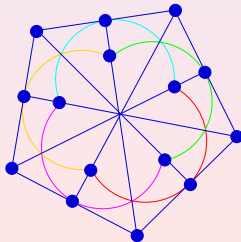


Groups and Geometry

Alice Niemeyer

The University of Western Australia

September 2004



Groups

Factor Groups

Classical Groups

Projective Spaces

Polar Spaces

Symplectic Groups

Generalised Quadrangles

Acknowledgements

Simple Groups

In Chemistry compounds are made up of elements.

Is there an analogue for finite groups?

YES! Simple groups are the building blocks.

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Normal Subgroups

A subgroup N of a group G is called *normal* if $g^{-1}ng \in N$ for all $g \in G$ and $n \in N$.

A group is called *simple* if its only normal subgroups are the group itself and the trivial subgroup.

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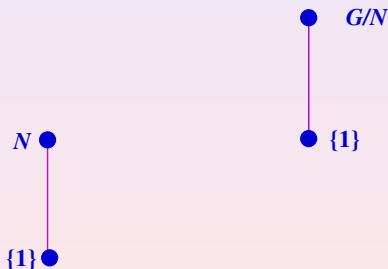
Factor Groups

The cosets of a normal subgroup N in G form a group, called the factor group G/N of G over N .

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Factor Groups

G is built up of N and G/N in a certain way.



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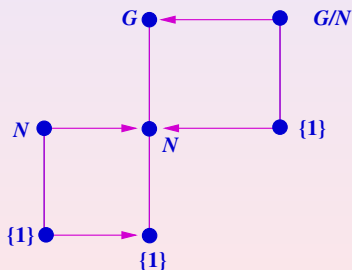
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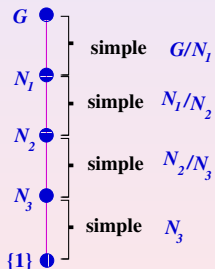
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Factor Groups

We can repeat this process.



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Groups of prime order

If $|G| = p$ and p is prime then G is simple.

$\langle(1, 2, 3)\rangle = \{(), (1, 2, 3), (1, 3, 2)\}$ is simple.

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Alternating Groups

S_n contains a subgroup A_n consisting of all permutations which can be expressed as an *even* number of 2-cycles.

A_n has half as many elements as S_n and is simple. S_n/A_n contains two elements and is also simple.

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Classical Groups

Another important class of finite simple groups arises from matrix groups.

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Classical Groups

\mathbb{F}_q finite field with q elements.
 V d -dimensional vector space over \mathbb{F}_q .
 $GL(d, q)$ group of invertible $d \times d$ matrices with entries in \mathbb{F}_q .

$GL(d, q)$ called **G**eneral **L**inear group.
It is the group of all invertible linear transformations from V to V .

General Linear Group $GL(2, 2)$

$$GL(2, 2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

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General Linear Group

How many elements does $GL(d, q)$ have?

$|GL(d, q)|$ equals the number of different bases $[v_1, \dots, v_d]$ for V .

- ▶ $q^d - 1$ choices for first vector
- ▶ $q^d - q$ choices for second vector
- ▶ $q^d - q^2$ choices for third
- ▶ \vdots

General Linear Group

How many elements does $GL(d, q)$ have?

$|GL(d, q)|$ equals the number of different bases $[v_1, \dots, v_d]$ for V .

$$\begin{aligned} |GL(d, q)| &= (q^d - 1)(q^d - q) \cdots (q^d - q^{d-1}) \\ &= q^{d(d-1)/2} \prod_{i=1}^d (q^i - 1). \end{aligned}$$

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General Linear Group

How many elements does $GL(d, q)$ have?

$|GL(d, q)|$ equals the number of different bases $[v_1, \dots, v_d]$ for V .

```
gap Size( GL(4, 3 ) );
```

```
> 24261120
```

```
gap 3^6 * (3-1)*(3^2-1)*(3^3-1)*(3^4-1);
```

```
> 24261120
```

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Action on V

$GL(d, q)$ maps vectors in V to vectors in V :

$$g \in GL(d, q) \text{ maps } \mathbf{v} \in V \text{ to } \mathbf{v}g.$$

We say $GL(d, q)$ acts on V .

