

Maximal subgroups of almost simple groups with socle $\mathrm{PSL}(2, q)$

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Abstract

We determine all maximal subgroups of the almost simple groups with socle $T = \mathrm{PSL}(2, q)$, that is, of all groups G such that $\mathrm{PSL}(2, q) \leq G \leq \mathrm{P}\Gamma\mathrm{L}(2, q)$, with $q \geq 4$.

1 Introduction

The problem of determining the maximal subgroups of the almost simple groups has a long history and has received much attention (see for example [1, 9, 12, 13, 14]). One of its most important applications is in determining all primitive permutation representations of such groups. All subgroups of $\mathrm{PSL}(2, q)$ were determined by Dickson in 1901 (see [5]) and from this one can easily read off the list of maximal subgroups. Recent combinatorial applications [4, 7] required a list of all maximal subgroups of any almost simple group with socle $\mathrm{PSL}(2, q)$, that is, of all groups G such that $\mathrm{PSL}(2, q) \leq G \leq \mathrm{P}\Gamma\mathrm{L}(2, q)$, with $q \geq 4$. There are statements in the literature about maximal subgroups of $\mathrm{PGL}(2, q)$ (for example [8, 9]) and for all such G with small values of q [3], but despite the fact that the general case is folklore, we are unaware of any general treatment in the current literature. The maximal subgroups of all low-dimensional classical groups will be determined in [2] using the framework of Aschacher's Theorem, and this will include all groups with socle $\mathrm{PSL}(2, q)$. The purpose of this note is to use

Dickson's classification of the maximal subgroups of $\text{PSL}(2, q)$ to determine the maximal subgroups of any almost simple group with socle $\text{PSL}(2, q)$. Our proof follows along the lines of the determination in [10] of all maximal subgroups of the almost simple groups with socle a Suzuki group $Sz(q)$ given Suzuki's classification [15] of the maximal subgroups of $Sz(q)$.

The group $\text{PGL}(2, q)$ is the group of all fractional linear transformations

$$t_{a,b,c,d} : z \mapsto \frac{az + b}{cz + d}$$

of the projective line $X = \{\infty\} \cup \text{GF}(q)$, where $a, b, c, d \in \text{GF}(q)$ with $ad - bc \neq 0$. Note that $t_{a,b,c,d} = t_{\lambda a, \lambda b, \lambda c, \lambda d}$ for all $\lambda \in \text{GF}(q)$. Then $\text{PSL}(2, q) = \{t_{a,b,c,d} \mid ad - bc \text{ is a square}\}$. Let ξ be a primitive element of $\text{GF}(q)$ and $\delta = t_{\xi, 0, 0, 1}$. Then $\text{PGL}(2, q) = \langle \text{PSL}(2, q), \delta \rangle$. For $q = p^f$, with p prime, we can also define the map $\phi : z \mapsto z^p$ on X . Then $\text{P}\Gamma\text{L}(2, q) = \langle \text{PGL}(2, q), \phi \rangle$ and we define $\text{P}\Sigma\text{L}(2, q) = \langle \text{PSL}(2, q), \phi \rangle$.

Let T be a finite nonabelian simple group and G be an almost simple group with socle T . Subgroups of G containing T correspond to subgroups of G/T so we concentrate on the subgroups of G which do not contain T . If M is a maximal subgroup of G then $M \cap T$ is not necessarily maximal in T . If $M \cap T$ is not maximal in T then M is called a *novelty*. It is the possible existence of novelties which requires that extra work needs to be done to determine the maximal subgroups of G . The following theorem lists all novelty maximal subgroups of almost simple groups with socle $\text{PSL}(2, q)$.

Theorem 1.1. *Let $T = \text{PSL}(2, q) \leq G \leq \text{P}\Gamma\text{L}(2, q)$ and let M be a maximal subgroup of G which does not contain T . Then either $M \cap T$ is maximal in T , or G and M are given in Table 1.*

Corollary 1.2. *Let $\text{PGL}(2, q) \leq G \leq \text{P}\Gamma\text{L}(2, q)$ and suppose that M is a maximal subgroup of G . Then $M \cap \text{PGL}(2, q)$ is maximal in $\text{PGL}(2, q)$.*

Theorem 1.1 is proved by combining Propositions 3.1, 3.2, 3.3 and 3.4. From these propositions we can also list the maximal subgroups of any almost simple group with socle $\text{PSL}(2, q)$. This provides the following two results.

