \subset C_G(x)$, therefore $T \cap s'(x) = 1$.

Conversely, suppose that x and $i(T)$ commute, $T \cap s'(x) = 1$, and that $x \notin T$. If $x = i(T)$, then $T = s(x)$, and $s(x) \cap s'(x) \cong C_3$, a contradiction. Hence $x \neq i(T)$ and x is one of the 6 non-central involutions in $C_G(i(T))$. Since $x \notin T = s(i(T))$, we have $x \in s'(i(T))$ and there exists a unique involution $t \in T$ commuting with x . Since $T \cap s'(x) = 1$ it follows that $t \in s(x)$. Now $s(x)$ and T are both in \mathcal{S} and the normaliser of an element in \mathcal{S} acts transitively on its 3 involutions. Thus there exists $g \in G$ mapping $s(x)$ to T and fixing t . Hence $g \in C_G(t)$ and g must map x onto $i(T)$. Moreover, x and $i(T)$ are two commuting involutions involutions of $C_G(t)$ other than t , which implies that one is in $s(t)$ and the other in $s'(t)$. In other words, they are not conjugate in $C_G(t)$, and we get a contradiction. Hence the claim is proved.

Of course, by symmetry, we also have that a vertex $x \in T' \in \mathcal{S}'$ if and only if x and $i(T')$ commute and $T' \cap s(x) = 1$.

Consider the map $s' : X \rightarrow \mathcal{S}'$. By the claim, the vertices x and y are adjacent in $\Gamma(G, X, \mathcal{S})$ if and only if there exists $T \in \mathcal{S}$ such that $i(T)$ commutes with both x and

y , and $T \cap s'(x) = T \cap s'(y) = 1$. Since $x = i(s'(x))$, $y = i(s'(y))$, and $s(i(T)) = T$, using the claim again implies that x and y are adjacent if and only if the involution $i(T)$ is contained in both $s'(x)$ and $s'(y)$. Therefore s' yields an isomorphism from $\Gamma(G, X, \mathcal{S})$ onto the dual graph of $\Gamma(G, X, \mathcal{S}')$. Since $\Gamma(G, X, \mathcal{S}) \cong \Gamma(G, X, \mathcal{S}')$, it follows that $\Gamma(G, X, \mathcal{S})$ is isomorphic to its dual graph. \square

We note that the proof given in [11] for the duality in the case where $q = 11$ is different to the one here. It relies on a geometrical description of the graph.

3.3 Automorphism groups

We now need more details on the different projective groups. The group $\text{PGL}(2, q)$ is the group of all fractional linear transformations

$$t_{a,b,c,d} : z \mapsto \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

of the projective line $L = \{\infty\} \cup \text{GF}(q)$ with the conventions $1/0 = \infty$ and $(a\infty + b)/(c\infty + d) = a/c$. Note that $t_{a,b,c,d} = t_{a',b',c',d'}$ if and only if $(a, b, c, d) = \lambda(a', b', c', d')$ for some $\lambda \neq 0$. The group $\text{PSL}(2, q)$ is then the set of all $t_{a,b,c,d}$ such that $ad - bc$ is a square in $\text{GF}(q)$. The Frobenius map $\phi : z \mapsto z^p$ also acts on L and $\phi^{-1}t_{a,b,c,d}\phi = t_{a^p, b^p, c^p, d^p}$. Then $\text{P}\Gamma\text{L}(2, q) = \langle \text{PGL}(2, q), \phi \rangle$ and $\text{P}\Sigma\text{L}(2, q) = \langle \text{PSL}(2, q), \phi \rangle$.

We split the determination of the full automorphism group into the cases where q is even and odd.

