# GAP workshop on Polar Spaces

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## A Generalised Quadrangle

Let $\mathbb{F} = GF(2)$ be the field with two elements and let $V$ be the 4-dimensional vector space over $\mathbb{F}$. This vector space can be generated in GAP by the command

```
gap> vs := VectorSpace( GF(2), IdentityMat( 4, GF(2) ) );
```

The symplectic group $\mathrm{Sp}(4,2)$ acts on $V$. This group leaves a symplectic form invariant which, with respect to the standard basis is described by the following matrix:

$$F = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

invariant. Note that since we are working over the field $GF(2)$ each 1-dimensional subspace has exactly one non-zero vector in it. Thus we can identify a 1-dimensional subspace of $V$ with its basis vector.

We define a projective space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ as follows:

- Points $\mathcal{P}$ are non-zero vectors in $V$.
- Lines $\mathcal{L}$ are sets $L$, where $L = U \backslash \{\mathbf{0}\}$ and $U$ is a 2-dimensional subspace $U = \langle \mathbf{u}, \mathbf{v} \rangle$ for which $\mathbf{u}^T F \mathbf{v} = 0$. Thus a line consists of the non-zero vectors of a 2-dimensional subspace whose basis vectors have a 0 inner product with respect to $F$. In GAP we can test whether the vectors $\mathbf{u}$ and $\mathbf{v}$ have inner product 0 by testing whether

```
u * F * v = Zero(GF(2));
```

The group $\mathrm{Sp}(4,2)$ can be created in GAP with the command

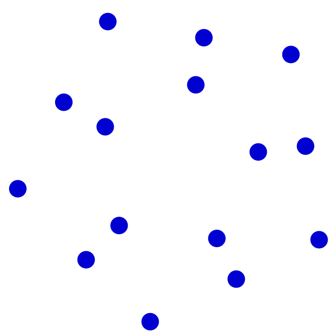
```
gap> grp := SP(4,2);
```

The group is returned as a group of $4 \times 4$ matrices with entries in $\mathbb{F}$.
All 2-dimensional subspaces of a vector-space $V$ can be generated in GAP with the command

```
gap> Subspaces( vs, 2 );
```

## Problems

1. Find a line $\ell_1$ which contains the point $(1, 0, 0, 0)$.
2. Find all lines of $\mathcal{S}$ as the orbit of $\ell_1$ under the action of $\mathrm{Sp}(4, 2)$ on 2-subspaces.
3. How many lines are there in $\mathcal{S}$?
4. How many points are on each line?
5. Is $\mathrm{Sp}(4, 2)$ transitive in its action on points?
6. Draw the graph of $\mathcal{S}$ using the supplied drawing of the points.



7. Using GAP show that not all lines intersect and find two that do not intersect. Inspect your graph to see whether your computation agrees with the drawing.
8. Find a line using GAP that meets both of the two non-intersecting lines.
9. Does the graph satisfy the Generalised Quadrangle Axiom?
10. Are there any triangles in the graph?
11. Construct a permutation group on $\Omega = \{1, \dots, n\}$, where $n$ is the number of points, which acts on $\Omega$ as $\mathrm{Sp}(4, 2)$ acts on $\mathcal{P}$. Use the GAP-functions `Action` and `OnRight`. Store the permutation group in a file for future use.
12. Write a GAP function `OnFlags` which implements the action of $\mathrm{Sp}(4, 2)$ on flags. (A flag is an incident point-line pair.)
13. Construct a permutation group on $\Delta = \{1, \dots, m\}$, where $m$ is the number of flags, which acts on $\Delta$ as $\mathrm{Sp}(4, 2)$ acts on flags.
14. Is $\mathrm{Sp}(4, 2)$ transitive on flags?