Theorem 1.3. *Let $G = \text{P}\Sigma\text{L}(2, q)$ for $q = p^f$ for p an odd prime and $f \geq 2$. Then the maximal subgroups of G which do not contain $\text{PSL}(2, q)$ are:*

- (1). *the stabiliser of a point of the projective line,*
- (2). *$N_G(D_{q-1})$ for $q \neq 9$,*
- (3). *$N_G(D_{q+1})$ for $q \neq 9$,*

G	M
$\text{PGL}(2, 7)$	$N_G(D_6) = D_{12}$
$\text{PGL}(2, 7)$	$N_G(D_8) = D_{16}$
$\text{PGL}(2, 9)$	$N_G(D_{10}) = D_{20}$
$\text{PGL}(2, 9)$	$N_G(D_8) = D_{16}$
M_{10}	$N_G(D_{10}) = C_5 \rtimes C_4$
M_{10}	$N_G(D_8) = C_8 \rtimes C_2$
$\text{PFL}(2, 9)$	$N_G(D_{10}) = C_{10} \rtimes C_4$
$\text{PFL}(2, 9)$	$N_G(D_8)$
$\text{PGL}(2, 11)$	$N_G(D_{10}) = D_{20}$
$\text{PGL}(2, q), q = p \equiv \pm 11, 19 \pmod{40}$	$N_G(A_4) = S_4$

Table 1: Novelty maximal subgroups

- (4). S_5 for $p \equiv \pm 3 \pmod{10}$ and $f = 2$,
- (5). $N_G(\text{PSL}(2, q_0))$ with $q = q_0^r$ for some prime r (2 conjugacy classes if $r = 2$).

Theorem 1.4. *Let $G = \text{PFL}(2, q)$ for $q \geq 4$ not a prime. Then the maximal subgroups of G which do not contain $\text{PSL}(2, q)$ are:*

- (1). *the stabiliser of a point of the projective line,*
- (2). $N_G(D_{2(q-1)})$,
- (3). $N_G(D_{2(q+1)})$,
- (4). $N_G(\text{PGL}(2, q_0))$ for $q = q_0^r$ with r prime, $q_0 \neq 2$ and r odd if q odd.

By [6, Theorem 2.1], if $G \leq \text{PFL}(2, q)$ is 3-transitive on the projective line then either G contains $\text{PGL}(2, q)$, or $q = p^f$ with p odd, f even and $G = M(s, q) = \langle \text{PSL}(2, q), \phi^s \delta \rangle$ for some divisor s of $f/2$. Note that $M(1, 9) = M_{10}$. We can now list the maximal subgroups of $M(s, q)$.

Theorem 1.5. *Let $G = M(s, q)$ with $q = p^f$ where p is an odd prime and s divides $f/2$. Then the maximal subgroups of G which do not contain $\text{PSL}(2, q)$ are:*

- (1). *the stabiliser of a point of the projective line,*
- (2). $N_G(D_{q-1})$,
- (3). $N_G(D_{q+1})$,
- (4). $N_G(\text{PSL}(2, q_0))$ where $q = q_0^r$ with r an odd prime.

2 Preliminaries

We begin by stating Dickson's result about the maximal subgroups of $\mathrm{PSL}(2, q)$. The result is divided according to the parity of q .

Theorem 2.1. *Let $q = 2^f \geq 4$. Then the maximal subgroups of $\mathrm{PSL}(2, q)$ are:*

- (1). $C_2^f \rtimes C_{q-1}$, that is, the stabiliser of a point of the projective line,
- (2). $D_{2(q-1)}$,
- (3). $D_{2(q+1)}$,
- (4). $\mathrm{PGL}(2, q_0)$, where $q = q_0^r$ for some prime r and $q_0 \neq 2$.