Theorem 3.9. *Let $G = \text{PSL}(2, q)$ with q odd, X the set of involutions in G , \mathcal{S} a conjugacy class of S_3 -subgroups, and let $\Gamma = \Gamma(G, X, \mathcal{S})$. If $q \equiv 1, 9, 11 \pmod{12}$, then $\text{Aut}(\Gamma) = \text{P}\Sigma\text{L}(2, q)$. If $q \equiv 5, 7 \pmod{12}$, then $\text{Aut}(\Gamma) = \text{P}\Gamma\text{L}(2, q)$.*

Proof. Let $A = \text{Aut}(\Gamma)$ and note that $G \leq A \leq \text{Sym}(V\Gamma)$. Using MAGMA [6] the result can be verified for $q = 5, 7, 9$, and 11. Thus we may assume that $q \geq 13$. For $q \equiv \pm 1 \pmod{4}$, we have the stabiliser in G of a vertex is $D_{q \mp 1}$ and $|V\Gamma| = q(q \pm 1)/2$. Moreover, by Theorem 3.3, Γ has valency $q \mp 1$ or $(q \mp 1)/2$ and so $\text{Alt}(V\Gamma)$ is not contained in A . Since $q \geq 13$, the subgroups D_{q-1} and D_{q+1} are maximal in G (see [12]) and hence both G and A act primitively on $V\Gamma$. Moreover, G and A share a common non-trivial orbital given by the edges of Γ . Using the main theorem of [19], we conclude that either $\text{soc}(A) = \text{PSL}(2, q)$ or $q \equiv 1 \pmod{4}$ and $\text{soc}(A) = A_{q+1}$. In this last case, A acts on $V\Gamma$ as on 2-sets of a $(q+1)$ -set. Therefore the stabiliser in A of a vertex has orbit sizes 1, $2(q-1)$ and $\binom{q-1}{2}$ on vertices. Since Γ has valency $q-1$ or $(q-1)/2$, we conclude that $\text{soc}(A) \neq A_{q+1}$. Hence $\text{soc}(A) = \text{PSL}(2, q)$ and so $G \leq A \leq \text{P}\Gamma\text{L}(2, q)$.

When $q \equiv 5, 7 \pmod{12}$, Table 1 states that G has a unique conjugacy class of S_3 -subgroups. Thus $\text{P}\Gamma\text{L}(2, q)$ fixes X and \mathcal{S} and so by Corollary 2.10, $A = \text{P}\Gamma\text{L}(2, q)$.

For $q \equiv 1, 9, 11 \pmod{12}$, Table 1 states that G has two conjugacy classes \mathcal{S} and \mathcal{S}' of S_3 -subgroups. These two classes are fused by any $g \in \text{PGL}(2, q) \setminus \text{PSL}(2, q)$ and so such elements do not induce automorphisms of Γ . Thus if $q = p$ then we have $A = G = \text{P}\Sigma\text{L}(2, q)$. When $q = p^f$ with $f \geq 2$ and $p \neq 3$, by Table 1, the $\text{PSL}(2, p)$ -subgroup centralised by ϕ contains an S_3 and so ϕ fixes \mathcal{S} and \mathcal{S}' . Thus again $A = \text{P}\Sigma\text{L}(2, q)$. Finally when $p = 3$, by Lemma 3.1, f is even and so ϕ centralises a $\text{PGL}(2, 3)$ -subgroup which contains an S_3 . Hence ϕ fixes \mathcal{S} and \mathcal{S}' and so $A = \text{P}\Sigma\text{L}(2, q)$. \square

To deal with the q even case we need the following lemma about the structure of the graph.

Lemma 3.10. *Let $G = \text{PSL}(2, q)$ for $q = 2^e \geq 4$, X be the unique conjugacy class of involutions of G and \mathcal{S} the unique class of S_3 -subgroups. Let $\Gamma = \Gamma(G, X, \mathcal{S})$. Then X can be partitioned into $q + 1$ blocks $X_\alpha = X \cap G_\alpha$ of size $q - 1$ ($\alpha \in \text{GF}(q) \cup \{\infty\}$), such that the subgraph induced by Γ on any two blocks is a matching between those blocks.*