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Example action on V

Let $\mathbb{F} = GF(2)$ and $d = 2$. Let

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \mathbf{v} = (1, 1). \text{ Then}$$

$$\mathbf{v}g = (1, 1) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = (1, 0)$$

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Special Linear Group

$SL(d, q)$ is the subgroup of $GL(d, q)$ consisting of all elements with determinant 1.

$$|GL(d, q)| = (q - 1)|SL(d, q)|.$$

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Action on subspaces of V

$GL(d, q)$ maps 1-dimensional subspaces of V to 1-dimensional subspaces:
 $\langle \mathbf{x} \rangle$ is mapped by g to $\langle \mathbf{x}g \rangle$.

$$P(V) = \{ \langle \mathbf{v} \rangle \mid \mathbf{v} \neq \mathbf{0} \}$$

is the set of 1-dimensional subspaces.

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Projective Spaces

A projective space consists of *points*, *lines*, *planes*, etc.

- ▶ Points: 1-dimensional subspaces
- ▶ Lines: 2-dimensional subspaces
- ▶ Planes: the 3-dimensional subspaces
- ▶ etc

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Projective Spaces

The point $\langle \mathbf{v} \rangle$ lies on the line $\langle \mathbf{x}, \mathbf{y} \rangle$ if

$$\langle \mathbf{v} \rangle \subseteq \langle \mathbf{x}, \mathbf{y} \rangle.$$

The line U is contained in the plane W if $U \subseteq W$.

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Projective Spaces

The incidence structure is defined by inclusion.

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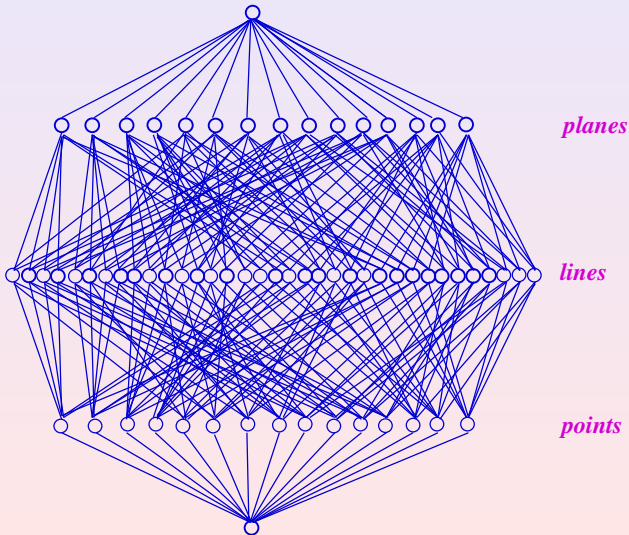
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$GF(2)^4$ generates $PG(3, 2)$



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$PG(d - 1, q)$

$PG(d - 1, q)$ is projective space generated by the vector space V of dimension d .

We say that the *projective dimension* of $PG(d - 1, q)$ is $d - 1$.

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Action on the Projective Space

$GL(d, q)$ acts on the projective space:

$g \in GL(d, q)$ maps the r -dimensional subspace $U = \langle \mathbf{u}_1, \dots, \mathbf{u}_r \rangle$ to another r -dimensional subspace, namely $Ug = \langle \mathbf{u}_1g, \dots, \mathbf{u}_rg \rangle$.

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Example

Let $\mathbb{F} = GF(2)$ and $d = 4$.

$$g = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

acts on the 2-dimensional subspace

$$U = \langle (0, 1, 1, 0), (1, 0, 0, 0) \rangle \text{ by}$$
$$Ug = \langle (1, 0, 1, 0), (0, 0, 1, 1) \rangle.$$

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Elements that do nothing

Elements $g = aI$ for $a \in \mathbb{F}^*$ fix each r -dimensional subspace. Hence they act trivially on the projective space.

All other elements act non-trivially.

The Centre of $GL(d, q)$

The center of $GL(d, q)$ is the set of all matrices that commute with all elements of $GL(d, q)$.

$$Z(GL(d, q)) = \{aI \mid a \in \mathbb{F}^*\},$$

so the centre consists of the elements that act trivially on the projective space.

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Projective Linear Groups

$$\mathrm{PGL}(d, q) = \mathrm{GL}(d, q) / Z(\mathrm{GL}(d, q))$$

is the *projective linear group*.

$$\mathrm{PSL}(d, q) = \mathrm{SL}(d, q) / Z(\mathrm{SL}(d, q)).$$

is the *projective special linear group*.

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Simplicity of Projective Special Linear Groups

For $d \geq 2$ and $(d, q) \neq (2, 2), (2, 3)$
 $\text{PSL}(d, q)$ is simple.