Theorem 2.2. *Let $q = p^f \geq 5$ with p an odd prime. Then the maximal subgroups of $\mathrm{PSL}(2, q)$ are:*

- (1). $C_p^f \rtimes C_{(q-1)/2}$, that is, the stabiliser of a point of a projective line,
- (2). D_{q-1} , for $q \geq 13$,
- (3). D_{q+1} , for $q \neq 7, 9$,
- (4). $\mathrm{PGL}(2, q_0)$, for $q = q_0^2$ (2 conjugacy classes),
- (5). $\mathrm{PSL}(2, q_0)$, for $q = q_0^r$ where r an odd prime,
- (6). A_5 , for $q \equiv \pm 1 \pmod{10}$, where either $q = p$ or $q = p^2$ and $p \equiv \pm 3 \pmod{10}$ (2 conjugacy classes),
- (7). A_4 , for $q = p \equiv \pm 3 \pmod{8}$ and $q \not\equiv \pm 1 \pmod{10}$,
- (8). S_4 , for $q = p \equiv \pm 1 \pmod{8}$ (2 conjugacy classes).

Let $T = \mathrm{PSL}(2, q)$ for $q \geq 4$. Then $\mathrm{Aut}(T) = \mathrm{P}\Gamma\mathrm{L}(2, q) = \langle T, \delta, \phi \rangle$. Now $\mathrm{Out}(T) = \mathrm{Aut}(T)/T = \langle \bar{\delta} \rangle \times \langle \bar{\phi} \rangle \cong C_{(2, q-1)} \times C_f$. We will frame our results in terms of the homomorphism $\rho : \mathrm{Out}(T) \rightarrow \langle \bar{\delta} \rangle$ defined by $(\bar{\delta}^i, \bar{\phi}^j) \mapsto \bar{\delta}^i$ for all i and j . If there is a unique conjugacy class of maximal subgroups of a given isomorphism type then ϕ and δ fix this class setwise. The following lemma deals with the case when there are two conjugacy classes of a given isomorphism type.

Lemma 2.3. *Let $T = \text{PSL}(2, q) \leq G \leq \text{P}\Gamma\text{L}(2, q)$ with q odd and suppose that T has two conjugacy classes of maximal subgroups of T of the same isomorphism type. Then these two classes are fused in G if and only if $\rho(G/T) \neq 1$.*

Proof. By [5], given the conditions on q in parts (4), (6) or (8) of Theorem 2.2, there is a unique conjugacy class of S_4 subgroups, A_5 subgroups and $\text{PGL}(2, q_0)$ subgroups in $\text{PGL}(2, q)$ and so δ fuses each pair of conjugacy classes.

The only cases where $\phi \neq 1$ and there are two classes of isomorphic maximal subgroups are $\text{PGL}(2, q_0)$ when $q = q_0^2$, and A_5 when $q = p^2$ for $p \equiv \pm 3 \pmod{10}$. The subgroup $\{t_{a,b,c,d} \mid a, b, c, d \in \text{GF}(q_0)\} \cong \text{PGL}(2, q_0)$ of T is clearly normalised by ϕ and so the two classes of $\text{PGL}(2, q_0)$ subgroups are fused in G if and only if $\rho(G/T) \neq 1$. Suppose now that $q = p^2$ for $p \equiv \pm 3 \pmod{10}$. Then T has two conjugacy classes C_1, C_2 of A_4 subgroups and two conjugacy classes D_1, D_2 of A_5 subgroups. Each pair of conjugacy classes is fused in $\text{PGL}(2, q)$. Since A_5 has only one conjugacy class of A_4 subgroups, it follows that $C_1 = \{R \leq H \mid H \in D_1, R \cong A_4\}$. As C_1 and C_2 are fused in $\text{PGL}(2, q)$, each A_4 subgroup of T is contained in an A_5 and so $C_2 = \{R \leq H \mid H \in D_2, R \cong A_4\}$. Hence two A_5 subgroups in different T -conjugacy classes do not meet in an A_4 . Let $H = A_5$ and $R \cong A_4$ be a subgroup of H . Then R is contained in some $\text{PGL}(2, p)$ subgroup S of G . Now S is centralised by some element $g\phi$ with $g \in T$ and so $g\phi$ centralises R . Since the only A_5 subgroups containing R are conjugate to H it follows that $H^{g\phi} = H^{g'}$ for some $g' \in T$. Thus ϕ does not fuse D_1 and D_2 and the result follows. \square

Given a group H and prime r we define $O_r(H)$ to be the largest normal r -subgroup of H . Note that $O_r(H)$ is characteristic in H . We say that H is *local* if it normalises an r -subgroup for some prime r , while we say that H is *nonlocal* otherwise. Let $T \triangleleft G$ be groups and H be a subgroup of H . We say that H *extends from T to G* if $G = TN_G(H)$. The following lemmas are combinations of [11, Lemmas 1.3.1, 1.3.2 and 1.3.3].