Proof. Let $\alpha, \beta \in \text{GF}(q) \cup \{\infty\}$ with $\alpha \neq \beta$. Then $G_\alpha \cong \text{AGL}(1, q) \cong C_2^e \rtimes C_{q-1}$ and $G_{\alpha, \beta} \cong C_{q-1}$. Thus each element of X lies in a unique point stabiliser of G . Moreover, since each G_α has a unique Sylow 2-subgroup and this subgroup is abelian, all involutions in X_α commute. Thus Γ is a multipartite graph with block set $\mathcal{P} = \{X_\alpha \mid \alpha \in \text{GF}(q) \cup \{\infty\}\}$. Now $G_{X_\alpha} = G_\alpha$ and $G_{\alpha, \beta} = G_{X_\alpha, X_\beta} \cong C_{q-1}$ acts regularly on both X_α and X_β . Since G acts arc-transitively on Γ (Lemma 2.8) it follows that each vertex in X_α is adjacent to at most one vertex in any other block. Using the fact from Theorem 3.3 that Γ has valency q , we can conclude that the subgraph induced on $X_\alpha \cup X_\beta$ is a matching. \square

Theorem 3.11. *Let $G = \text{PSL}(2, q)$ for $q \geq 4$ even, X be the unique conjugacy class of involutions of G and \mathcal{S} the unique class of S_3 -subgroups. Let $\Gamma = \Gamma(G, X, \mathcal{S})$. Then $\text{Aut}(\Gamma) = \text{P}\Gamma\text{L}(2, q)$.*

Proof. Let $A = \text{Aut}(\Gamma)$. Since G has a unique conjugacy class of involutions and of S_3 -subgroups it follows from Corollary 2.10 that $\text{P}\Gamma\text{L}(2, q) \leq A$. By Lemma 3.10, Γ is a multipartite graph with block set $\mathcal{P} = \{X_\alpha \mid \alpha \in \text{GF}(q) \cup \{\infty\}\}$, where $X_\alpha = X \cap G_\alpha$. Let $x \in X_\infty$. By Lemma 3.10, for each $\alpha \neq \infty$, $|\Gamma(x) \cap X_\alpha| = 1$. By Corollary 3.5, each edge of Γ lies in a unique triangle. Hence, A_x preserves a partition of \mathcal{P} into blocks of size 2 given by the triangles containing x .

Let $y \in X_0$ be the unique element of X_0 adjacent to x and let $\{x, y, z\}$ be a triangle. Then $z \in X_\alpha$ for some $\alpha \in \text{GF}(q) \setminus \{0\}$. Note that $G_{X_\infty, X_0, X_\alpha} = 1$. Now let $x' \in X_\infty$ with $x' \neq x$ and let $y' \in X_0$ adjacent to x' . Since G is arc-transitive, there exists $g \in G$ such that $(x, y)^g = (x', y')$. Moreover, $\{x', y', z^g\}$ is the unique triangle of Γ containing $\{x', y'\}$. Since $G_{X_\infty, X_0, X_\alpha} = 1$ and g fixes X_∞ and X_0 , it follows that $z^g \notin X_\infty \cup X_0 \cup X_\alpha$. Hence if $h \in A_{X_\infty, X_0, X_\alpha}$ then $h \in A_x$. Since x was arbitrary, if K is the kernel of the action of A on \mathcal{P} , it follows that K fixes X_∞ pointwise. Since $K \triangleleft A$ and A acts transitively on \mathcal{P} we conclude that $K = 1$, that is, A acts faithfully on \mathcal{P} . Thus $\text{P}\Gamma\text{L}(2, q) \leq A \leq S_{q+1}$. By [18] it follows that either $\text{soc}(A) = \text{PSL}(2, q)$ or $A_{q+1} \leq A$. If $q = 4$ then $\text{PSL}(2, 4) = A_5$ and $\text{P}\Gamma\text{L}(2, 4) = S_5$. Hence $\text{Aut}(\Gamma) = \text{P}\Gamma\text{L}(2, 4)$ in this case. Suppose now that $q \geq 8$ and $A_{q+1} \leq A$. Then there exists $h \in A$ which induces a 3-cycle on \mathcal{P} and fixes X_∞, X_0 and X_α . As we have seen, this implies that $h \in A_x$ and so preserves a partition of \mathcal{P} into blocks of size 2, contradicting h inducing a 3-cycle on \mathcal{P} . Hence A_{q+1} is not contained in A and so $\text{soc}(A) = \text{PSL}(2, q)$. As $\text{P}\Gamma\text{L}(2, q) \leq A$ it follows that $A = \text{P}\Gamma\text{L}(2, q)$. \square

Lemma 3.10 also enables us to determine the diameter of the S_3 -involvement graph of $\text{PSL}(2, q)$ for q even.