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Collineations

A permutation g of the points is a *collineation* of a projective space, if it maps lines to lines.

Collineations map subspaces to subspaces.

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Automorphism Groups of Projective Spaces

The *automorphism group* of $PG(d - 1, q)$ is the set of all collineations.

Note that $PGL(d, q)$ is a subgroup of the full automorphism group of $PG(d - 1, q)$ for $d \geq 3$.

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Full automorphism group

The full automorphism group of $PG(d-1, q)$ for $d \geq 3$ is $P\Gamma L(d, q)$ and it is not much larger than $PGL(d, q)$.

Note $P\Gamma L(d, q) = PGL(d, q)$ when q is prime.

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How do we prove this?

Base and Stabiliser Chain argument.

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Groups and Projective Spaces

- ▶ Through the study of projective spaces we learn about their automorphism groups.
- ▶ Through the study of the automorphism groups we learn about the underlying geometries.

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Other subgroups

$GL(d, q)$ has other important subgroups.

These arise as subgroups which preserve certain types of inner products.

Here we only look closely at one example, namely the symplectic group.

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Polar Spaces

We define a new incidence structure from $PG(d - 1, q)$ by deleting various subspaces.

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Symplectic Forms

$f : V \times V \rightarrow \mathbb{F}$ is *symplectic* if for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a \in \mathbb{F}$

- ▶ $f(\mathbf{u} + \mathbf{v}, \mathbf{w}) = f(\mathbf{u}, \mathbf{w}) + f(\mathbf{v}, \mathbf{w})$,
- ▶ $f(\mathbf{u}, \mathbf{v} + \mathbf{w}) = f(\mathbf{u}, \mathbf{v}) + f(\mathbf{u}, \mathbf{w})$,
- ▶ $f(a\mathbf{u}, \mathbf{v}) = f(\mathbf{u}, a\mathbf{v}) = af(\mathbf{u}, \mathbf{v})$,

bilinear form

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Symplectic Forms

And also

- ▶ $f(\mathbf{u}, \mathbf{v}) = -f(\mathbf{v}, \mathbf{u})$,
- ▶ $f(\mathbf{x}, \mathbf{y}) = 0$ for all $\mathbf{x} \in V$ implies $\mathbf{y} = 0$,
(non-degenerate)
- ▶ $f(\mathbf{x}, \mathbf{x}) = 0$ for all $\mathbf{x} \in V$.

We write $\mathbf{u} \perp \mathbf{v}$ if $f(\mathbf{u}, \mathbf{v}) = 0$ and say \mathbf{u} is *orthogonal* to \mathbf{v} .

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Example of Symplectic Form

Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Define a form f on $V = GF(2)^4$ by

$$f(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T A \mathbf{v}.$$

Then f is a symplectic form.

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Symplectic Groups

$g \in \text{GL}(d, q)$ *preserves* symplectic form f if

$$f(\mathbf{u}g, \mathbf{v}g) = f(\mathbf{u}, \mathbf{v}) \text{ for all } \mathbf{u}, \mathbf{v} \in V.$$

Define

$$\text{Sp}(d, q) = \{g \in \text{GL}(d, q) \mid g \text{ preserves } f\}.$$

$\text{Sp}(d, q)$ is the *symplectic group*.

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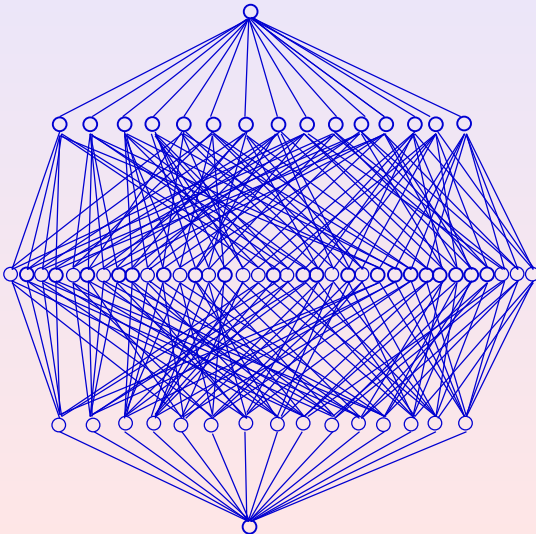
Symplectic Polar Spaces