Lemma 2.4. *Let T be a nonabelian simple group and $T \leq G \leq \text{Aut}(T)$. Suppose that M is a maximal subgroup of G with M not containing T and $M_0 = T \cap M$. Then*

- (1). $M/M_0 \cong G/T$;
- (2). $M = N_G(M_0)$;
- (3). $O_r(M_0) \triangleleft M$;

(4). If $1 < K \leq M_0$ and $K \triangleleft M$ then $M_0 = N_T(K)$;

(5). If M_0 is nonlocal then $C_T(\text{soc}(M_0)) = 1$.

Lemma 2.5. *Let T be a nonabelian simple group and $T \leq G \leq \text{Aut}(T)$. Then the following hold.*

(1). *Suppose that H is a maximal subgroup of T . Then H extends from T to G if and only if $N_G(H)$ is a maximal subgroup of G .*

(2). *A subgroup H of T extends to G if and only if the T -conjugacy class of H is the G -conjugacy class of H .*

(3). *Suppose $1 < H \leq K < T$, and that K extends from T to G and H extends from K to $N_G(K)$. If H is self-normalising in T then $N_G(H) \leq N_G(K)$.*

3 Determining the maximal subgroups

We first deal with the case where $M_0 = M \cap T$ is nonlocal. We have the following proposition.

Proposition 3.1. *Let $T = \text{PSL}(2, q) \leq G \leq \text{P}\Gamma\text{L}(2, q)$ for $q \geq 4$ and let M be a maximal subgroup of G not containing T such that M is nonlocal. Then one of the following holds.*

(1). $q = p \equiv \pm 1 \pmod{10}$, $G = T$ and $M = A_5$. (2 classes)

(2). $q = p^2$, $p \equiv \pm 3 \pmod{10}$, $G = T$ and $M = A_5$. (2 classes)

(3). $q = p^2$, $p \equiv \pm 3 \pmod{10}$, $G = \text{P}\Sigma\text{L}(2, q)$ and $M = S_5$. (2 classes)

(4). q even, $M = N_G(\text{PGL}(2, q_0))$ where $q = q_0^r$ for some prime r and $q_0 \neq 2$.

(5). q odd, $M = N_G(\text{PGL}(2, q_0))$, $q = q_0^2$, and $G \leq \text{P}\Sigma\text{L}(2, q)$. (2 classes)

(6). q odd, $M = N_G(\text{PSL}(2, q_0))$, $q = q_0^r$ for some odd prime r .

In particular, M is not a novelty. Conversely, each case listed is in fact a maximal subgroup.

Proof. Letting $M_0 = M \cap T$ and looking at the list of maximal subgroups of $\text{PSL}(2, q)$, we see that either $M_0 = A_5$ is maximal in T , or $M_0 \leq K$, where $K = \text{PSL}(2, q_0)$ or $\text{PGL}(2, q_0)$ is maximal in T . Suppose first that $M_0 = A_5$ is maximal in T . Then either $q = p \equiv \pm 1 \pmod{10}$ or $q = p^2$ and $p \equiv \pm 3 \pmod{10}$. In the first instance $\text{Out}(T) = C_2$ and the two classes of maximal A_5 subgroups are fused in $\text{PGL}(2, q)$ by Lemma 2.3. Hence we have case (1). In the second case $\text{Out}(T) = C_2^2$. If $G = T$ then we are in case (2). By Lemma 2.3, if $\rho(G/T) \neq 1$ then the two classes of maximal A_5 subgroups are fused in G and so by Lemma 2.5, $N_G(A_5)$ is not maximal in G . This leaves us to consider $G = \text{P}\Sigma\text{L}(2, q)$. Since ϕ fixes both of the T -conjugacy classes of A_5 subgroups it follows from Lemma 2.5 that $N_G(A_5)$ is maximal in G . Moreover, as A_5 is not in a subfield group, ϕ does not centralise A_5 . Hence $N_G(A_5) \cong S_5$ and so case (3) holds.