Theorem 3.12. *Let $G = \text{PSL}(2, q)$ for $q \geq 4$ even, X be the unique conjugacy class of involutions of G and \mathcal{S} the unique class of S_3 -subgroups. Let $\Gamma = \Gamma(G, X, \mathcal{S})$. Then Γ has diameter 3.*

Proof. For each $\alpha \in \text{GF}(q) \cup \{\infty\}$ let $X_\alpha = X \cap G_\alpha$. By Lemma 3.10, Γ is multipartite with blocks $\{X_\alpha \mid \alpha \in \text{GF}(q) \cup \{\infty\}\}$ such that the graph induced between any two blocks is a complete matching. Let $x \in X$. Without loss of generality we may assume $x \in X_\infty$. By Theorem 3.3, x has q neighbours and these are each in a different X_α , $\alpha \in \text{GF}(q)$. By Corollary 3.5, each of these q vertices is adjacent to exactly one other neighbour of x . Hence each is adjacent to $q - 2$ vertices at distance 2 from x , and, by Lemma 3.10, none of these are in X_∞ . Moreover, we claim that Γ contains no 4-cycle, and so there are exactly $q(q - 2)$ vertices at distance 2 from x . Therefore, the vertices at distance at most 2 from x cover exactly the involutions not in X_∞ . Since the subgraph induced on two blocks of the partition is a matching, all vertices distinct from x in X_∞ are at distance 3 from x . Therefore any vertex is at distance at most 3 from x .

It remains to prove that there is no 4-cycle in Γ . Suppose (x, y, z, t) is a 4-cycle. By Lemma 3.10, the four vertices are in four distinct blocks of the partition. Since G has one orbit on $V\Gamma$ (Lemma 2.8), we may assume $x = t_{1,1,0,1} \in X_\infty$. A calculation shows that the unique involution in X_i ($i \in \text{GF}(q)$) adjacent to x is $t_{i+1,i^2,1,i+1}$. Moreover, for $j \in \text{GF}(q) \setminus \{i\}$, the image of the edge $\{t_{1,1,0,1}, t_{0,1,1,0}\}$ under $t_{i^2+ij,i^2+ij+j,i+j,i+j+1} \in G$ is $\{t_{i+1,i^2,1,i+1}, t_{i^2+j^2+j,j^2,1,i^2+j^2+j}\}$. Thus the unique involution in X_j ($j \in \text{GF}(q) \setminus \{i\}$) adjacent to $t_{i+1,i^2,1,i+1}$ is $t_{i^2+j^2+j,j^2,1,i^2+j^2+j}$. Hence $y = t_{i+1,i^2,1,i+1}$ and $t = t_{i'+1,i'^2,1,i'+1}$ for distinct $i, i' \in \text{GF}(q)$. We know $z \in G_j$ for some $j \in \text{GF}(q) \setminus \{i, i'\}$. Since z is adjacent to both y and t , we have $z = t_{i^2+j^2+j,j^2,1,i^2+j^2+j} = t_{i'^2+j^2+j,j^2,1,i'^2+j^2+j}$. Hence $i^2 + j^2 + j = i'^2 + j^2 + j$, and so $i = i'$, which contradicts the fact that y and t are distinct. \square

Acknowledgments: We thank Jean Doyen, Bernhard Mühlherr and Tim Pentilla for their helpful conversations. We are also grateful to the referee whose comments greatly improved the paper.