Let $f : V \times V \rightarrow \mathbb{F}$ be a *symplectic* form.
The symplectic polar space $SP(d - 1, q)$
consists of those subspaces S of
 $PG(d - 1, q)$ for which

$$f(\mathbf{u}, \mathbf{v}) = 0 \text{ for all } \mathbf{u}, \mathbf{v} \in S.$$

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$SP(4, 2)$ where $V = GF(2)^4$



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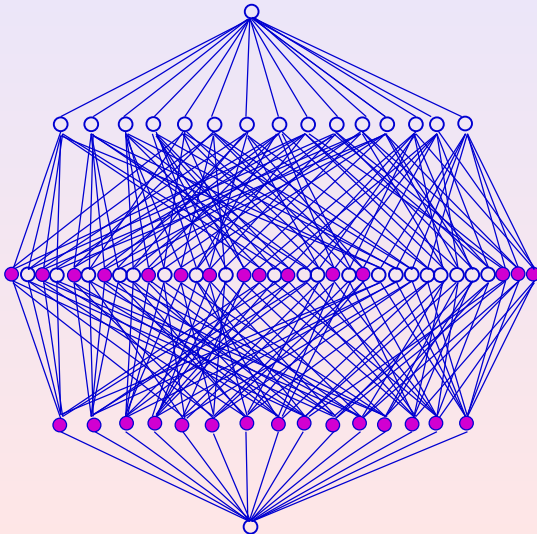
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$SP(4, 2)$ where $V = GF(2)^4$



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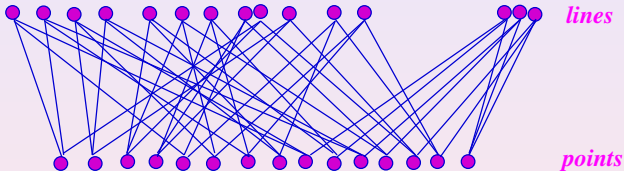
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$SP(4, 2)$ where $V = GF(2)^4$



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Rank of a polar space

The *rank* of a polar space is the number of different types of objects (points, lines, planes) we have.

By a theorem of Witt the rank of a polar space is at most $d/2$.

rank of $SP(4, 2)$ is 2.

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Generalised Quadrangles

Generalised Quadrangles are polar spaces of rank 2.

Axioms for Generalised Quadrangles

A *Generalised Quadrangle* of order (s, t) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ consisting of a point set \mathcal{P} and a line set \mathcal{L} such that

- ▶ Each point lies on $t + 1$ lines;
- ▶ Two distinct points are on at most one line;
- ▶ Each line has $s + 1$ points;
- ▶ Two distinct lines have at most one point in common;

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The GQ-Axiom

For a line ℓ and a point P not on ℓ there is a *unique* line m such that

- ▶ P is on m ;
- ▶ m intersects ℓ in exactly one point.

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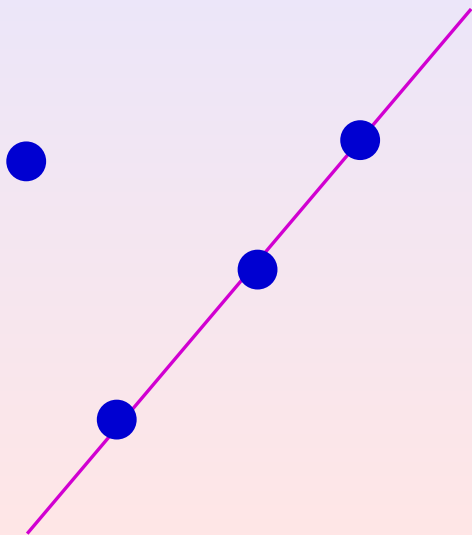
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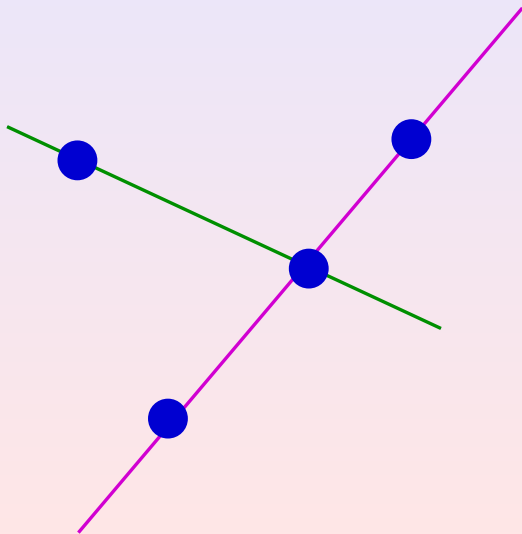
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