Suppose next that $M_0 \leq K$, where $K = \text{PSL}(2, q_0)$ or $\text{PGL}(2, q_0)$, with K maximal in T . Then $K = C_T(a)$ for some outer automorphism a of T . Since $\text{Out}(T)$ is abelian, $[M, a] \leq T$. Moreover, $[M, a]$ commutes with M_0 and so $[M, a] \leq C_T(M_0)$. By Lemma 2.4, $C_T(\text{soc}(M_0)) = 1$ and so $[M, a] = 1$. Since M is maximal in G it follows that $M = C_G(a)$. Thus $M_0 = C_T(a)$ and so $M_0 = K$ and M is not a novelty. If q is even or $q = q_0^r$ for r an odd prime, then there is a unique conjugacy class of maximal subgroups $M_0 = \text{PSL}(2, q_0)$. Thus Lemma 2.5 implies that $M = N_G(M_0)$ is maximal and we get cases (4) and (6). If $q = q_0^2$ with q odd then there are two classes of maximal $\text{PGL}(2, q_0)$ subgroups in T and by Lemma 2.3, these are fused in G if and only if $\rho(G/T) \neq 1$. Thus $N_G(\text{PGL}(2, q_0))$ is maximal in G if and only if $G \leq \text{P}\Sigma\text{L}(2, q)$. This gives case (5). \square

When $M_0 = M \cap T$ is local, there is some prime r such that $O_r(M_0) \neq 1$. Then $O_r(M_0)$ has nontrivial centre Z , which is characteristic in M_0 . Moreover, Z has a unique maximal elementary abelian subgroup (the group generated by all elementary abelian subgroups of Z) and so this is also characteristic in M_0 . Hence when M_0 is local there is an elementary abelian r -subgroup E of M_0 such that $E \triangleleft M$. There are three cases to consider: $r = p$, $r = 2$ and r is an odd prime dividing $q \pm 1$.

Proposition 3.2. *Let $T = \text{PSL}(2, q) \leq G \leq \text{P}\Gamma\text{L}(2, q)$ with $q = p^f \geq 4$ for some prime p , and suppose that M is a maximal subgroup of G which normalises an elementary abelian p -subgroup E of $M_0 = T \cap M$. Then M is the stabiliser of a point of the projective line. In particular, M is not a novelty. Conversely, the stabiliser in G of a point of the projective line is maximal.*

Proof. By Lemma 2.4, $M = N_G(E)$. Moreover, E is contained in a Sylow p -

subgroup P of T and so is contained in some stabiliser $K = P \rtimes C_{(q-1)/(2,q-1)}$ in T of a point of the projective line. Since P is abelian $P \leq M_0$, and since the only maximal subgroup containing P is K we have $M_0 \leq K$. Moreover, as K/P is abelian it follows that $M_0 \triangleleft K$. Since $M = N_G(M_0)$ we have $M_0 = K$ and so $M = N_G(K)$. Thus M is the stabiliser of a point of the projective line. Moreover, since K is maximal in T it follows that M is not a novelty and as T has only one conjugacy class of subgroups isomorphic to K , Lemma 2.5 implies that the stabiliser in G of a point of the projective line is maximal. \square

Proposition 3.3. *Let $T = \text{PSL}(2, q) \leq G \leq \text{P}\Gamma\text{L}(2, q)$ with $q \geq 4$ and let r be an odd prime dividing $q \pm 1$. If M is a maximal subgroup of G which normalises an elementary abelian r -subgroup E of $M_0 = M \cap T$ for some odd prime r dividing $q \pm 1$ then $M = N_G(D_{(2,q-1)(q\pm 1)})$. Conversely, $N_G(D_{(2,q-1)(q\pm 1)})$ is maximal in G except*

- (1). $N_G(D_4)$ when $G = \text{PSL}(2, 5)$ or $\text{PGL}(2, 5)$,
- (2). D_8 or D_6 when $G = \text{PSL}(2, 7)$,
- (3). $N_G(D_{10})$ or $N_G(D_8)$ when $G = \text{PSL}(2, 9)$ or $\text{P}\Sigma\text{L}(2, 9)$,
- (4). D_{10} when $G = \text{PSL}(2, 11)$.