The first author is a Collaborateur Scientifique F.R.S.-FNRS (Belgium). The second author holds an Australian Research Council Australian Research Fellowship. This paper was initiated during a visit to UWA by the first author which was supported by the FNRS.

References

- [1] Michael Aschbacher, 3-transposition groups. Cambridge Tracts in Mathematics, 124. Cambridge University Press, Cambridge, 1997.
- [2] C. Bates, D. Bundy, S. Hart and P. Rowley, Commuting involution graphs for sporadic simple groups, *J. Algebra* 316 (2007) 849–868.
- [3] C. Bates, D. Bundy, S. Perkins and P. Rowley, Commuting involution graphs in special linear groups, *Comm. Algebra* 32 (2004) 4179–4196.

- [4] C. Bates, D. Bundy, S. Perkins and P. Rowley, Commuting involution graphs for symmetric groups, *J. Algebra* 266 (2003) 133–153.
- [5] A. Blokhuis, On subsets of $\text{GF}(q^2)$ with square differences, *Indag. Math.* 46 (1984) 369–372.
- [6] Wieb Bosma, John Cannon and Catherine Playoust, The Magma algebra system I: The user language, *J. Symb. Comp.* 24 3/4 (1997) 235–265. Also see the MAGMA home page at <http://www.maths.usyd.edu.au:8000/u/magma/>.
- [7] I. Broere, D. Döman and J. N. Ridley, The clique numbers and chromatic numbers of certain Paley graphs, *Quaestiones Math.* 11 (1988) 91–93.
- [8] A. E. Brouwer, A. M. Cohen and A. Neumaier, Distance-regular graphs. *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*, 18. Springer-Verlag, Berlin, 1989.
- [9] Francis Buekenhout, La géométrie des groupes de Fischer, unpublished notes, Université Libre de Bruxelles, 1974.
- [10] H. Cuyper and J. I. Hall, The 3-transposition groups with trivial centre, *J. Algebra* 178 (1995) 149–193.
- [11] Alice Devillers, Michael Giudici, Cai Heng Li and Cheryl E. Praeger, Some graphs related to the small Mathieu groups, submitted.
- [12] Leonard Eugene Dickson, *Linear groups: With an exposition of the Galois field theory*, Dover Publications Inc., New York, 1958.
- [13] Bernd Fischer, Finite groups generated by 3-transpositions. I. *Invent. Math.* 13 (1971) 232–246.
- [14] Bernd Fischer, *Finite groups generated by 3-transpositions*, University of Warwick, Lecture Notes, 1969.
- [15] Michael Giudici, Cai Heng Li and Cheryl E. Praeger, Symmetrical Covers, Decompositions and Factorisations of Graphs, to appear in: *Applications to Group Theory and Combinatorics*. Eds: Jack Koolen, Jin Ho Kwak and Mingyao Xu. A. A. Balkema Publishers (Taylor & Francis).
- [16] J. I. Hall, The general theory of 3-transposition groups, *Math. Proc. Cambridge Philos. Soc.* 114 (1993) 269–294.
- [17] Oliver H. King, The subgroup structure of finite classical groups in terms of geometric configurations, in *Surveys in combinatorics 2005*, in *London Math. Soc. Lecture Note Ser.* 327, pp. 29–56 Cambridge Univ. Press, Cambridge, 2005.
- [18] Martin W. Liebeck, Cheryl E. Praeger and Jan Saxl, A classification of the maximal subgroups of the finite alternating and symmetric groups. *J. Algebra* 111 (1987) 365–383.

- [19] Martin W. Liebeck, Cheryl E. Praeger and Jan Saxl, Primitive permutation groups with a common suborbit, and edge-transitive graphs. *Proc. London Math. Soc.* (3) 84 (2002) 405–438.
- [20] Peter Lorimer, Vertex-transitive graphs: symmetric graphs of prime valency. *J. Graph Theory* 8 (1984) 55–68.
- [21] George Maxwell, The normal subgroups of finite and affine Coxeter groups. *Proc. London Math. Soc.* (3) 76 (1998) 359–382.