Proof. Since r is an odd prime dividing $q \pm 1$ it follows that $N_T(E) = D_{(2,q-1)(q\pm 1)}$. Hence $M_0 = D_{(2,q-1)(q\pm 1)}$ and $M = N_G(M_0)$. For $q \neq 7, 9$, Theorems 2.1 and 2.2 imply that $D_{(2,q-1)(q\pm 1)}$ is maximal in T and since there is a unique conjugacy class in T of such subgroups, Lemma 2.5 implies that $N_G(D_{(2,q-1)(q\pm 1)})$ is maximal in G . Similarly, when $q \geq 13$ then $N_G(D_{(2,q-1)(q-1)})$ is maximal in G . The assertions about the maximality of $N_G(D_{(2,q-1)(q\pm 1)})$ for small values of q can then be checked in [3]. \square

Proposition 3.4. *Let $T = \text{PSL}(2, q) \leq G \leq \text{P}\Gamma\text{L}(2, q)$ with $q \geq 5$ odd, and let M be a maximal subgroup of G which normalises an elementary abelian 2-subgroup E of $M_0 = T \cap M$. Then one of the following holds:*

- (1). $|E| = 2$ and $M_0 = D_{q\pm 1}$ where 2 divides $\frac{q\pm 1}{2}$.
- (2). $|E| = 4$ and $M_0 = A_4$ with $q = p \equiv \pm 3 \pmod{8}$. Conversely, $N_G(A_4)$ is maximal in G for these values of q except when $G = \text{PSL}(2, q)$ and $q \equiv \pm 1 \pmod{10}$.
- (3). $|E| = 4$, $M_0 = \text{PSL}(2, 3) \cong A_4$ and $q = 3^r$ for r an odd prime. Conversely, $N_G(M_0)$ is maximal in G in this case.

- (4). $|E| = 4$, $M_0 = \text{PGL}(2, 3) \cong S_4$, $q = 9$ and $G = \text{PSL}(2, 9)$ or $\text{P}\Sigma\text{L}(2, 9)$.
(2 classes) Conversely, $N_G(M_0)$ is maximal in G .
- (5). $|E| = 4$, $M_0 = S_4$, $q = p \equiv \pm 1 \pmod{8}$ and $G = T$ (2 classes).
Conversely M_0 is maximal in G .

Proof. Looking at the list of maximal subgroups of T we note that $|E| = 2$ or 4. If $|E| = 2$ then $M_0 = N_T(E) = D_{q\pm 1}$ where 2 divides $\frac{q\pm 1}{2}$. Thus we have case (1).

If $|E| = 4$ then $M_0 = A_4$ when $q \equiv \pm 3 \pmod{8}$, while $M_0 = S_4$ when $q \equiv \pm 1 \pmod{8}$. Suppose first that M_0 is contained in a subfield group of T , that is $M_0 \leq C_T(a)$ for some field automorphism a . Since $\text{Out}(T)$ is abelian, then $[M, a] \leq C_T(M_0) = 1$. Thus $M = C_G(a)$ and $M_0 = C_T(a)$. Hence q is a power of 3 and $C_T(a) = \text{PSL}(2, 3) \cong A_4$ or $\text{PGL}(2, 3) \cong S_4$. Since M is maximal, $C_G(a)$ is not contained in the centraliser of any other automorphism and so $q = 3^r$ for some prime r . If r is odd then $M_0 = \text{PSL}(2, 3) \cong A_4$ and there is a unique such class. Moreover, $\text{PSL}(2, 3)$ is maximal in $\text{PSL}(2, 3^r)$ for r odd and so by Lemma 2.5, $N_G(M_0)$ is maximal in G . Thus we have case (3). If $r = 2$ then $M_0 = \text{PGL}(2, 3) \cong S_4$ and there are two classes of such subgroups. These classes are maximal in T and by Lemma 2.3 are fused in G if and only if $\rho(G/T) \neq 1$. Hence by Lemma 2.5, $N_G(M_0)$ is maximal in G if and only if $\rho(G/T) = 1$. Thus we have case (4).

Suppose now that M_0 is not contained in a subfield group. Then by Theorem 2.2, either $M_0 = A_4$ and $q = p \equiv \pm 3 \pmod{8}$, or $M_0 = S_4$ and $q = p \equiv \pm 1 \pmod{8}$. Note that the only possibilities for G are then $\text{PSL}(2, q)$ and $\text{PGL}(2, q)$. If $q = p \equiv \pm 1 \pmod{8}$ then there are two classes of S_4 subgroups and these are maximal in T . By Lemma 2.3, they are fused in $\text{PGL}(2, q)$ and so do not extend to maximal subgroups of $\text{PGL}(2, q)$. Hence we have case (5). If $q = p \equiv \pm 3 \pmod{8}$ then there is a unique conjugacy class of A_4 subgroups and $N_{\text{PGL}(2, q)}(A_4) = S_4$ by [5]. Now A_4 is maximal in T if and only if $q \not\equiv \pm 1 \pmod{10}$. If $q \not\equiv \pm 1 \pmod{10}$ then Lemma 2.5 implies that $N_{\text{PGL}(2, q)}(A_4)$ is maximal in $\text{PGL}(2, q)$. If $q \equiv \pm 1 \pmod{10}$ then M_0 is contained in an A_5 . However, by Lemma 2.3, $\text{PGL}(2, q)$ interchanges the two classes of maximal A_5 subgroups of T while there is only one class of A_4 subgroups. Hence in this case we also have $N_{\text{PGL}(2, q)}(A_4)$ is maximal in $\text{PGL}(2, q)$. Thus case (2) holds. \square

Note that the maximality of $N_G(D_{q\pm 1})$ was determined in Proposition 3.3. Collating the results of Propositions 3.1, 3.3 and 3.4 we can deduce Theorems 1.1, 1.3, 1.4 and 1.5 follow. We also obtain the following well known list of maximal subgroups of $\text{PGL}(2, q)$ for q odd.

Theorem 3.5. *Let $G = \mathrm{PGL}(2, q)$ with $q = p^f > 3$ for some odd prime p . Then the maximal subgroups of G not containing $\mathrm{PSL}(2, q)$ are:*

- (1). $C_p^f \rtimes C_{q-1}$.
- (2). $D_{2(q-1)}$, for $q \neq 5$.
- (3). $D_{2(q+1)}$.
- (4). S_4 for $q = p \equiv \pm 3 \pmod{8}$.
- (5). $\mathrm{PGL}(2, q_0)$ for $q = q_0^r$ with r an odd prime.

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References

- [1] M. Aschbacher, On the maximal subgroups of the finite classical groups. *Invent. Math.* 76 (1984), 469–514.
- [2] J. N. Bray, D. F. Holt and C. M. Roney-Dougal, *The maximal subgroups of the low-dimensional classical groups*, in preparation.
- [3] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups* (Clarendon Press, Oxford, 1985).
- [4] A. Devillers, M. Giudici, C. H. Li and C. E. Praeger, Primitive decompositions of Johnson graphs, *in preparation*.
- [5] L. E. Dickson, *Linear groups: With an exposition of the Galois field theory*, Dover Publications Inc., New York, 1958).
- [6] A. Gardiner, C. E. Praeger and S. Zhou, Cross ratio graphs. *J. London Math. Soc. (2)* 64 (2001), 257–272.
- [7] M. Giudici and C. H. Li, On finite edge-primitive graphs, *in preparation*.
- [8] B. Huppert, *Endliche Gruppen. I*, in *Die Grundlehren der Mathematischen Wissenschaften, Band 134* (Springer-Verlag, Berlin, 1967).
- [9] O. H. King, ‘The subgroup structure of finite classical groups in terms of geometric configurations’, in *Surveys in combinatorics 2005*, in *London Math. Soc. Lecture Note Ser.* **327**, pp. 29–56 (Cambridge Univ. Press, Cambridge, 2005).

- [10] P. B. Kleidman, *The subgroup structure of some finite simple groups* (Ph.D. thesis, Cambridge University, 1987).
- [11] P. B. Kleidman, The maximal subgroups of the finite 8-dimensional orthogonal groups $P\Omega_8^+(q)$ and of their automorphism groups. *J. Algebra* 110 (1987), 173–242.
- [12] P. B. Kleidman and M. W. Liebeck, *The subgroup structure of the finite classical groups* in *London Math. Soc. Lecture Note Ser.* **129** (Cambridge University Press, Cambridge, 1990.)
- [13] M. W. Liebeck, C. E. Praeger and J. Saxl, A classification of the maximal subgroups of the finite alternating and symmetric groups. *J. Algebra* 111 (1987), 365–383.
- [14] M. W. Liebeck, G. M. Seitz, ‘A survey of maximal subgroups of exceptional groups of Lie type’, in *Groups, combinatorics & geometry (Durham, 2001)*, pp139–146, (World Sci. Publ., River Edge, NJ, 2003.)
- [15] M. Suzuki, On a class of doubly transitive groups, *Ann. of Math.* (2) 75 1962 105–145.